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POSITIVE SOLUTIONS FOR A BOUNDARY VALUE PROBLEM OF THE DISCRETE BEAM EQUATION

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ABSTRACT. We study a boundary value problem for the discrete beam equation. Some upper and lower estimates for positive solutions of the problem are obtained. Sufficient conditions for the existence and nonexistence of positive solutions are established.

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1. INTRODUCTION

Boundary value problems are important in theory and have wide applications in physical sciences. For example, the boundary value problem

$$u^{(4)}(t) = g(t)f(u(t)), \quad 0 \le t \le 1, \tag{1.1}$$

$$u(0) = u'(0) = u'(1) = u(1) = 0 \tag{1.2}$$

arises in the study of elasticity and has definite physical meanings. The equation (1.1) is often referred to as the beam equation. It describes the deflection of a beam under a force. The boundary conditions (1.2) mean that the beam is embedded at both ends $t = 0$ and $t = 1$. Related is the boundary value problem that consists of the equation (1.1) and the boundary conditions

$$u(0) = u'(0) = u'(1) = u'''(1) = 0. \tag{1.3}$$

It is well known that the problem (1.1), (1.3) arises in the study of symmetric solutions to the problem (1.1), (1.2).

In 2005, Yang [10] studied the problem (1.1), (1.3) and proved the following theorem.

Theorem 1.1. *If $u \in C^4[0,1]$ satisfies (1.3) and is such that*

$$u''''(t) \ge 0, \quad 0 \le t \le 1,$$

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then $u(t) \geq 0$ *for* $0 \leq t \leq 1$, *and*

$$(3t^2 - 2t^3)u(1) \leq u(t) \leq (2t - t^2)u(1), \quad 0 \leq t \leq 1.$$

In this paper, we consider the fourth order discrete boundary value problem

$$\Delta^4 y_{i-2} = g_i f(y_i), \quad 1 \leq i \leq n, \tag{1.4}$$

$$y_0 = \Delta y_{-1} = \Delta y_n = \Delta^3 y_{n-1} = 0. \tag{1.5}$$

Here the forward difference operator Δ is defined as

$$\Delta y_i = y_{i+1} - y_i.$$

Problem (1.4), (1.5) can be considered a discrete analogue of problem (1.1), (1.3). It is important to study boundary value problems of difference equations from a computational point of view. An example is included at the end of Section 3 to show how we discretize a boundary value problem of fourth order differential equation and then solve it numerically.

The main purpose of this paper is to prove some upper and lower estimates for positive solutions of the problem (1.4), (1.5). These upper and lower estimates are essential in finding sufficient conditions for the existence of positive solutions. We refer the reader to [1, 3, 6, 8, 9, 11] for some recent works on boundary value problems of discrete beam equations. Throughout the paper, we assume that

(H) $n \geq 4$ is a fixed integer, $g_i \geq 0$ for $1 \leq i \leq n$,

$$\sum_{i=1}^{n} g_i > 0,$$

and $f : [0, +\infty) \to [0, +\infty)$ is a continuous function.

Definition 1.2. By a positive solution to the problem (1.4), (1.5), we mean a sequence

$$(y_{-1}, y_0, y_1, \ldots, y_n, y_{n+1}, y_{n+2})$$

which satisfies the difference equation (1.4), the boundary conditions (1.5), and the inequalities

$$y_i > 0, \quad 1 \leq i \leq n.$$

Throughout the paper, we let $\mathbb{R}$ denote the set of real numbers. For $1 \leq i \leq n$ and $1 \leq j \leq n$, we define

$$G_{i,j} = \begin{cases} \frac{j}{6}(j+1)(1+3i-j) - \frac{ij(i+1)(j+1)}{4(n+1)}, & j \leq i, \\ \frac{i}{6}(i+1)(1+3j-i) - \frac{ij(i+1)(j+1)}{4(n+1)}, & i < j. \end{cases}$$

Then, $G_{i,j}$ is the Green function for the problem (1.4), (1.5), in the sense that

(1) First, if

$$(y_{-1}, y_0, y_1, \ldots, y_n, y_{n+1}, y_{n+2})$$

is a solution to the problem (1.4), (1.5), then

$$(y_1, y_2, \ldots, y_n)$$

solves the sum equation

$$y_i = \sum_{j=1}^{n} G_{i,j} g_j f(y_j), \quad 1 \le i \le n; \tag{1.6}$$

(2) Secondly, if

$$(y_1, y_2, \ldots, y_n) \in \mathbb{R}^n$$

solves the sum equation (1.6), then

$$(0, 0, y_1, y_2, \ldots, y_{n-1}, y_n, y_n, y_{n-1})$$

is a solution to the problem (1.4), (1.5).

In other words, problem (1.4), (1.5) is equivalent to the sum equation (1.6). Hence, in order to solve the problem (1.4), (1.5), it suffices to solve the sum equation (1.6). We leave it to the reader to verify that

$$G_{i,j} > 0, \quad 1 \le i, j \le n,$$

and

$$G_{i,j} = G_{j,i}, \quad 1 \le i, j \le n.$$

Throughout the paper, we define the norm $\|\cdot\|$ on $\mathbb{R}^n$ as

$$\|y\| = \max_{1 \le i \le n} |y_i|, \quad \forall y = (y_1, \ldots, y_n) \in \mathbb{R}^n.$$

We define the subset Y of $\mathbb{R}^n$ as

$$Y = \{y = (y_1, \ldots, y_n) \in \mathbb{R}^n : y_i \ge 0, 1 \le i \le n\}.$$

It is obvious that Y is a positive cone of $\mathbb{R}^n$. For each $y = (y_1, \ldots, y_n) \in Y$, we define $Ty \in \mathbb{R}^n$ as

$$Ty = \left(\sum_{j=1}^{n} G_{1,j} g_j f(y_j), \sum_{j=1}^{n} G_{2,j} g_j f(y_j), \ldots, \sum_{j=1}^{n} G_{n,j} g_j f(y_j) \right),$$

or equivalently,

$$(Ty)_i = \sum_{j=1}^{n} G_{i,j} g_j f(y_j), \quad 1 \le i \le n.$$

It is easy to see that $T : Y \to Y$ is a continuous operator.

To prove some of our results, we will need the following fixed point theorem, which is due to Krasnosel'skii [7].

Theorem 1.3. *Let $(X, \|\cdot\|)$ be a Banach space over the reals, and let $P \subset X$ be a cone in X. Let H_1 and H_2 be real numbers such that $H_2 > H_1 > 0$, and let*

$$\Omega_i = \{v \in X \mid \|v\| < H_i\}, \quad i = 1, 2.$$

If

$$L : P \cap (\overline{\Omega_2} - \Omega_1) \to P$$

is a completely continuous operator such that, either

(K1) $\|Lv\| \le \|v\|$ *if* $v \in P \cap \partial\Omega_1$, *and* $\|Lv\| \ge \|v\|$ *if* $v \in P \cap \partial\Omega_2$, *or*
(K2) $\|Lv\| \ge \|v\|$ *if* $v \in P \cap \partial\Omega_1$, *and* $\|Lv\| \le \|v\|$ *if* $v \in P \cap \partial\Omega_2$.

Then L has a fixed point in $P \cap (\overline{\Omega_2} - \Omega_1)$.

The rest of this paper is organized as follows. In Section 2, we prove some upper and lower estimates for positive solutions of the problem (1.4), (1.5). In Section 3, we state and prove the existence and nonexistence results for positive solutions of the problem (1.4), (1.5). In Section 4, we give estimates for the first eigenvalue of the linear boundary eigenvalue problem associated with the problem (1.4), (1.5).

2. UPPER AND LOWER ESTIMATES

We fix some notations first. For $1 \le i \le n$, we define

$$a_i = \frac{i(i+1)(3n+2-2i)}{n(n+1)(n+2)}$$

and

$$b_i = \frac{i(2n+1-i)}{n(n+1)}.$$

It is easy to verify that $(a_1, \dots, a_n) \in Y$ and $(b_1, \dots, b_n) \in Y$.

Lemma 2.1. *We have*

$$0 < a_1 < a_2 < a_3 < \cdots < a_n = 1,$$

$$0 < b_1 < b_2 < b_3 < \cdots < b_n = 1,$$

and $a_i \le b_i$ for $1 \le i \le n$.

Proof. It is easy to see that $a_0 > 0$, $b_0 > 0$, and $a_n = b_n = 1$. For $1 \le i \le n-1$, we have

$$a_{i+1} - a_i = \frac{6(i+1)(n-i)}{n(n+1)(n+2)} > 0,$$

$$b_{i+1} - b_i = \frac{2(n-i)}{n(n+1)} > 0.$$

For $1 \le i \le n$, we have

$$b_i - a_i = \frac{2i(n-i)(n+1-i)}{n(n+1)(n+2)} \ge 0.$$

The proof of the lemma is now complete. $\square$

We have the following estimates for the Green function.

Lemma 2.2. *For $1 \le i \le n$ and $1 \le j \le n$, we have*

$$a_i G_{n,j} \le G_{i,j} \le b_i G_{n,j}. \tag{2.1}$$

Proof. We take two cases to prove the first inequality in (2.1). If $1 \le i < j \le n$, then

$$\begin{aligned} &G_{i,j} - a_i G_{n,j} \\ &\quad = \frac{i(i+1)(n-j)(n-j+1)}{6n(n+1)(n+2)}[(j-i)n + 2j(n-i) + 2(j-i) + n + 2] \\ &\quad \ge 0. \end{aligned}$$

If $1 \le j \le i \le n$, then

$$\begin{aligned} &G_{i,j} - a_i G_{n,j} \\ &\quad = \frac{j(j+1)(n-i)(n-i+1)}{6n(n+1)(n+2)}[(i-j)n + 2i(n-j) + 2(i-j) + n + 2] \\ &\quad \ge 0. \end{aligned}$$

Thus, we proved the first inequality in (2.1).

It also takes two cases to prove the second inequality in (2.1). If $1 \le i < j \le n$, then

$$\begin{aligned} &b_i G_{n,j} - G_{i,j} \\ &\quad = \frac{i}{6n(n+1)}\big[(j-1)(j+1)(n-j)(n+1-j) + (j-i)^2 n(n+1) \\ &\qquad + j(j-i)(n^2 - j^2 + n + 1)\big] \\ &\quad \ge 0. \end{aligned}$$

If $1 \le j \le i \le n$, then

$$b_i G_{n,j} - G_{i,j} = \frac{(j-1)j(j+1)(n-i)(n-i+1)}{6n(n+1)} \ge 0.$$

Thus, we proved the second inequality in (2.1). The proof of the lemma is now complete. □

Lemma 2.3. *If $y = (y_1, \dots, y_n) \in Y$, then*

$$a_i (Ty)_n \le (Ty)_i \le b_i (Ty)_n, \quad 1 \le i \le n.$$

In particular, if $y = (y_1, \dots, y_n)$ is a non-negative solution to the sum equation (1.6), *then $y \in Y$ and*

$$a_i y_n \le y_i \le b_i y_n, \quad 1 \le i \le n.$$

Proof. Let $y \in Y$. For $1 \le i \le n$, we have

$$(Ty)_i = \sum_{j=1}^{n} G_{i,j} g_j f(y_j) \le b_i \sum_{j=1}^{n} G_{n,j} g_j f(y_j) = b_i (Ty)_n.$$

In a similar way, we can show that

$$(Ty)_i \ge a_i (Ty)_n, \quad 1 \le i \le n.$$

The proof is now complete. □

3. EXISTENCE AND NONEXISTENCE OF POSITIVE SOLUTIONS

First, we define some important constants:

$$A = \sum_{j=1}^{n} G_{n,j} g_j a_j, \quad B = \sum_{j=1}^{n} G_{n,j} g_j b_j,$$

$$F_0 = \limsup_{x \to 0^+} \frac{f(x)}{x}, \quad f_0 = \liminf_{x \to 0^+} \frac{f(x)}{x},$$

$$F_\infty = \limsup_{x \to +\infty} \frac{f(x)}{x}, \quad f_\infty = \liminf_{x \to +\infty} \frac{f(x)}{x}.$$

Throughout the rest of the paper, we let

$$P = \{v = (v_1, \ldots, v_n) \in Y \mid v_n \ge 0,\ a_i v_n \le v_i \le b_i v_n \ \text{ for } \ 1 \le i \le n\}.$$

Clearly P is a positive cone in $\mathbb{R}^n$. The next lemma shows that if $u \in P$, then the greatest component of u is u_n.

Lemma 3.1. *If $u \in P$, then $u_n = \|u\|$.*

Proof. If $u \in P$, then we have

$$0 \le u_i \le b_i u_n \le b_n u_n = u_n, \quad \text{for} \quad 1 \le i \le n.$$

The proof is complete. □

We can restate Lemma 2.3 as follows.

Lemma 3.2. *We have $T(P) \subset P$. And, if $u \in \mathbb{R}^n$ is a nonnegative solution to the sum equation* (1.6)*, then $u \in P$.*

It is clear that the sum equation (1.6) is equivalent to the equality

$$Tu = u, \quad u \in P.$$

In order to find a positive solution to the problem (1.4), (1.5), we need only to find a fixed point u of T such that $u \in P$ and $u_n = \|u\| > 0$.

Now, we are ready to prove some sufficient conditions for the existence of at least one positive solution to the problem (1.4), (1.5).

Theorem 3.3. *Suppose that (H) holds. If $BF_0 < 1 < Af_\infty$, then the problem* (1.4), (1.5) *has at least one positive solution.*

Proof. First, we choose $\varepsilon > 0$ such that $(F_0 + \varepsilon)B \le 1$. From the definition of F_0 we see that there exists $H_1 > 0$ such that

$$f(x) \le (F_0 + \varepsilon)x \quad \text{for} \quad 0 < x \le H_1.$$

For each $u \in P$ with $\|u\| = H_1$, we have

$$\begin{aligned}(Tu)_n &= \sum_{j=1}^{n} G_{n,j} g_j f(u_j) \le (F_0 + \varepsilon) \sum_{j=1}^{n} G_{n,j} g_j u_j \\ &\le (F_0 + \varepsilon)\|u\| \sum_{j=1}^{n} G_{n,j} g_j b_j \\ &= (F_0 + \varepsilon)\|u\| B \le \|u\|,\end{aligned}$$

which means $\|Tu\| \le \|u\|$. Thus, if we let $\Omega_1 = \{u \in \mathbb{R}^n \mid \|u\| < H_1\}$, then

$$\|Tu\| \le \|u\| \quad \text{for} \quad u \in P \cap \partial\Omega_1.$$

To construct Ω_2, we choose $\delta > 0$ such that

$$(f_\infty - \delta) \sum_{j=1}^{n} G_{n,j} g_j a_j \ge 1.$$

There exists $H_3 > 0$ such that $f(x) \ge (f_\infty - \delta)x$ for $x \ge H_3$. Let $H_2 = H_3/a_1 + H_1$. Then $H_2 > H_1$. If $u \in P$ is such that $\|u\| = H_2$, then, for $1 \le i \le n$, we have

$$u_i \ge u_n a_i \ge H_2 a_1 \ge H_3.$$

Therefore, for each $u \in P$ with $\|u\| = H_2$, we have

$$\begin{aligned}(Tu)_n &= \sum_{j=1}^{n} G_{n,j} g_j f(u_j) \ge (f_\infty - \delta) \sum_{j=1}^{n} G_{n,j} g_j u_j \\ &\ge (f_\infty - \delta)\|u\| \sum_{j=1}^{n} G_{n,j} g_j a_j \ge \|u\|,\end{aligned}$$

which means $\|Tu\| \ge \|u\|$. Thus, if we let $\Omega_2 = \{u \in \mathbb{R}^n \mid \|u\| < H_2\}$, then $\overline{\Omega_1} \subset \Omega_2$, and

$$\|Tu\| \ge \|u\| \quad \text{for} \quad u \in P \cap \partial\Omega_2.$$

Now, since the condition (K1) of Theorem 1.3 is satisfied, there exists a fixed point of T in $P \cap (\overline{\Omega_2} - \Omega_1)$. The proof is now complete. □

In a similar way, we can prove the next existence result.

Theorem 3.4. *Suppose that (H) holds. If $BF_\infty < 1 < Af_0$, then the problem* (1.4), (1.5) *has at least one positive solution.*

The next two theorems provide sufficient conditions for the nonexistence of positive solutions to the problem (1.4), (1.5).

Theorem 3.5. *Suppose that (H) holds. If $Bf(x) < x$ for all $x \in (0, +\infty)$, then the problem* (1.4), (1.5) *has no positive solutions.*

Proof. Assume to the contrary that $u \in P$ is a positive solution of the sum equation (1.6). Then, $u_i > 0$ for $1 \le i \le n$, and

$$\begin{aligned} u_n &= \sum_{j=1}^{n} G_{n,j} g_j f(u_j) < B^{-1} \sum_{j=1}^{n} G_{n,j} g_j u_j \\ &\le B^{-1} u_n \sum_{j=1}^{n} G_{n,j} g_j b_j = B^{-1} B u_n = u_n, \end{aligned}$$

which is a contradiction. The proof is complete. □

Theorem 3.6. *Suppose that (H) holds. If $Af(x) > x$ for all $x \in (0, +\infty)$, then the problem* (1.4), (1.5) *has no positive solutions.*

The proof of Theorem 3.6 is quite similar to that of Theorem 3.5 and is therefore omitted.

Example 3.7. Consider the problem

$$\Delta^4 y_{i-2} = \lambda g_i f(y_i), \quad 1 \le i \le 10, \tag{3.1}$$

$$y_0 = \Delta y_{-1} = \Delta y_{10} = \Delta^3 y_9 = 0. \tag{3.2}$$

Here $\lambda > 0$ is a parameter, $g_i = (1 + i + i^2)/1000000$ for $1 \le i \le 10$, and

$$f(u) = \frac{\lambda u(1 + 2u)}{1 + u}, \quad u \ge 0, \tag{3.3}$$

It is easy to see that $f_0 = F_0 = \lambda$, $f_\infty = F_\infty = 2\lambda$, and

$$\lambda u < f(u) < 2\lambda u, \quad \text{for} \quad u > 0.$$

By direct calculation, we find that

$$A = \frac{68539}{2000000} \quad \text{and} \quad B = \frac{23067}{625000}.$$

By Theorem 3.3, if

$$14.59 \approx \frac{1}{2A} < \lambda < \frac{1}{B} \approx 27.09,$$

then the problem (3.1), (3.2) has at least one positive solution.

From Theorems 3.5 and 3.6, we see that if

$$\lambda \ge \frac{1}{A} \approx 29.18 \quad \text{or} \quad \lambda \le \frac{1}{2B} \approx 13.547,$$

then the problem (3.1), (3.2) has no positive solutions.

This example shows that our existence and nonexistence conditions are quite sharp.

Example 3.8. We consider the boundary value problem

$$u''''(t) = p(t) \cdot (7 + 3u(t)), \quad 0 < t < 1, \tag{3.4}$$

$$u(0) = u'(0) = u'(1) = u'''(1) = 0, \tag{3.5}$$

where

$$p(t) = \frac{240t}{49 + 75t^2 - 60t^3 + 6t^5}.$$

This boundary value problem has a positive solution

$$\hat{u}(t) = (25t^2 - 20t^3 + 2t^5)/7.$$

Note that $\max_{0 \le t \le 1} \hat{u}(t) = 1$.

For illustrative purposes, we now try to solve problem (3.4), (3.5) numerically. First, we let $n = 500$, $h = 1/n$, and, for each $i = -1, 0, 1, 2, 3, \ldots, n, n+1, n+2$, we let

$$t_i = (i - 1/2)h.$$

This means that we divide the interval $[0, 1]$ into $n = 500$ subintervals of equal length and we let $t_1, t_2, \ldots, t_n$ be the midpoints of these subintervals. The discretization of the boundary value problem (3.4), (3.5) is now

$$\left(u(t_{i-2}) - 4u(t_{i-1}) + 6u(t_i) - 4u(t_{i+1}) + u(t_{i+2})\right)/h^4 = p(t_i)(7 + 3u(t_i)), \quad i = 1, 2, 3, \ldots, n, \tag{3.6}$$

$$\left.\begin{array}{l} u(t_0) = 0, \quad u(t_0) - u(t_{-1}) = 0, \\ u(t_{n+1}) - u(t_n) = 0, \quad u(t_{n+2}) - 3u(t_{n+1}) + 3u(t_n) - u(t_{n-1}) = 0. \end{array}\right\} \tag{3.7}$$

If we denote $u(t_i)$ by u_i for each $i = -1, 0, 1, \ldots, n, n+1, n+2$, then we can rewrite problem (3.6), (3.7) as

$$\Delta^4 u_{i-2} = h^4 p(t_i)(7 + 3u_i), \quad i = 1, 2, 3, \ldots, n, \tag{3.8}$$

$$u_0 = \Delta u_{-1} = \Delta u_n = \Delta^3 u_{n-1} = 0. \tag{3.9}$$

The boundary value problem (3.8), (3.9) is equivalent to the sum equation

$$u_i = \sum_{j=1}^{n} G(i, j) h^4 p(t_j)(7 + 3u_j), \quad i = 1, 2, 3, \ldots, n. \tag{3.10}$$

By using the contraction mapping theorem, we can show that the sum equation (3.10) has a unique solution.

Next, we construct an approximate solution to the sum equation (3.10) through iteration. Let

$$u_i^{(0)} = 1, \quad i = 1, 2, \ldots, n.$$

For $m \ge 1$, we let

$$u_i^{(m)} = \sum_{j=1}^{n} G(i, j) h^4 p(t_j) \left(7 + 3u_j^{(m-1)}\right), \quad i = 1, 2, 3, \ldots, n.$$

This iteration converges very fast, and for $m \geq 6$, $\left\{u_i^{(m)}\right\}_{i=1}^{n}$ is a very nice approximation for $\hat{u}(t)$. Calculations using *MATLAB* indicate that

$$\max_{1\leq i\leq n} \left|u_i^{(6)} - \hat{u}(t_i)\right| < 0.0103,$$

which is a very small error if compared with the size of $\hat{u}(t)$.

4. ESTIMATES FOR THE FIRST EIGENVALUE

In this section, we consider the boundary eigenvalue problem

$$\Delta^4 y_{i-2} = \lambda g_i y_i, \quad 1 \leq i \leq n, \tag{4.1}$$

$$y_0 = \Delta y_{-1} = \Delta y_n = \Delta^3 y_{n-1} = 0. \tag{4.2}$$

If we let

$$D = \begin{pmatrix} 6 & -4 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ -4 & 6 & -4 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ 1 & -4 & 6 & -4 & 1 & \cdots & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -4 & 6 & -4 & \cdots & 0 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 1 & -4 & 6 & -4 & 1 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 & -4 & 6 & -3 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & -3 & 2 \end{pmatrix},$$

$$K = \mathrm{diag}(g_1, \ldots, g_n),$$

$$y = (y_1, y_2, \ldots, y_n),$$

then we can write the problem (4.1), (4.2) into the form

$$Dy^T = \lambda K y^T,$$

or equivalently

$$y^T = \lambda G K y^T, \tag{4.3}$$

where

$$G = (G_{i,j})_{n\times n}.$$

Let M be the number of nonzero components in the vector $(g_1, \ldots, g_n)$. In 2006, Ji and Yang [6] proved the following result.

Theorem 4.1. *The eigenvalue problem* (4.3) *has* M *positive eigenvalues. Each of the* M *eigenvalues is simple. The smallest eigenvalue* λ_1 *corresponds to a positive eigen-solution*

$$y = (y_1, \ldots, y_n) \in Y.$$

By using the upper and lower estimates for positive solutions that we obtained in Section 2, we can prove the following theorem.

Theorem 4.2. *If λ_1 is the smallest positive eigenvalue of the problem* (4.3)*, then*

$$1/B \leq \lambda_1 \leq 1/A.$$

Proof. Let $z = (z_1, \ldots, z_n) \in Y$ be an eigen-solution corresponding to λ_1. Then it is obvious that $z \in P$, $z_n > 0$. And, we have

$$z_n \;=\; \lambda_1 (GKz)_n = \lambda_1 \sum_{j=1}^{n} G_{n,j} g_j z_j \geq \lambda_1 z_n \sum_{j=1}^{n} G_{n,j} g_j a_j = \lambda_1 z_n A.$$

This implies that $\lambda_1 \leq 1/A$. In a very similar way, we can show that $\lambda_1 \geq 1/B$. The proof is complete. □

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REFERENCES

[1] D. R. Anderson and F. Minhós, A discrete fourth-order Lidstone problem with parameters, *Appl. Math. Comput.* **214** (2009), no. 2, 523–533.

[2] J. M. Davis, P. W. Eloe, and J. Henderson, Comparison of eigenvalues for discrete Lidstone boundary value problems, *Dynam. Systems Appl.* **8** (1999), no. 3–4, 381–388.

[3] J. R. Graef, L. Kong, and M. Wang, Multiple solutions to a periodic boundary value problem for a nonlinear discrete fourth order equation, *Adv. Dyn. Syst. Appl.* **8** (2013), no. 2, 203–215.

[4] J. R. Graef and B. Yang, Existence and nonexistence of positive solutions of fourth order nonlinear boundary value problems, *Appl. Anal.* **74** (2000), no. 1–2, 201–214.

[5] J. Henderson, Multiple symmetric positive solutions for discrete Lidstone boundary value problems, *Dynam. Contin. Discrete Impuls. Systems* **7** (2000), no. 4, 577–585.

[6] J. Ji and B. Yang, Eigenvalue comparisons for boundary value problems of the discrete beam equation, *Adv. Difference Equ.*, **2006** (2006), Art. ID 81025, 9 pp.

[7] M. A. Krasnosel'skii, *Positive Solutions of Operator Equations*, Noordhoff, Groningen, 1964.

[8] R. Ma and Y. Xu, Existence of positive solution for nonlinear fourth-order difference equations, *Comput. Math. Appl.* **59** (2010), no. 12, 3770–3777.

[9] Y. Sang, Z. Wei, and W. Dong, Existence and uniqueness of positive solutions for discrete fourth-order Lidstone problem with a parameter, *Adv. Difference Equ.* **2010** (2010), Art. ID 971540, 18 pp.

[10] B. Yang, Positive solutions for the beam equation under certain boundary conditions, *Electronic J. of Differential Equations*, **2005** (2005), no. 78, 1–8.

[11] C. Yuan, D. Jiang, and D. O'Regan, Existence and uniqueness of positive solutions for fourth-order nonlinear singular continuous and discrete one-dimension p-Laplacian problems, *Panamer. Math. J.* **18** (2008), no. 3, 31–46.