Communications in Applied Analysis **19** (2015), 623–642

SOME PEROV'S AND KRASNOSEL'SKII TYPE FIXED POINT RESULTS AND APPLICATION

ABDELGHANI OUAHAB

Laboratory of Mathematics, Univ Sidi Bel Abbes
PoBox 89, 22000 Sidi-Bel-Abbès, Algeria
E-mail: agh_ouahab@yahoo.fr

ABSTRACT. In this paper, we establish a single and multivalued version of a Perov type fixed point theorem. Also in generalized Banach spaces, we extend the Krasnosel'skii type fixed point theorem for the sum of $B+A$, where B is an expansive operator and A is a continuous map. Finally, our results are used to prove the existence of solutions for impulsive differential inclusions.

AMS (MOS) Subject Classification. 47H10,47H30, 54H25.

1. Introduction

The classical Banach contraction principle was extended for contractive maps on spaces endowed with vector-valued metrics by Perov in 1964 [25] and Perov and Kibenko [27]. Until now, there have been a number of attempts to generalize the Perov fixed point theorem in several directions and also there have been a number of applications in various fields of nonlinear analysis, system of ordinary differential and semilinear differential equations. In 1958, Krasnosel'skii [20] established that the equation

$$Bu + Au = u, \quad u \in M, \tag{1.1}$$

has a solution in $M \subseteq E$ where E is a Banach space, A and B satisfy:

(i) $Ax + By \in M$ for all $x, y \in M$.
(ii) A is continuous on M and $\overline{A(M)}$ is a compact set in E.
(iii) B is a k-contraction on X.

That result combined the Banach contraction principle and Schauder's fixed point theorem. The existence of fixed points for the sum of two operators has attracted tremendous interest, and their applications are frequent in nonlinear analysis. Many improvements of Krasnosel'skii's theorem have been established in the literature in the course of time by modifying the above assumptions; see, for example, [2, 4, 7, 11, 12, 13, 16, 18, 23, 32]. Very recently, in generalized Banach spaces, Krasnosel'skii

Received December 15, 2014

type fixed point theorems of single and multivalued operator were studied by Petre [26] and Petre and Petruşel [28].

The paper is organized as follows: in Section 2 we collect some definitions and facts which will be needed in the sequel. Section 3 contains fixed point theorems for expansive maps acting in generalized metric spaces. Section 4 is devoted to establishing a multivalued version type of Perov fixed point theorem. The aim of Section 5 is to prove a Krasnosel'skii theorem by expansive operator perturbation. Finally, our results are used to prove the existence of solution for system of impulsive differential inclusions.

2. Preliminaries

In this section, we recall from the literature some notations, definitions, and auxiliary results which will be used throughout this paper.

Definition 2.1. Let X be a nonempty set. By a vector-valued metric on X we mean a map $d : X \times X \to \mathbb{R}^n$ with the following properties:

(i) $d(u, v) \geq 0$ for all $u, v \in X$; if $d(u, v) = 0$ then $u = v$;
(ii) $d(u, v) = d(v, u)$ for all $u, v \in X$;
(iii) $d(u, v) \leq d(u, w) + d(w, v)$ for all $u, v, w \in X$.

We call the pair (X, d) a generalized metric space. For $r = (r_1, \ldots, r_n) \in \mathbb{R}^n_+$, we will denote by

$$B(x_0, r) = \{x \in X : d(x_0, x) < r\}$$

the open ball centered in x_0 with radius r and

$$\overline{B(x_0, r)} = \{x \in X : d(x_0, x) \leq r\}$$

the closed ball centered in x_0 with radius r. We mention that for generalized metric spaces, the notation of open subset, closed set, convergence, Cauchy sequence and completeness are similar to those in usual metric spaces. If, $x, y \in \mathbb{R}^n$, $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n)$, by $x \leq y$ we mean $x_i \leq y_i$ for all $i = 1, \ldots, n$. Also $|x| = (|x_1|, \ldots, |x_n|)$ and $\max(x, y) = \max(\max(x_1, y_1), \ldots, \max(x_n, y_n))$. If $c \in \mathbb{R}$, then $x \leq c$ means $x_i \leq c$ for each $i = 1, \ldots, n$.

Definition 2.2. A square matrix of real numbers is said to be convergent to zero if and only if its spectral radius $\rho(M)$ is strictly less than 1. In other words, this means that all the eigenvalues of M are in the open unit disc (i.e., $|\lambda| < 1$ for every $\lambda \in \mathbb{C}$ with $det(A - \lambda I) = 0$, where I denote the unit matrix of $\mathcal{M}_{n\times n}(\mathbb{R})$).

Theorem 2.3 ([29]). *Let $M \in \mathcal{M}_{n\times n}(\mathbb{R}_+)$. The following assertions are equivalent:*

(i) *M is convergent to zero;*

(ii) $M^k \to 0$ *as* $k \to \infty$;

(iii) *The matrix* $(I - M)$ *is nonsingular and*

$$(I - M)^{-1} = I + M + M^2 + \cdots + M^k + \cdots;$$

(iv) *The matrix* $(I - M)$ *is nonsingular and* $(I - M)^{-1}$ *has nonnegative elements.*

Definition 2.4. We say that a non-singular matrix $A = (a_{ij})_{1 \le i,j \le n} \in \mathcal{M}_{n \times n}(\mathbb{R})$ has the absolute value property if

$$A^{-1}|A| \le I,$$

where

$$|A| = (|a_{ij}|)_{1 \le i,j \le n} \in \mathcal{M}_{n \times n}(\mathbb{R}_+).$$

Some examples of matrices convergent to zero, $A \in \mathcal{M}_{n \times n}(\mathbb{R})$, which also satisfies the property $(I - A)^{-1}|I - A| \le I$ are:

1) $A = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$, where $a, b \in \mathbb{R}_+$ and $\max(a, b) < 1$

2) $A = \begin{pmatrix} a & -c \\ 0 & b \end{pmatrix}$, where $a, b, c \in \mathbb{R}_+$ and $a + b < 1$, $c < 1$

3) $A = \begin{pmatrix} a & -a \\ b & -b \end{pmatrix}$, where $a, b, c \in \mathbb{R}_+$ and $|a - b| < 1$, $a > 1$, $b > 0$.

Definition 2.5. Let (X, d) be a generalized metric space. An operator $N : X \to X$ is said to be contractive if there exists a convergent to zero matrix M such that

$$d(N(x), N(y)) \le M d(x, y) \quad \text{for all } x, y \in X.$$

Theorem 2.6 ([25])**.** *Let* (X, d) *be a complete generalized metric space and* $N : X \to X$ *a contractive operator with Lipschitz matrix* M. *Then* N *has a unique fixed point* x_* *and for each* $x_0 \in X$ *we have*

$$d(N^k(x_0), x_*) \le M^k (I - M)^{-1} d(x_0, N(x_0)) \quad \textit{for all } k \in \mathbb{N}.$$

Denote by $\mathcal{P}(X) = \{Y \subset X : Y \neq \emptyset\}$, $\mathcal{P}_{cl}(X) = \{Y \in \mathcal{P}(X) : Y \text{ closed}\}$, $\mathcal{P}_b(X) = \{Y \in \mathcal{P}(X) : Y \text{ bounded}\}$. Let (X, d_*) be a metric space, we will denote by H_{d_*} the Hausdorff pseudo-metric distance on $\mathcal{P}(X)$, defined as

$$H_{d_*} : \mathcal{P}(X) \times \mathcal{P}(X) \longrightarrow \mathbb{R}_+ \cup \{\infty\},\ H_{d_*}(A, B) = \max \left\{ \sup_{a \in A} d_*(a, B), \sup_{b \in B} d_*(A, b) \right\},$$

where $d_*(A, b) = \inf_{a \in A} d_*(a, b)$ and $d_*(a, B) = \inf_{b \in B} d_*(a, b)$. Then $(\mathcal{P}_{b,cl}(X), H_{d_*})$ is a metric space and $(\mathcal{P}_{cl}(X), H_{d_*})$ is a generalized metric space. In particular, H_{d_*} satisfies the triangle inequality.

Let (X,d) be a generalized metric space with

$$d(x,y) := \begin{pmatrix} d_1(x,y) \\ \cdots \\ d_n(x,y) \end{pmatrix}.$$

Notice that d is a generalized metric space on X if and only if d_i, $i = 1,\dots,n$, are metrics on X. Consider the generalized Hausdorff pseudo-metric distance

$$H_d : \mathcal{P}(X) \times \mathcal{P}(X) \longrightarrow \mathbb{R}^n_+ \cup \{\infty\}$$

defined by

$$H_d(A,B) := \begin{pmatrix} H_{d_1}(A,B) \\ \cdots \\ H_{d_n}(A,B) \end{pmatrix}.$$

Definition 2.7. Let (X,d) be a generalized metric space. A multivalued operator $N : X \to \mathcal{P}_{cl}(X)$ is said to be contractive if there exists a matrix $M \in \mathcal{M}_{n\times n}(\mathbb{R}_+)$ such that

$$M^k \to 0 \quad \text{as } k \to \infty$$

and

$$H_d(N(u), N(v)) \le M d(u,v), \quad \text{for all } u, v \in X.$$

Remark 2.8. In generalized metric spaces in Perov's sense, the notions of convergence sequence, Cauchy sequence, completeness, open subset and closed subset are similar to those for usual metric spaces.

Definition 2.9. A multi-valued map $F : [a,b] \to \mathcal{P}(Y)$ is said to be measurable provided for every open $U \subset Y$, the set $F_+^{-1}(U)$ is Lebesgue measurable.

Lemma 2.10 ([8, 15])**.** *The mapping $F : J \to \mathcal{P}_{cl}(Y)$ is measurable if and only if for each $x \in Y$, the function $\zeta : J \to [0,+\infty)$ defined by*

$$\zeta(t) = \mathrm{dist}(x, F(t)) = \inf\{\|x-y\| : y \in F(t)\}, \quad t \in J,$$

is Lebesgue measurable.

The following lemma, needed in this paper, is the celebrated Kuratowski-Ryll-Nardzewski selection theorem.

Lemma 2.11 ([15], Theorem 19.7)**.** *Let Y be a separable metric space and $F : [a,b] \to \mathcal{P}(Y)$ a measurable multi-valued map with nonempty closed values. Then F has a measurable selection.*

3. Perov Type Fixed Point Theorem

Our first purpose here is to establish a Perov fixed point theorem type for expansive and nonexpansive operators.

Definition 3.1. Let E be a vector space on $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. By a vector-valued norm on E we mean a map $\|\cdot\| : X \to \mathbb{R}^n$ with the following properties:

(i) $\|x\| \geq 0$ for all $x \in E$; if $\|x\| = 0$ then $x = (0, \ldots, 0)$;
(ii) $\|\lambda x\| = \lambda \|x\|$ for all $x \in E$ and $\lambda \in \mathbb{K}$;
(iii) $\|x + y\| \leq \|x\| + \|y\|$ for all $x \in E$.

The pair $(E, \|\cdot\|)$ is called a generalized normed space. If the generalized metric generated by $\|\cdot\|$ (i.e., $d(x, y) = \|x - y\|$) is complete then the space $(E, \|\cdot\|)$ is called a generalized Banach space. Denote by $\mathcal{P}_{cv}(E) = \{Y \in \mathcal{P}(E) \colon\ Y \text{ convex}\}$.

Theorem 3.2 ([9]). *Let E be a generalized Banach space, let $C \in \mathcal{P}_{cv}(E)$ and $f : C \to C$ be a continuous operator with relatively compact range. Then f has at least one fixed point in C.*

Theorem 3.3. *Let E be a generalized Banach space, $Y \subseteq E$ a nonempty convex compact subset of E and $f : Y \to Y$ be a single valued map. Assume that*

$$d(f(x), f(y)) \leq d(x, y) \quad \textit{for all } x, y \in Y.$$

Then f has a fixed point.

Proof. For every $m \in \mathbb{N}$, we have $\frac{I}{2^m} \in \mathcal{M}_{n\times n}(\mathbb{R}_+)$ and

$$\frac{I}{2^{mk}} \to 0 \quad \text{as } k \to \infty.$$

Thus, for some $x_0 \in Y$ the mapping $f_m : Y \to Y$ defined by

$$f_m(x) = \left(1 - \frac{1}{2^m}\right) f(x) + \frac{1}{2^m} x_0 \in Y \quad \text{for all } x \in Y.$$

Hence, we get

$$d(f_m(x), f_m(y)) \leq \frac{I}{2^m} d(x, y) \quad \text{for all } x, y \in Y.$$

From Theorem 2.6 there exists unique $x_m \in Y$ such that

$$x_m = f_m(x_m), \quad m \in \mathbb{N}.$$

Since Y is compact, then there exists subsequence of $(x_m)_{m\in\mathbb{N}}$ that converges to $x \in Y$. Now we show that $x = f(x)$.

$$\begin{aligned} d(x, f(x)) &= \begin{pmatrix} d_1(x, f(x)) \\ \cdots \\ d_n(x, f(x)) \end{pmatrix} \\ &\le d(x, x_m) + d(x_m, f(x_m)) + d(f(x_m), f(x)) \\ &\le 2Id(x, x_m) + d(x_m, f(x_m)) \end{aligned}$$

and

$$\begin{aligned} d(x_m, f(x_m)) &= \begin{pmatrix} d_1(x_m, f(x_m)) \\ \cdots \\ d_n(x_m, f(x_m)) \end{pmatrix} \\ &= \begin{pmatrix} \|x_m - f(x_m)\|_1 \\ \cdots \\ \|x_m - f(x_m)\|_n \end{pmatrix} \\ &= \begin{pmatrix} \|\frac{f(x_m)}{2^m} - \frac{1}{2^m}x_0\|_1 \\ \cdots \\ \|\frac{f(x_m)}{2^m} - \frac{1}{2^m}x_0\|_n \end{pmatrix} \\ &\le \frac{1}{2^m}d(x_m, x) + \frac{1}{2^m}d(f(x), x_0). \end{aligned}$$

Hence

$$d(x, f(x)) \le \left(2 + \frac{1}{2^m}\right) Id(x, x_m) + \frac{1}{2^m}d(f(x), x_0) \to 0 \quad \text{as } m \to \infty.$$

□

Definition 3.4. Let (X, d) be a generalized metric space and C be a subset of X. The mapping $B : C \to X$ is said to be expansive, if there exists a constant $k \in \mathbb{R}$, $k > 1$ such that

$$d(B(x), B(y)) \ge kd(x, y) \quad \text{for all } x, y \in C.$$

Lemma 3.5. *Let X be a generalized metric space and $C \subseteq X$. Assume the mapping $B : C \to X$ is expansive with constant $k > 1$. Then the inverse of $B : C \to B(C)$ exists and*

$$d(B^{-1}(x), B^{-1}(y)) \le \frac{I}{k}d(x, y), \quad x, y \in B(C).$$

Proof. Let $x, y \in C$ and $B(x) = B(y)$, then

$$d(B(x), B(y)) \ge kd(x, y) \Rightarrow d(x, y) = 0 \Rightarrow x = y.$$

Thus $B : C \to B(C)$ is invertible. Let $x, y \in B(C)$, then there exist $a, b \in C$ such that

$$B(a) = x, \quad B(b) = y.$$

Hence

$$d(a,b) = d(B^{-1}(x), B^{-1}(y)) \quad \text{and} \quad d(x,y) = d(B(a), B(b)) \geq kd(a,b).$$

Therefore

$$d(B^{-1}(x), B^{-1}(y)) \leq \frac{I}{k} d(x,y) \quad \text{for all } x, y \in C.$$

□

As a consequence of Perov's Theorem we have the following result:

Theorem 3.6. *Let X be a complete generalized metric space and C be a closed subset of X. Assume $B : C \to X$ is expansive and $C \subseteq B(C)$. Then there exists a unique point $x \in C$ such that $x = B(x)$.*

Proof. Since B is expansive there exists $k > 1$ such that

$$d(B(x), B(y)) \geq kd(x,y) \quad \text{for all } x, y \in C.$$

From Lemma 3.5 the operator $B : C \to C$ is invertible and

$$d(B^{-1}(x); B^{-1}(y)) \leq \frac{I}{k} d(x,y), \quad x, y \in C.$$

Hence B^{-1} is contractive. By Theorem 2.6 there exists unique $x \in C$ such that

$$B^{-1}(x) = x \Rightarrow x = B(x).$$

□

Lemma 3.7. *Let $B : X \to X$ be a map such that B^m (m-power) is an expansive map for some $m \in \mathbb{N}$. Assume further that there exists a closed subset C of X such that C is contained $B(C)$. Then there exists a unique fixed point of B.*

Proof. Since B^m is an expansive map and $C \subseteq B^m(C)$, it follows from Theorem 3.6 that there exists unique fixed point of B^m. Let $x \in C$ be a fixed point of B^m. Using the fact that B^m is an expansive map, then there exists $k > 1$ such that

$$d(B^m(x), B^m(y)) \geq kd(x,y) \quad \text{for all } x, y \in C.$$

Hence

$$d(x, B(x)) = d(B^m(x), B^{m+1}(x)) \geq kd(x, B(x)) \Rightarrow d(x, B(x)) = 0.$$

Then B has a unique fixed point in C. □

Theorem 3.8. *Let X be a complete generalized metric space and C be a closed subset of X and let $B : C \to X$ is expansive operator in the sense that there exists $M \in \mathcal{M}_{n \times n}(\mathbb{R}_+)$ such that*

$$M^k \to M_* \neq 0, \quad \textit{as } k \to \infty$$

and

$$d(B(x), B(y)) \geq M_* d(x, y) \quad \text{for all } x, y \in C.$$

Assume that for each $x, y \in \mathbb{R}^n_+$

$$M_* x \leq M_* y \Rightarrow x \leq y \quad \text{and} \quad (I - M_*)x \leq (I - M_*)y \Rightarrow x \leq y$$

and M_* *is invertible such that*

$$(M_*^{-1})^k \to 0, \quad \text{as } k \to \infty.$$

If $C \subseteq B(C)$*. Then* B *has unique point in* C*.*

Proof. Let $x, y \in C$ such that if $B(x) = x$, and $B(y) = y$ then

$$d(B(x), B(y)) \geq M_* d(x, y) \Rightarrow (I - M_*) d(x, y) \leq 0.$$

Hence $x = y$ and $B^{-1} : C \to C$ exists and is continuous. Now we show that B^{-1} is contractive. Let $x, y \in C$. Then there exist $x_1, x_2 \in C$ such that

$$B(x_1) = x, \quad B(x_2) = y.$$

Hence

$$d(x, y) \geq M_* d(x_1, x_2) \Rightarrow M_* d(x_1, x_2) \leq M_* M_*^{-1} d(x, y).$$

Thus

$$d(B^{-1}(x), B^{-1}(y)) \leq M_*^{-1} d(x, y) \quad \text{for all } x, y \in C.$$

Then B^{-1} is contractive, by Theorem 2.6 there exists unique $x \in C$ such that

$$B^{-1}(x) = x \Rightarrow B(x) = x.$$

□

4. Multivalued Fixed Points

In this section, we will provide a multivalued version of Perov's fixed point theorem.

Theorem 4.1. *Let* (X, d) *be a complete generalized metric space and* $F : X \to \mathcal{P}_{cl,b}(X)$ *a contractive multivalued operator with Lipschitz matrix* M*. Then* N *has at least one fixed point.*

Theorem 4.2. *Let* (X, d) *be a generalized complete metric space, and let* $F : X \to \mathcal{P}_{cl}(X)$ *be a multivalued map. Assume that there exist* $A, B, C \in \mathcal{M}_{n\times n}(\mathbb{R}_+)$ *such that*

$$H_d(F(x), F(y)) \leq A d(x, y) + B d(y, F(x)) + C d(x, F(x)) \tag{4.1}$$

where $A + C$ *is convergent to zero. Then there exists* $x \in X$ *such that* $x \in F(x)$*.*

Proof. Let $x \in X$ and

$$D(x, d(x, F(x)) = \{y \in X : d(x, y) \leq d(x, F(x))\}.$$

Since $F(x)$ is closed, then

$$D(x) \cap F(x) \neq \emptyset.$$

So we can select $x_1 \in F(x)$ such that

$$d(x, x_1) \leq d(x, F(x)) \leq Ad(x, x_1) + Bd(x_1, F(x)) + Cd(x, F(x));$$

thus

$$d(x, x_1) \leq (A + C)d(x, F(x)). \tag{4.2}$$

For $x_2 \in F(x_1)$ we have

$$\begin{aligned} d(x_2, x_1) &\leq d(x_1, F(x)) + H_d(F(x), F(x_1)) \\ &\leq Ad(x, x_1) + Cd(x, F(x)) \\ &\leq (A + C)d(x, x_1). \end{aligned}$$

Then

$$d(x_2, x_1) \leq (A + C)^2 d(x, F(x)). \tag{4.3}$$

Continuing this procedure we can find a sequence $(x_n)_{n \in \mathbb{N}}$ of X such that

$$d(x_n, x_{n+1}) \leq (A + C)^{n+1} d(x, F(x)), \quad n \in \mathbb{N}.$$

Let $p \in \mathbb{N}$. Since d is metric we have

$$d(x_n, x_{n+p}) \leq d(x_n, x_{n+1}) + \cdots + d(x_{n+p-1}, x_{n+p}).$$

Hence, for all $n, p \in \mathbb{N}$, the following estimation holds:

$$d(x_n, x_{n+p}) \leq (A + C)^{n+1}(I + (A + C) + (A + C)^2 + \cdots + (A + C)^{p-1})d(x, F(x)).$$

Therefore

$$d(x_n, x_{n+p}) \to 0 \quad \text{as } n \to \infty;$$

so $(x_n)_{n \in \mathbb{N}}$ is a Cauchy sequence in the complete generalized metric space X. Then there exists $x_* \in X$ such that

$$d(x_n, x_*) \to 0 \quad \text{as } n \to \infty.$$

From (4.1) we obtain

$$\begin{aligned} d(x_*, F(x_*)) &\le d(x_*, x_n) + H_d(F(x_{n+1}), F(x_*)) \\ &\le d(x_n, x_*) + Ad(x_{n+1}, x_*) + Bd(x_*, F(x_{n+1})) \\ &\quad + Cd(x_{n+1}, F(x_{n+1})) \\ &\le d(x_n, x_*) + Ad(x_{n+1}, x_*) + Bd(x_*, F(x_{n+1})) \\ &\quad + Cd(x_{n+1}, F(x_{n+1})) \\ &\le d(x_n, x_*) + Ad(x_{n+1}, x_*) + Bd(x_*, x_n) \\ &\quad + Cd(x_{n+1}, x_n) \to 0 \quad \text{as } n \to \infty. \end{aligned}$$

This implies that $x_* \in F(x_*)$. □

Lemma 4.3. *Let (X, d) be a generalized Banach space and $F : X \to \mathcal{P}_{cl}(Y)$ be a multivalued map. Assume that there exist $p \in \mathbb{N}$ and $M \in \mathcal{M}_{n\times n}(\mathbb{R}_+)$ convergent to zero such that*

$$H_d(F^p(x), F^p(y)) \le Md(x, y), \quad \text{for each } x, y \in X$$

and

$$\sup_{a\in F^{p+1}(y)} d(a, F(x)) \le d(y, F(x)).$$

Then there exists $x \in X$, such that $x \in F(x)$.

Proof. By Theorem 4.2, there exists $x \in X$ such that $x \in F^p(x)$. Now we show that $x \in F(x)$.

$$\begin{aligned} d(x, F(x)) &\le d(x, F^{p+1}(x)) + H_d(F^{p+1}(x), F(x)) \\ &\le H_d(F^p(x), F^{p+1}(x)) \\ &\le Md(x, F(x)). \end{aligned}$$

Hence

$$d(x, F(x)) \le M^k d(x, F(x)) \to 0 \quad \text{as } k \to \infty \Rightarrow d(x, F(x)) = 0.$$

□

Theorem 4.4. *Let (X, d) be a complete generalized metric space and $B(x_0, r_0) = \{x \in X : d(x, x_0) < r_0\}$ be the open ball in X with radius r_0 and centered at some point $x_0 \in X$. Assume that $F : B(x_0, r_0) \to \mathcal{P}_{cl}(X)$ is a contractive multivalued map such that*

$$H_d(x_0, F(x_0)) < (I - M)r_0,$$

where $M \in \mathcal{M}_{n\times n}(\mathbb{R}_+)$ is the matrix contractive for F. Then F has at least one fixed point.

Proof. Let $r_1 \in \mathbb{R}^n_+$ be such that

$$d(x_0, F(x_0)) \le (I - M)r_1 < (I - M)r_0.$$

Set

$$K(x_0, r_1) = \{x \in X : d(x, x_0) \le r_1\}.$$

It is clear that $K(x_0, r_1)$ is complete generalized metric space. Let us define a multivalued map

$$F_*(x) = F(x) \quad \text{for all } x \in K(x_0, r_1).$$

In view of Theorem 4.2, for the proof it is sufficient to show that

$$F_*(K(x_0, r_1)) \subseteq K(x_0, r_1).$$

Let $x \in K(x_0, r_1)$. Then we have:

$$d(x_0, y) \le \sup_{z \in F(x)} d(x_0, z) = H_d(x_0, F(x)), \quad \text{for all } y \in F(x).$$

Thus

$$\begin{aligned} d(x_0, y) &\le H_d(x_0, F(x_0)) + H_d(F(x_0), F(y)) \\ &\le (I - M)r_1 + Md(x_0, y) \le (I - M)r_1 + r_1 M = r_1, \end{aligned}$$

and the proof is completed. $\square$

Lemma 4.5. *Let E be a generalized Banach space, $Y \subseteq E$ a nonempty convex compact subset of E and $F : X \to \mathcal{P}_{cl}(Y)$ a multivalued map such that*

$$H_d(F(x), F(y)) \le d(x, y), \quad \textit{for each } x, y \in X.$$

Then there exists $x \in X$, such that $x \in F(x)$.

Proof. For every $m \in \mathbb{N}$, we have $\frac{I}{2^m} \in \mathcal{M}_{n\times n}(\mathbb{R}_+)$ and

$$\frac{I}{2^{mk}} \to 0 \quad \text{as } k \to \infty.$$

Thus, for some $x_0 \in Y$ the mapping $f_m : Y \to Y$ defined by

$$F_m(x) = \left(1 - \frac{1}{2^m}\right) F(x) + \frac{1}{2^m} x_0 \in Y \quad \text{for all } x \in Y.$$

Then

$$H_d(F_m(x), F_m(y)) \le \frac{I}{2^m} d(x, y) \quad \text{for all } x, y \in Y.$$

From Theorem 4.2 there exists $x_m \in Y$ such that

$$x_m \in F_m(x_m), \quad m \in \mathbb{N}.$$

Since Y is compact, then there exists a subsequence of $(x_m)_{m\in\mathbb{N}}$ that converges to $x \in Y$. Now we show that $x \in F(x)$.

$$\begin{aligned} d(x, F(x)) &= \begin{pmatrix} d_1(x, F(x)) \\ \cdots \\ d_n(x, F(x)) \end{pmatrix} \\ &\leq d(x, x_m) + d(x_m, F(x_m)) + H_d(F(x_m), F(x)) \\ &\leq 2I d(x, x_m) + d(x_m, F(x_m)) \end{aligned}$$

and

$$\begin{aligned} d(x_m, F(x_m)) &= \begin{pmatrix} d_1(x_m, F(x_m)) \\ \cdots \\ d_n(x_m, F(x_m)) \end{pmatrix} \\ &= \begin{pmatrix} \|x_m - F(x_m)\|_1 \\ \cdots \\ \|x_m - F(x_m)\|_n \end{pmatrix} \\ &\leq d(x_m, F(x)) + H_d(F(x), F(x_m)) \\ &\leq d(x_m, z_m) + d(x, x_m) \\ &\leq d(x_m, x) + \frac{1}{2^m} d(z_m, x_0) \end{aligned}$$

where

$$z_m \in F(x) \quad \text{and} \quad x_m = \left(1 - \frac{1}{2^m}\right) z_m + \frac{1}{2^m} x_0.$$

Since $x_m \to x$ as $m \to \infty$, then

$$z_m \to x \quad \text{as } m \to \infty.$$

Hence

$$d(x, F(x)) \leq 3I d(x, x_m) + \frac{1}{2^m} d(z_m, x_0) \to 0 \quad \text{as} \quad m \to \infty.$$

□

5. Krasnosel'skii's Theorem Type

In this section we present a Krasnosel'skii fixed point type theorem by using the expansive operator combined with continuous operator.

Lemma 5.1. *Let E be a generalized normed space and $C \subseteq E$. Assume the mapping $B : C \to X$ is expansive with constant $k > 1$. Then the inverse of $I - B : C \to (I - B)(C)$ exists and*

$$d((I-B)^{-1}(x), (I-B)^{-1}(y)) \leq \frac{1}{k-1} d(x, y), \quad x, y \in (I-B)(C).$$

Proof. Let $x, y \in C$ and $x - B(x) = y - B(y)$, then

$$\begin{aligned} 0 = d(x-B(x), y-B(y)) &= \begin{pmatrix} \|x-B(x)-y+B(y)\|_1 \\ \cdots \\ \|x-B(x)-y+B(y)\|_n \end{pmatrix} \\ &\geq \begin{pmatrix} \|B(y)-B(x)\|_1 - \|x-y\|_1 \\ \cdots \\ \|B(y)-B(x)\|_n - \|x-y\|_n \end{pmatrix} \\ &\geq \begin{pmatrix} k\|y-x\|_1 - \|x-y\|_1 \\ \cdots \\ k\|y-x\|_n - \|x-y\|_n \end{pmatrix} \\ &= (k-1)Id(x,y). \end{aligned}$$

Thus $I - B : C \to (I-B)(C)$ is invertible. Let $x, y \in (I-B)(C)$, then there exist $a, b \in C$ such that

$$a - B(a) = x, \quad b - B(b) = y.$$

Hence

$$d(a,b) = d((I-B)^{-1}(x), (I-B)^{-1}(y)) \quad \text{and } d(x,y) \geq kd(a,b) - d(a,b).$$

Therefore

$$d(I-B)^{-1}(x), (I-B)^{-1}(y)) \leq \frac{I}{k-1} d(x,y) \quad \text{for all } x, y \in (I-B)(C).$$

□

Theorem 5.2. *Let E be a generalized Banach space and C be a compact convex subset of E. Assume that $A : M \to X$ is continuous and $B : C \to E$ is a continuous expansive map satisfying*

$(\mathcal{H}_1)$ *for each $x, y \in C$ such that*

$$x = B(x) + A(y) \Rightarrow x \in C.$$

Then there exists $y \in C$ such that $y = B(y) + A(y)$.

Proof. Let $y \in C$. Let $F_y : C \to X$ be a operator defined by

$$F_y(x) = B(x) + A(y), \quad x \in C.$$

From Theorem 3.6 there exists a unique $x(y) \in C$ such that

$$x(y) = B(x(y)) + A(y).$$

By Lemma 5.1, $I - B$ is invertible. Moreover, $(I-B)^{-1}$ is continuous. Let us define $N : C \to C$ by

$$y \to N(y) = (I-B)^{-1}A(y).$$

Let $x \in C$ and $N(x) = (I - B)^{-1}(A(x))$. Then

$$N(x) = (I - B)^{-1}(A(x)) \Rightarrow N(x) = B(N(x)) + A(x),$$

and thus $(\mathcal{H}_1)$ implies that $N(x) \in C$. Let $\{y_m : m \in \mathbb{N}\} \subseteq C$ be a sequence converging to y in C; we show that $N(y_m)$ converges to $N(y)$. Set $x_m = (I - B)^{-1}A(y_m)$, then

$$(I - B)(x_m) = A(y_m), \quad m \in \mathbb{N}.$$

Since C is compact, there exists a subsequence of $\{x_m\}$ converging to some $x \in C$. Then

$$(I - B)(x_m) \to (I - B)(x) \quad \text{as } m \to \infty.$$

Hence

$$A(y_m) \to (I - B)(x) \quad \text{as } m \to \infty.$$

Therefore

$$N(y_m) \to N(y) \quad \text{as } m \to \infty.$$

Hence from Theorem 3.2, there exists $y \in C$ such that $y = (I - B)^{-1}A(y)$, and we deduce that $B + G$ has a fixed point in C. □

6. Applications: Impulsive Differential Inclusions

Differential equations with impulses were considered for the first time by Milman and Myshkis [24] and then followed by a period of active research which culminated with the monograph by Halanay and Wexler [17]. The dynamics of many processes in physics, population dynamics, biology, medicine may be subject to abrupt changes such as shocks or perturbations (see for instance [1, 21] and the references therein). These perturbations may be seen as impulses. For instance, in the periodic treatment of some diseases, impulses correspond to the administration of a drug treatment. In environmental sciences, impulses correspond to seasonal changes of the water level of artificial reservoirs. Their models are described by impulsive differential equations and inclusions. Important contributions to the study of the mathematical aspects of such equations have been undertaken in [3, 5, 10, 14, 22, 31] among others. In this section we consider the following system of differential inclusions with impulse effects,

$$x'(t) \in F_1(t, x(t), y(t)), \quad y'(t) \in F_2(t, x(t), y(t)), \quad \text{a.e. } t \in [0, 1] \tag{6.1}$$

$$x(\tau^+) - x(\tau^-) = I_1(x(\tau), y(\tau)), \quad y(\tau^+) - y(\tau^-) = I_2(x(\tau), y(\tau)) \tag{6.2}$$

$$x(0) = x_0, \quad y(0) = y_0, \tag{6.3}$$

where $0 < \tau < 1$, $J := [0, 1]$, $F_i : J \times \mathbb{R} \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$, $i = 1, 2$ are multifunctions, and $I, \overline{I} \in C(\mathbb{R} \times \mathbb{R}, \mathbb{R})$. The notations $x(\tau^+) = \lim_{h \to 0^+} x(\tau + h)$ and $x(\tau^-) = \lim_{h \to 0^+} x(t - h)$ stand for the right and the left limits of the function y at $t = \tau$, respectively.

In order to define a solution for Problem (6.1)–(6.3), consider the space of piecewise continuous functions:

$$PC([0,1],\mathbb{R} = \{y : [0,1] \to \mathbb{R}, y \in C(J\backslash\{\tau\},\mathbb{R}) \text{ such that } y(\tau^-) \text{ and } y(\tau^+) \text{ exist and satisfy } y(\tau^-) = y(\tau)\}.$$

Endowed with the norm

$$\|y\|_{PC} = \sup\{|y(t)| : t \in J\},$$

PC is a Banach space.

In the proof of the existence result for the problem we can easily prove the following auxiliary lemma.

Lemma 6.1. *Let $f_1, f_2 \in L^1(J,\mathbb{R})$. Then y is a solution of the impulsive system*

$$x'(t) = f_1(t), \quad y'(t) = f_2(t), \qquad a.e.\ t \in [0,1] \tag{6.4}$$

$$x(\tau^+) - x(\tau^-) = I_1(x(\tau), y(\tau)), \quad y(\tau^+) - y(\tau^-) = I_2(x(\tau), y(\tau)) \tag{6.5}$$

$$x(0) = x_0, \quad y(0) = y_0, \tag{6.6}$$

if and only if y is a solution of the impulsive integral equation

$$\begin{cases} x(t) &= x_0 + g_1(t) + I_1(x(\tau), y(\tau)), \quad a.e.\ t \in [0,1] \\ y(t) &= y_0 + g_2(t) + I_2(x(\tau), y(\tau)), \quad a.e.\ t \in [0,1], \end{cases} \tag{6.7}$$

where $g_i(t) = \int_0^t f_i(s)ds$, $i = 1, 2$.

In this section we assume the following conditions:

($\mathcal{H}_1$) $F_i : [0,1] \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathcal{P}_{cp}(\mathbb{R})$; $t \longmapsto F_i(t,u,v)$ are measurable for each $u, v \in \mathbb{R}$, $i = 1, 2$.

($\mathcal{H}_2$) There exist functions $l_i \in L^1(J,\mathbb{R}^+)$, $i = 1, \ldots, 3$, such that

$$H_d(F_i(t,u,v), F_i(t,\overline{u},\overline{v})) \le l_i(t)|u - \overline{u}| + l_i(t)|v - \overline{v}|, \quad t \in J \text{ for all } u, \overline{u}, v, \overline{v} \in \mathbb{R}$$

and

$$H_d(0, F_i(t,0,0)) \le l_i(t) \quad \text{for a.e. } t \in J, \quad i = 1, 2.$$

($\mathcal{H}_3$) There exist constants $a_i, b_i \ge 0$, $i = 1, 2$ such that

$$|I_1(u,v) - I_1(\overline{u},\overline{v})| \le a_1|u - \overline{u}| + a_2|v - \overline{v}|, \quad \text{for all } u, \overline{u}, v, \overline{v} \in \mathbb{R}$$

and

$$|I_2(u,v) - I_2(\overline{u},\overline{v})| \le b_1|u - \overline{u}| + b_2|v - \overline{v}|, \quad \text{for all } u, \overline{u}, v, \overline{v} \in \mathbb{R}.$$

Theorem 6.2. *Assume that ($\mathcal{H}_1$)–($\mathcal{H}_3$) are satisfied and the matrix*

$$M = \begin{pmatrix} \|l_1\|_{L^1} + a_1 & \|l_2\|_{L^1} + a_2 \\ \|l_3\|_{L^1} + b_1 & \|l_4\|_{L^1} + b_2 \end{pmatrix}$$

is convergent to zero. Then the problem (6.1)–(6.3) has at least one solution.

Proof. Consider the operator $N : PC \times PC \to \mathcal{P}(PC \times PC)$ defined by

$$N(x,y) = \left\{ (h_1,h_2) \in PC \times PC : \begin{pmatrix} h_1(t) \\ h_2(t) \end{pmatrix} = \left\{ \begin{array}{ll} x_0 + \int_0^t f_1(s)ds & \\ +I_1(x(\tau), y(\tau)), & t \in J \\ & \\ y_0 + \int_0^t f_2(s)ds & \\ +I_2(x(\tau), y(\tau)), & t \in J \end{array} \right\} \right.$$

where $f_i \in S_{F_i,x,y} = \{f \in L^1(J,\mathbb{R}) : f(t) \in F_i(t,x(t),y(t)),\ \text{a.e. } t \in J\}$. Clearly, fixed points of the operator N are solutions of Problem (6.1)–(6.3). Let

$$N_i(x,y) = \left\{ h \in PC : h(t) = x_i + \int_0^t f_i(s)ds + I_i(x(\tau), y(\tau)),\ t \in J \right\},$$

where $x_1 = x_0,\ x_2 = y_0,\ f_i \in S_{F_i,x,y} = \{f \in L^1(J,\mathbb{R}) : f(t) \in F_i(t,x(t),y(t)),$ a.e. $t \in J\}$. We show N satisfies the assumptions of Theorem 4.2.

Let $(x,y), (\overline{x},\overline{y}) \in PC \times PC$ and $(h_1,h_2) \in N(x,y)$. Then there exist $f_i \in S_{F_i,x,y}$, $i = 1,2$ such that

$$(h_1(t), h_2(t)) = \begin{cases} x_0 + \int_0^t f_1(s)ds + I_1(x(\tau), y(\tau)),\ t \in J \\ y_0 + \int_0^t f_2(s)ds + I_2(x(\tau), y(\tau)),\ t \in J. \end{cases}$$

$(\mathcal{H}_2)$ implies that

$$H_{d_1}(F_1(t,x(t),y(t)), F_1(t,\overline{x}(t),\overline{y}(t))) \le l_1(t)|x(t) - \overline{x}(t)| + l_2(t)|y(t) - \overline{y}(t)|, \quad t \in J$$

and

$$H_{d_2}(F_2(t,x(t),y(t)), F_2(t,\overline{x}(t),\overline{y}(t))) \le l_3(t)|x(t) - \overline{x}(t)| + l_4(t)|y(t) - \overline{y}(t)|, \quad t \in J.$$

Hence, there is some $(w,\bar{w}) \in F_1(t,\overline{x}(t),\overline{y}(t)) \times F_2(t,\overline{x}(t),\overline{y}(t))$ such that

$$|f_1(t) - w| \le l_1(t)|x(t) - \overline{x}(t)| + l_2(t)|y(t) - \overline{y}(t)|, \quad t \in J,$$

and

$$|f_2(t) - \bar{w}| \le l_3(t)|x(t) - \overline{x}(t)| + \bar{l}_4(t)|y(t) - \overline{y}(t)|, \quad t \in J.$$

Consider the multi-valued maps $U_i : J \to \mathcal{P}(\mathbb{R})$, $i = 1,2$ defined by

$$U_1(t) = \{v \in F_1(t,x(t),y(t)) : |f_i(t) - w| \le l_1(t)|x(t) - \overline{x}(t)| + l_2(t)|y(t) - \overline{y}(t)|,$$
$$\text{a.e.} \quad t \in J\}$$

and

$$U_2(t) = \{v \in F_2(t,\bar{x}(t),\bar{y}(t)) : |f_2(t) - w| \le l_3(t)|x(t) - \bar{x}(t)| + l_4(t)|y(t) - \bar{y}(t)|,$$
$$\text{a.e.} \quad t \in J\}$$

Then, for $i = 1,2$, $U_i(t)$ is a nonempty set and Theorem III.4.1 in [8] tells us that U_i is measurable. Moreover, the multi-valued intersection operator $V_i(\cdot) = U_i(\cdot) \cap$

$F_i(\cdot, \overline{x}(t), \overline{y}(\cdot))$ is measurable. Therefore, by Lemma 2.11, there exists a function $t \mapsto \overline{f}_i(t)$, which is a measurable selection for V_i, that is $\overline{f}_i(t) \in F(t, \overline{x}(t), \overline{y}(t))$ and

$$|f_1(t) - \overline{f}_1(t)| \leq l_1(t)|x(t) - \overline{x}(t)| + l_2(t)|y(t) - \overline{y}(t)|, \quad \text{a.e. } t \in J.$$

and

$$|f_2(t) - \bar{f}_2(t)| \leq l_3(t)|x(t) - \overline{x}(t)| + l_4(t)|y(t) - \overline{y}(t)|$$

Define $\overline{h}_1, \overline{h}_2$ by

$$\overline{h}_1(t) = x_0 + \int_0^t \bar{f}_1(s)ds + I_1(x(\tau), y(\tau)), \quad t \in J.$$

and

$$\overline{h}_2(t) = y_0 + \int_0^t \bar{f}_2(s)ds + I_2(x(\tau), y(\tau)), \quad t \in J.$$

Then we have, for $t \in J$,

$$|h_1(t) - \overline{h}_1(t)| \leq (\|l_1\|_{L^1} + a_1)\|x - \overline{x}\|_{PC} + (\|l_2\|_{L^1} + a_2)\|_{L^1}\|y - \overline{y}\|_{PC}.$$

Thus

$$\|h_1 - \overline{h}_1\|_{PC} \leq (\|l_1\|_{L^1} + a_1)\|x - \overline{x}\|_{PC} + (\|l_2\|_{L^1} + a_2)\|y - \overline{y}\|_{PC}.$$

By an analogous relation, obtained by interchanging the roles of y and $\overline{y}$, we finally arrive at the estimate

$$H_{d_1}(N_1(x,y), N_1(\overline{x}, \overline{y})) \leq (\|l_1\|_{L^1} + a_1)\|x - \overline{x}\|_{PC} + (\|l_2\|_{L^1} + a_2)\|_{L^1}\|y - \overline{y}\|_{PC}.$$

Similarly we have

$$H_{d_2}(N_2(x,y), N_2(\overline{x}, \overline{y})) \leq (\|l_3\|_{L^1} + b_1)\|x - \overline{x}\|_{PC} + (\|l_4\|_{L^1} + b_2)\|_{L^1}\|y - \overline{y}\|_{PC}.$$

Therefore

$$H_d(N(x,y), N(\overline{x}, \overline{y})) \leq M \begin{pmatrix} \|x - \overline{x}\|_{PC} \\ \|y - \overline{y}\|_{PC} \end{pmatrix}, \quad \text{for all } (x,y), (\overline{x}, \overline{y}) \in PC \times PC.$$

Hence, by Theorem 4.2, the operator N has at least one fixed point which is solution of (6.1)–(6.3). □

REFERENCES

[1] Z. Agur, L. Cojocaru, G. Mazaur, R.M. Anderson and Y.L. Danon, Pulse mass measles vaccination across age cohorts, *Proc. Nat. Acad. Sci. USA.* **90** (1993), 11698–11702.

[2] C. S. Barroso, Krasnosel'skii's fixed point theorem for weakly continuous maps, *Nonlinear Anal.* **55** (2003), 25–31.

[3] D. D. Bainov and P. S. Simeonov, *Systems with Impulse Effect*, Ellis Horwood Ltd., Chichister, 1989.

[4] C. S. Barroso and E. V. Teixeira, A topological and geometric approach to fixed points results for sum of operators and applications, *Nonlinear Anal.* **60** (2005), 625–650.

[5] M. Benchohra, J. Henderson, and S. K. Ntouyas, *Impulsive Differential Equations and Inclusions* Contemporary Mathematics and its Applications, 2. Hindawi Publishing Corporation, New York, 2006.

[6] M. Boriceanu, Krasnosel'skii-type theorems for multivalued operators, *Fixed Point Theory* **9** (2008), 35–45.

[7] T. A. Burton and C. Kirk, A fixed point theorem of Krasnoselskii-Schaefer type, *Math. Nachr.* **189** (1998), 23–31.

[8] C. Castaing and M. Valadier, *Convex Analysis and Measurable Multifunctions,* Lecture Notes in Mathematics, Springer-Verlag, Vol. 580, Berlin-Heidelberg-New York, 1977.

[9] R. Cristescu, *Order Structures in Normed Vector Spaces*, Editura Ştiinţifică şi Enciclopedică, Bucureşti, 1983 (in Romanian).

[10] S. Djebali, L. Gorniewicz and A. Ouahab, *Solutions Sets for Differential Equations and Inclusions*, De Gruyter Series in Nonlinear Analysis and Applications **18**. Berlin: de Gruyter, 2013.

[11] J. Garcia-Falset, Existence of fixed points for the sum of two operators, *Math. Nachr.* **12** (2010), 1726–1757.

[12] J. Garcia-Falset, K. Latrach, E. Moreno-Gálvez and M. A Taoudi, Schaefer-Krasnosel'skii fixed point theorems using a usual measure of weak noncompactness, *J. Differential Equations* **252** (2012), 3436–3452.

[13] J. Garcia-Falset and O. Muñiz-Pérez, Fixed point theory for 1-set weakly contractive and pseudocontractive mappings, *Appl. Math. Comput.* **219** (2013), 6843–6855.

[14] J. R. Graef, J. Henderson and A. Ouahab, *Impulsive Differential Inclusions. A Fixed Point Approach*, De Gruyter Series in Nonlinear Analysis and Applications 20, Berlin: de Gruyter, 2013.

[15] L. Górniewicz, *Topological Fixed Point Theory of Multivalued Mappings,* Mathematics and its Applications, Kluwer Academic Publishers, 495, Dordrecht, 1999.

[16] L. Górniewicz and A. Ouahab, Some fixed point theorems of a Krasnosel'skii type and application to differential inclusions, *Fixed Point Theory*, to appear.

[17] A. Halanay and D. Wexler, *Teoria Calitativa a Systeme cu Impulduri*, Editura Republicii Socialiste Romania, Bucharest, 1968.

[18] J. Henderson and A. Ouahab, Some multivalued fixed point theorems in topological vector spaces, *Journal of Fixed Point Theory*, to appear.

[19] Sh. Hu and N. Papageorgiou, *Handbook of Multivalued Analysis, Volume I: Theory*, Kluwer, Dordrecht, 1997.

[20] M. A. Krasnosel'skii, Some problems of nonlinear analysis, *Amer. Math. Soc. Transl. Ser.* (2) **10** (1958), 345–409.

[21] E. Kruger-Thiemr, Formal theory of drug dosage regiments, *J. Theoret. Biol.* **13** (1966), 212–235.

[22] V. Lakshmikantham, D. D. Bainov and P. S. Simeonov, *Theory of Impulsive Differential Equations*, World Scientific, Singapore, 1989.

[23] Y. Liu and Z. Li, Krasnosel'skii-type fixed point theorems, *Proc. Amer. Math. Soc.*, **136** (2008), 1213–1220.

[24] V. D. Milman and A. A. Myshkis, On the stability of motion in the presence of impulses, *Sib. Math. J.* (in Russian) **1** (1960) 233–237.

[25] A. I. Perov, On the Cauchy problem for a system of ordinary differential equations, *Pviblizhen. Met. Reshen. Differ. Uvavn.,* **2**, (1964), 115–134 (in Russian).

[26] I. R. Petre, A multivalued version of Krasnosel'skii's theorem in generalized Banach spaces, *An. Şt. Univ. Ovidius Constanţa*, **22** (2014), 177–192.

[27] A. I. Perov, A. V. Kibenko, On a certain general method for investigation of boundary value problems, *Izv. Akad. Nauk SSSR, Ser. Mat.,* **30** 1966, 249–264 (in Russian).

[28] I. R. Petre and A. Petruşel, Krasnosel'skii's theorem in generalized Banach spaces and applications, *Electron. J. Qual. Theory Differ. Equ.* (2012), no. 85, 20 pp.

[29] I. A. Rus, The theory of a metrical fixed point theorem: theoretical and applicative relevances, *Fixed Point Theory* **9** (2008), 541–559.

[30] R. S. Varga, *Matrix Iterative Analysis*, Springer Series in Computational Mathematics, 27, Springer-Verlag, Berlin, 2000.

[31] A. M. Samoilenko and N. A. Perestyuk, *Impulsive Differential Equations,* World Scientific, Singapore, 1995.

[32] T. Xiang and R. Yuan, A class of expansive-type Krasnosel'skii fixed point theorems, *Nonlinear Anal.* **71** (2009), 3229–3239.