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EXISTENCE AND UNIQUENESS OF SOLUTIONS OF CAPUTO FRACTIONAL DIFFERENTIAL EQUATIONS WITH INTEGRAL BOUNDARY CONDITIONS

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ABSTRACT. In this paper we develop the monotone iterative technique for Caputo fractional differential equations with integral boundary conditions. In order to prove the corresponding results we use the weakened assumtion of C^q-continuity in place of local Hölder continuity.

KEYWORDS. Caputo fractional differential equations, integral boundary conditions, Monotone iterative technique, Existence.

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1. INTRODUCTION

The theory of Fractional differential equations has been growing independently and is emerging as tool in modelling the real world problems in a wide range of areas, as the fractional calculus generalizes and includes the concepts of classical derivative and integral of integer order and consequently its theory is much richer than the theory of ordinary differential equations.

Although numerous theoretical applications of the fractional calculus operators have been found during their long history, many mathematicians and applied researchers have tried to model real processes using Fractional Differential Equations and it was realised that the derivatives of the arbitrary order provide an excelent frame work for modelling the problems in a variety of disciplines such as Control Theory of Dynamical Systems, Electro-Chemistry of Corrosion, Optics and Signal Processing, Network Traffic, Atmospheric Diffusion of Polution and many more. There is a vast amount of literature available on this important area and some of important publications are [2,4,5,6,9] .

In this paper we develop the Monotone iterative technique [3] which is a theoretical and a constructive method to obtain existence of solutions in a sector. While developing this technique we consider the weakened hypothesis of C^q-continuity.

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2. PRELIMINARIES

In this section we present all the results that are necessary to prove our main result. We begin with a Lemma [11] with the weakened hypothesis of C_p-continuity. This Lemma is essential in proving the basic differential inequality results for fractional differential inequalities.

As observed above, the comparison theorems [4] in the fractional differential equations set-up require Hölder continuity. Although this requirement is used to develop iterative techniques such as the monotone iterative technique [5,6,7,8,10] and the method of quasilinearization [12,13], there is no feasible way to check whether the functions involved are Hölder continuous. To avoid this situation, we used comparison results of [11] under the weaker condition of C_p-continuity. Basically, Lemma 2.3.1 in [4] is essential in establishing the comparison theorems, a detailed proof of this result under the weakened hypothesis was taken from [11].

We begin with the definition of the class $C_p[[t_0, T], \mathbb{R}]$, and proceed to state the results from [11].

Definition 2.1. m is said to be C_p-continuous, (i.e) $m \in C_p[[t_0, T], \mathbb{R}]$, if $m \in C[(t_0, T], \mathbb{R}]$ and $(t - t_0)^p m(t) \in C[[t_0, T], \mathbb{R}]$ with $p + q = 1$.

Definition 2.2. For $m \in C_p[[t_0, T], \mathbb{R}]$, the Riemann-Liouville derivative of $m(t)$ is defined as

$$D^q m(t) = \frac{1}{\Gamma(p)} \frac{d}{dt} \int_{t_0}^{t} (t-s)^{p-1} m(s) ds. \tag{2.1}$$

We next give the proof of the following lemma from [11] that is vital for our main result.

Lemma 2.3. *Let $m \in C_p[[t_0, T], \mathbb{R}]$. Suppose that for any $t_1 \in (t_0, T]$, we have $m(t_1) = 0$ and $m(t) < 0$ for $t_0 < t < t_1$, $m(t)(t-t_0)^p\big|_{t=t_0} < 0$ then it follows that*

$$D^q m(t_1) \geq 0.$$

Proof. Consider $m \in C_p[[t_0, T], \mathbb{R}]$, such that $m(t_1) = 0$ and $m(t) < 0$ for $t_0 < t < t_1$. Then $m(t)$ is continuous on $(t_0, T]$ and $m(t)(t - t_0)^p$ is continuous on $[t_0, T]$. Since $m(t)$ is continuous on $(t_0, T]$, given any t_1 such that $t_0 < t_1 \leq T$, there exists a $k(t_1) > 0$ such that

$$-k(t_1)(t_1 - s) \leq m(t) - m(s) \leq k(t_1)(t_1 - s) \tag{2.2}$$

for $t_0 < t_1 - h \leq s \leq t_1 + h < T$, $h > 0$. We require to prove $D^q m(t_1) \geq 0$. We have $D^q m(t) = \frac{1}{\Gamma(p)} \frac{d}{dt} \int_{t_0}^{t} (t-s)^{p-1} m(s) ds$, set $H(t) = \int_{t_0}^{t} (t-s)^{p-1} m(s) ds$, then, $H(t_1) - H(t_1 - h) = \int_{t_0}^{t_1 - h} [(t_1 - s)^{p-1} - (t_1 - h - s)^{p-1}] m(s) ds + \int_{t_1 - h}^{t_1} (t_1 - s)^{p-1} m(s) ds$. Let $I_1 = \int_{t_0}^{t_1-h} [(t_1 - s)^{p-1} - (t_1 - h - s)^{p-1}] m(s) ds$ and $I_2 = \int_{t_1-h}^{t_1} (t_1 - s)^{p-1} m(s) ds$.

Since $t_1 - s > t_1 - h - s$ and $p-1<0$, we have $(t_1-s)^{p-1} < (t_1-h-s)^{p-1}$. Also by using the fact that $m(t) < 0$ for $t_0 \leq t < t_1$, we get $I_1 \geq 0$. Now, consider $I_2 = \int_{t_1-h}^{t_1} (t_1-s)^{p-1} m(s)ds$. Using (2.2) and the fact that $m(t_1) = 0$, for $s \in (t_1-h, t_1+h)$ we obtain, $m(s) \geq -k(t_1)(t_1-s)$, and $I_2 \geq -k(t_1)\int_{t_1-h}^{t_1}(t_1-s)^p ds = -k(t_1)\frac{h^{p+1}}{p+1}$. Thus we have $H(t_1) - H(t_1-h) \geq -k(t_1)\frac{h^{p+1}}{p+1}$. Now by dividing with h and apllying the limits as $h \to 0$, we have $\lim_{h\to 0}[\frac{H(t_1)-H(t_1-h)}{h} + \frac{k(t_1)h^{p+1}}{(p+1)h}] \geq 0$. Since $p \in (0,1)$, we conclude that $\frac{dH(t_1)}{dt} \geq 0$, which implies that $D^q m(t_1) \geq 0$. □

We now define a C^q-continuous function.

Definition 2.4. u is said to be C^q-continuous, (i.e) $u \in C^q[[t_0,T],\mathbb{R}]$ if the Caputo derivative of u denoted by ${}^cD^q u$ exists and u satisfies

$$ {}^cD^q u(t) = \frac{1}{\Gamma(1-q)} \int_{t_0}^{t} (t-s)^{-q} u'(s) ds. \tag{2.3} $$

We note that the Caputo and Riemann-Liouville derivatives are related as follows.

$$ {}^cD^q x(t) = D^q[x(t) - x(t_0)]. \tag{2.4} $$

It is convenient to work with the Caputo fractional derivative, since the initial conditions for fractional differential equations are of the same form as those of ordinary differential equations. Further, the Caputo fractional derivative of a constant is zero, which is useful in our work.

Consider the IVP for the Caputo fractional differential equation with integral boundary condition given by

$$ \begin{cases} {}^cD^q x = f(t,x), \quad t \in [t_0,T], \\ x(t_0) = \lambda \int_{t_0}^{T} x(s)ds + d, \qquad \text{where } d \geq 0, \lambda = 1 \text{ or } -1 \end{cases} \tag{2.5} $$

for $0 < q < 1$, $f \in C[[t_0,T]\times\mathbb{R},\mathbb{R}]$. If $x \in C^q[[t_0,T],\mathbb{R}]$ satisfies (2.5), then it also satisfies the Volterra fractional integral

$$ x(t) = \lambda \int_{t_0}^{T} x(s)ds + d + \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t-s)^{q-1} f(s,x(s))ds, \tag{2.6} $$

where $d \geq 0$, $\lambda = 1$ or -1 and $t \in [t_0,T]$.

3. MONOTONE ITERATIVE TECHNIQUE FOR THE CASE $\lambda = 1$

The monotone iterative technique is a flexible mechanism that guarantees the existence of a solution in a sector. This technique has been developed for fractional differential equations in [7,8,10], and for differential equations with integral boundary conditions in [14], R-L fractional differential equations with integral boundary conditions in [17] with the hypothesis of local Hölder continuity and in [15, 16] for a HCFDE with a weakened hypothesis of C^q continuity. In this section we develop the monotone iterative technique for Caputo fractional differential equations (CFDE) involving integral boundary conditions for $\lambda = 1$, with the weakened hypothesis of C^q-continuity.

Consider the Caputo fractional differential equation given by

$$\begin{cases} {}^cD^q x = f(t,x), \quad t \in [t_0, T], \\ x(t_0) = \int_{t_0}^{T} x(s)ds + d, \quad d \geq 0. \end{cases} \tag{3.1}$$

for $0 < q < 1$, $f \in C[[t_0, T] \times \mathbb{R}, \mathbb{R}]$.

We begin with the definition of lower and upper solutions for (3.1).

Definition 3.1. $v, w \in C^q[[t_0, T], \mathbb{R}]$ are said to be lower and upper solutions of IVP of CFDE (3.1), if and only if, they satisfy the following inequalities

$$\begin{cases} {}^cD^q v(t) \leq f(t, v(t)), \quad t \in [t_0, T], \\ v(t_0) \leq \int_{t_0}^{T} v(s)ds + d, \quad d \geq 0 \end{cases} \tag{3.2}$$

and

$$\begin{cases} {}^cD^q w(t) \geq f(t, w(t)), \quad t \in [t_0, T], \\ w(t_0) \ \geq \int_{t_0}^{T} w(s)ds + d, \quad d \geq 0 \end{cases} \tag{3.3}$$

respectively.

We now present the fundamental results relative to strict and non strict fractional differential inequalities in Caputo fractional derivative set up, with a weakend hypothesis of C^q-continuity, for $\lambda = 1$. As the result for strict inequalities is very similar to that of Theorem 2.4 of [11], we omit.

Theorem 3.2. *Let $v, w \in C^q[[t_0, T], \mathbb{R}]$, $f \in C[[t_0, T] \times \mathbb{R}, \mathbb{R}]$ and*

$$(i)\ {}^cD^q v(t) \leq f(t, v(t))$$

and

$$(ii) {}^cD^q w(t) \geq f(t, w(t)),$$

$t_0 < t \leq T$, with one of the inequalities (i) or (ii) being strict. Then $v(t_0) < w(t_0)$, where $v(t_0) \leq \int_{t_0}^{T} v(s)ds + d$ and $w(t_0) \geq \int_{t_0}^{T} w(s)ds + d$ implies that

$$v(t) < w(t), \quad t_0 \leq t \leq T. \tag{3.4}$$

Theorem 3.3. *Let $v, w \in C^q[[t_0, T], \mathbb{R}]$, $f \in C[[t_0, T] \times \mathbb{R}, \mathbb{R}]$ and*

$$(i)\ {}^cD^q v(t) \leq f(t, v(t))$$

and

$$(ii) {}^cD^q w(t) \geq f(t, w(t)),$$

$t_0 < t \le T$. *Further assume that* f *satisfies the Lipschitz condition*

$$f(t,x) - f(t,y) \le L(x-y), \quad x \ge y, L > 0. \tag{3.5}$$

Then, $v(t_0) \le w(t_0)$, *where* $v(t_0) \le \int_{t_0}^{T} v(s)ds + d$ *and* $w(t_0) \ge \int_{t_0}^{T} w(s)ds + d$ *implies* $v(t) \le w(t)$, $t \in [t_0, T]$.

Proof. We set

$$w_\epsilon(t) = w(t) + \epsilon\lambda(t),$$

where ϵ is a small positive number and $\lambda(t) = E_{q,q}(2L(t-t_0)^q)$ now $w_\epsilon(t_0) = w(t_0) + \epsilon\lambda(t_0)$

$$\begin{aligned}
w_\epsilon(t_0) &= w(t_0) + \epsilon\left[\int_{t_0}^{T} \lambda(s)ds + d\right] \\
&= w(t_0) + \epsilon[\int_{t_0}^{T} \sum_{k=0}^{\infty} \frac{2L(t-t_0)^{qk}}{\Gamma(qk+q)} ds + d] \\
&= w(t_0) + \epsilon d + \epsilon \sum_{k=0}^{\infty} \int_{t_0}^{T} \frac{2L(s-t_0)^{qk}}{\Gamma(qk+q)} ds \\
&= w(t_0) + \epsilon d + \epsilon \sum_{k=0}^{\infty} \frac{2L(s-t_0)^{qk+1}}{\Gamma(qk+q)(qk+1)}\Big|_{t_0}^{T} \\
&= w(t_0) + \epsilon d + \epsilon \sum_{k=0}^{\infty} \frac{(qk+1)(T-t_0)^{qk+1}}{\Gamma(qk+q)(qk+1)(T-t_0)}
\end{aligned}$$

by choosing

$$\begin{aligned}
2L &= \frac{(qk+1)}{(T-t_0)} \\
&= w(t_0) + \epsilon d + \epsilon \sum_{k=0}^{\infty} \frac{(T-t_0)^{qk}}{\Gamma(qk+q)} \\
&= w(t_0) + \epsilon d + \epsilon E_{q,q}((T-t_0)^q) \\
&> w(t_0) \ge v(t_0)
\end{aligned}$$

and $w_\epsilon(t) > w(t)$.

Employing the Lipschitz condition (3.5) , we see that

$$\begin{aligned}
{}^cD^q w_\epsilon(t) &= {}^cD^q w(t) + {}^cD^q \epsilon\lambda(t) \\
&\ge f(t,w(t)) - f(t,w_\epsilon(t)) + f(t,w_\epsilon(t)) + \epsilon[2L\lambda(t)] \\
&\ge L(-\epsilon\lambda(t)) + f(t,w_\epsilon(t)) + \epsilon(2L\lambda(t)) \\
&> f(t,w_\epsilon(t))
\end{aligned}$$

Here we employed the fact that $\lambda(t)$ is the solution of linear Caputo Fractional Differential Equation

$$\begin{cases} {}^cD^q\lambda(t) = 2L\lambda(t), \quad t_0 \le t \le T, \\ \lambda(t_0) = \int_{t_0}^{T} \lambda(s)ds + d > 0. \end{cases}$$

Now applying the Theorem 3.2 to $v(t)$, $w_\epsilon(t)$ to get $v(t) < w_\epsilon(t)$, $t_0 \le t \le T$ As $\epsilon \to 0$, we arrive at the desired conclusion $v(t) \le w(t)$, $t_0 \le t \le T$. □

Parallel to Theorem 2.4.3 in [4], we state the comparison theorem for the Caputo fractional differential equation with integral boundary conditions for $\lambda = 1$ using the same weaker hypothesis. As the proof is similar to that of Theorem 2.4.3 in [4], we omit it.

Theorem 3.4. *Assume that $m \in C^q[[t_0, T], \mathbb{R}]$ and*

$$ {}^cD^q m(t) \le g(t, m(t)), \quad t_0 \le t \le T,$$

where $g \in C[[t_0, T] \times \mathbb{R}, \mathbb{R}]$. Let $r(t)$ be the maximal solution of the IVP

$$\begin{cases} {}^cD^q u = g(t, u), \\ u(t_0) = \int_{t_0}^{T} u(s)ds + d \ge 0 \end{cases} \tag{3.6}$$

existing on $[t_0, T]$ such that $m(t_0) = \int_{t_0}^{T} m(s)ds + d \le u(t_0)$. Then we have $m(t) \le r(t)$, $t_0 \le t \le T$.

Lemma 3.5. *The linear nonhomogeneous Caputo Fractional Differential Equation (CFDE)*

$$\begin{cases} {}^cD^q x(t) = f(t, y) - M(x - y), \quad t \in [t_0, T], \\ x(t_0) = \int_{t_0}^{T} x(s)ds + d, \quad d \ge 0 \end{cases} \tag{3.7}$$

has a unique solution [2,4] in the interval $[t_0, T]$, is given by

$$x(t) = \left(\int_{t_0}^{T} x(s)ds + d\right) E_q[-M(t-t_0)^q] + \int_{t_0}^{t} (t-s)^{q-1} E_{q,q}(-M(t-s)^q) f(s, y(s))ds.$$

As the proof is similar to given in [2,4] hence we omit.

Lemma 3.6. *The linear homogeneous Caputo fractional differential equation (CFDE)*

$$\begin{cases} {}^cD^q p(t) = -Mp(t), \quad t \in [t_0, T], \\ p(t_0) = \int_{t_0}^{T} p(s)ds + d, \quad d \ge 0 \end{cases} \tag{3.8}$$

has a unique solution [2,4] in the interval $[t_0, T]$, is given by

$$p(t) = \left(\int_{t_0}^{T} p(s)ds + d\right) E_q[-M(t - t_0)^q], \quad t \in [t_0, T].$$

Corollary 3.7. *From the above lemma we can also have*

$$\begin{cases} {}^cD^q p(t) \leq -Mp(t), & t \in [t_0, T], \\ p(t_0) \;\; = \int_{t_0}^T p(s)ds + d, & d \geq 0 \end{cases} \tag{3.9}$$

has a unique solution in the interval $[t_0, T]$, *is given by*

$$p(t) \leq \left(\int_{t_0}^T p(s)ds + d \right) E_q[-M(t-t_0)^q], \quad t \in [t_0, T]$$

Lemma 3.8. *Suppose that*

(i) $v_0(t), w_0(t)$ *be the lower and upper solutions of the IVP of CFDE* (3.1), *and* $v_0(t) \leq w_0(t)$, $t \in [t_0, T]$,

(ii) $v_1(t), w_1(t)$ *be the unique solutions of the linear nonhomogeneous Caputo fractional differential equations*

$$\begin{cases} {}^cD^q v_1 &= f(t, v_0) - M(v_1 - v_0), \quad t \in [t_0, T], \\ v_1(t_0) &= \int_{t_0}^T v_0(s)ds + d, \quad d \geq 0 \end{cases} \tag{3.10}$$

$$\begin{cases} {}^cD^q w_1 = f(t, w_0) - M(w_1 - w_0), & t \in [t_0, T], \\ w_1(t_0) = \int_{t_0}^T w_0(s)ds + d, & d \geq 0 \end{cases} \tag{3.11}$$

respectively,

(iii) $f \in C[[t_0, T] \times \mathbb{R}, \mathbb{R}]$ *and* $f(t, x) - f(t, y) \leq -M(x - y)$ *for* $v_0 \leq x \leq y \leq w_0$, $M > 0$.

Then $v_0(t) \leq v_1(t) \leq w_1(t) \leq w_0(t)$, $t \in [t_0, T]$.

Proof. Suppose that $v_0(t)$ be a lower solution of IVP of CFDE (3.1), and $v_1(t)$ be the unique solution of (3.10). Now, set $p(t) = v_0(t) - v_1(t)$, where $t \in [t_0, T]$. Then we have ${}^cD^q p(t) = {}^cD^q v_0(t) - {}^cD^q v_1(t) \leq f(t, v_0) - [f(t, v_0) - M(v_1 - v_0)] \leq -Mp(t)$. Thus ${}^cD^q p(t) \leq -Mp(t)$ and $p(t_0) = v_0(t_0) - v_1(t_0) \leq \int_{t_0}^T v_0(s)ds + d - \int_{t_0}^T v_0(s)ds - d = 0$. Then from the Corollary 3.7 we have $p(t) \leq p(t_0)E_q(-M(t - t_0)^q)$. Which yields $p(t) \leq 0$, $t \in [t_0, T]$ since $p(t_0) \leq 0$. Thus we have $v_0(t) \leq v_1(t)$, $t \in [t_0, T]$. Similarly suppose that $w_0(t)$ be an upper solution of IVP of CFDE (3.1) and $w_1(t)$ be the unique solution of (3.11). Now, set $p(t) = w_1(t) - w_0(t)$, where $t \in [t_0, T]$. Then we have ${}^cD^q p(t) = {}^cD^q w_1(t) - {}^cD^q w_0(t) \leq [f(t, w_0) - M(w_1 - w_0)] - f(t, w_0) \leq -Mp(t)$. Thus ${}^cD^q p(t) \leq -Mp(t)$ and $p(t_0) = w_1(t_0) - w_0(t_0) \leq \int_{t_0}^T w_0(s)ds + d - \int_{t_0}^T w_0(s)ds - d = 0$. Then from the Corollary 3.7 we have $p(t) \leq p(t_0)E_q(-M(t-t_0)^q)$. Which yields $p(t) \leq 0$, $t \in [t_0, T]$ since $p(t_0) \leq 0$. Thus we have $w_1(t) \leq w_0(t)$, $t \in [t_0, T]$. To prove $v_1 \leq w_1$ set $p(t) = v_1(t) - w_1(t)$, where $t \in [t_0, T]$. Then we have ${}^cD^q p(t) = {}^cD^q v_1(t) - {}^cD^q w_1(t) = f(t, v_0(t)) - M[v_1(t) - v_0(t))] - f(t, w_0(t)) + M[w_1(t) - w_0(t)] \leq -M[v_0(t) - w_0(t)] - M[v_1(t) - v_0(t))] + M[w_1(t) - w_0(t)] \leq -Mp(t)$. Thus ${}^cD^q p(t) \leq -Mp(t)$ and $p(t_0) = v_1(t_0) - w_1(t_0) = \int_{t_0}^T v_0(s)ds + d - \int_{t_0}^T w_0(s)ds -$

$d \le 0$ since $v_o(t) \le w_0(t)$, $t \in [t_0, T]$. Then from the Corollary 3.7 we have $p(t) \le p(t_0)E_q(-M(t-t_0)^q)$. Which yields $p(t) \le 0$, $t \in [t_0, T]$ since $p(t_0) \le 0$. Thus we have $v_1(t) \le w_1(t)$, $t \in [t_0, T]$, which completes the proof. □

We now state and prove the main result.

Theorem 3.9. *Assume that*

i) v_0, w_0 *be the lower and upper solutions of IVP of CFDE* (3.1) *such that* $v_0(t) \le w_0(t)$ *for* $t \in [t_0, T]$,
ii) $f \in C[[t_0, T] \times \mathbb{R}, \mathbb{R}]$ *and* $f(t,x) - f(t,y) \le -M(x-y)$ *for* $v_0 \le x \le y \le w_0$, $M > 0$.

Then there exist monotone sequences $\{v_n\}, \{w_n\}$ *such that* $v_n \to \rho$, $w_n \to r$ *as* $n \to \infty$ *uniformly and monotonically on* $[t_0, T]$ *and* (ρ, r) *are minimal and maximal solutions of the IVP of CFDE* (3.1) *respectively.*

Proof. Consider

$$\begin{cases} {}^cD^q v_{i+1}(t) = f(t, v_i(t)) - M(v_{i+1}(t) - v_i(t)), \quad t \in [t_0, T], \\ v_{i+1}(t_0) = \int_{t_0}^T v_i(s)ds + d, \quad d \ge 0 \end{cases} \tag{3.12}$$

and

$$\begin{cases} {}^cD^q w_{i+1}(t) = f(t, w_i(t)) - M(w_{i+1}(t) - w_i(t)), \quad t \in [t_0, T], \\ w_{i+1}(t_0) = \int_{t_0}^T w_i(s)ds + d, \quad d \ge 0 \end{cases} \tag{3.13}$$

For $i = 0$, Lemma 3.8 shows that

$$v_0(t) \le v_1(t) \le w_1(t) \le w_0(t), \quad t \in [t_0, T] \tag{3.14}$$

Next for $i > 1$, we assume that,

$$v_{i-1}(t) \le v_i(t) \le w_i(t) \le w_{i-1}(t), \quad t \in [t_0, T] \tag{3.15}$$

and prove that

$$v_i(t) \le v_{i+1}(t) \le w_{i+1}(t) \le w_i(t), \quad t \in [t_0, T]. \tag{3.16}$$

Since, $v_i(t)$ is a lower solution of IVP of CFDE (3.1) and $v_{i+1}(t)$ is the unique solution of linear nonhomogeneous CFDE (3.12) by applying Lemma 3.8 we obtain that $v_i(t) \le v_{i+1}(t)$ on $t \in [t_0, T]$. Similarly $w_i(t)$ is an upper solution of IVP of CFDE (3.1) and $w_{i+1}(t)$ is the unique solution of linear nonhomogeneous CFDE (3.13), application of Lemma 3.8 yields $w_{i+1}(t) \le w_i(t)$, $t \in [t_0, T]$. To prove $v_{i+1}(t) \le w_{i+1}(t)$, $t \in [t_0, T]$. Set $p(t) = v_{i+1}(t) - w_{i+1}(t)$, on $t \in [t_0, T]$. Then

$$\begin{aligned} {}^cD^q p(t) &= {}^cD^q v_{i+1}(t) - {}^cD^q w_{i+1}(t) \\ &= [f(t, v_i(t)) - M(v_{i+1}(t) - v_i(t))] - [f(t, w_i(t)) - M(w_{i+1}(t) - w_i(t))] \\ &\le -Mp(t), \end{aligned}$$

and $p(t_0) = v_{i+1}(t_0) - w_{i+1}(t_0) = \int_{t_0}^{T} v_i(s)ds + d - \int_{t_0}^{T} w_i(s)ds - d \leq 0$, since $v_i(t) \leq w_i(t)$ for all $t \in [t_0, T]$. Thus we have

$$\begin{cases} {}^cD^q p(t) \leq -Mp(t), \\ p(t_0) \leq 0. \end{cases}$$

From the Corollary 3.7 we get

$$p(t) \leq p(t_0)E_q(-M(t-t_0)^q), \quad t \in [t_0, T].$$

This yields $p(t) \leq 0$, $t \in [t_0, T]$, since $p(t_0) \leq 0$. Thus we have $v_{i+1}(t) \leq w_{i+1}(t)$, $t \in [t_0, T]$ Thus (3.16) is proved and therefore by the principle of mathematical induction, we see that

$$v_0 \leq v_1 \leq v_2 \leq \cdots \leq v_n \leq w_n \leq \cdots \leq w_2 \leq w_1 \leq w_0, \quad t \in [t_0, T].$$

Thus the sequence of functions $\{v_n\}$,$\{w_n\}$ are continuous and uniformly bounded on $[t_0, T]$, also we can conclude that the sequence of functions $\{v_n\}$ and $\{w_n\}$ are equicontinuous by using Lemma 2.3.2 in [4] and the relation between the solution of Caputo fractional differential equations and the solution of R-L fractional differential equations [1]. Thus by using Arzela-Ascoli's Theorem, we obtain that the entire sequence $\{v_n\}$ converges uniformly and monotonically to $\rho(t)$ on $[t_0, T]$, and $\{w_n\}$ converges uniformly and monotonically to $r(t)$ on $[t_0, T]$, i.e

$$\lim_{n\to\infty} v_n = \rho \text{ and } \lim_{n\to\infty} w_n = r,$$

$t \in [t_0, T]$.

To show that ρ and r are the solutions of the IVP of CFDE (3.1) using the corresponding Volterra's integral equations

$$v_{n+1}(t) = \int_{t_0}^{T} v_n(s)ds + d + \frac{1}{\Gamma(q)}\int_{t_0}^{t}(t-s)^{q-1}h(s, v_n(s))ds, \quad d \geq 0 \tag{3.17}$$

and

$$w_{n+1}(t) = \int_{t_0}^{T} w_n(s)ds + d + \frac{1}{\Gamma(q)}\int_{t_0}^{t}(t-s)^{q-1}h(s, w_n(s))ds, \quad d \geq 0 \tag{3.18}$$

where $h(s, v_n(s)) = f(t, v_n(s)) - M(v_{n+1}(s) - v_n(s))$ and $h(s, w_n(s)) = f(t, w_n(s)) - M(w_{n+1}(s) - w_n(s))$. Now by taking the limits as $n \to \infty$, and using the uniform continuity of f and the uniform convergence of the sequences $\{v_n\}$ and $\{w_n\}$, $t \in [t_0, T]$, we get

$$\begin{cases} {}^cD^q \rho(t) = f(t, \rho(t)), \quad t \in [t_0, T], \\ \rho(t_0) = \int_{t_0}^{T} x(s)ds + d, \quad d \geq 0 \end{cases} \tag{3.19}$$

and

$$\begin{cases} {}^cD^q r(t) = f(t, r(t)), \quad t \in [t_0, T], \\ r(t_0) = \int_{t_0}^{T} x(s)ds + d, \quad d \geq 0 \end{cases} \tag{3.20}$$

for $t \in [t_0, T]$. Further,

$$v_0(t) \leq \rho(t) \leq r(t) \leq w_0(t)$$

$t \in [t_0, T]$. To prove $\rho(t)$, $r(t)$ are respectively minimal and maximal solutions of IVP of CFDE (3.1), we have to show that if $x(t)$ is any solution of IVP of CFDE (3.1) such that $v_0(t) \leq x(t) \leq w_0(t)$, $t \in [t_0, T]$ then

$$v_0(t) \leq \rho(t) \leq x(t) \leq r(t) \leq w_0(t)$$

$t \in [t_0, T]$. To prove this, since $v_n(t)$, $w_n(t)$ satisfy

$${}^cD^q v_n(t) = f(t, v_{n-1}(t)) - M(v_n(t) - v_{n-1}(t)), \quad v_n(t_0) = \int_{t_0}^{T} v_{n-1}(s)ds + d, \quad d \geq 0$$

and

$${}^cD^q w_n(t) = f(t, w_{n-1}(t)) - M(w_n(t) - w_{n-1}(t)), \quad w_n(t_0) = \int_{t_0}^{T} w_{n-1}(s)ds + d, \quad d \geq 0$$

respectively, suppose that for some n, $v_n(t) \leq x(t) \leq w_n(t)$, $t \in [t_0, T]$ and set $p(t) = v_{n+1}(t) - x(t)$ so that,

$$\begin{aligned} {}^cD^q p(t) &= {}^cD^q v_{n+1}(t) - {}^cD^q x(t) = [f(t, v_n(t)) - M(v_{n+1}(t) - v_n(t)] - f(t, x) \\ &\leq -M(v_n(t) - x(t)) - M(v_{n+1}(t) - v_n(t)) \\ &= -Mp(t), \quad t \in [t_0, T]. \end{aligned}$$

Also $p(t_0) = \int_{t_0}^{T} v_n(s)ds + d - \int_{t_0}^{T} x(s)ds - d \leq 0$, $d \geq 0$ since $v_n(t) \leq x(t)$, $t \in [t_0, T]$. Thus

$$\begin{cases} {}^cD^q p(t) \leq -Mp(t), & t \in [t_0, T], \\ p(t_0) \leq 0 \end{cases}$$

Now by applying the Corollary 3.7 we get $v_{n+1}(t) \leq x(t)$, $t \in [t_0, T]$. Similar process yields that $x(t) \leq w_{n+1}(t)$, $t \in [t_0, T]$. Thus we have $v_{n+1}(t) \leq x(t) \leq w_{n+1}(t)$ on $[t_0, T]$, whenever $v_n(t) \leq x(t) \leq w_n(t)$ on $[t_0, T]$. Now taking the limits, we obtain that $\rho(t) \leq x(t) \leq r(t)$ for $t \in [t_0, T]$. □

Corollary 3.10. *If in addition to the assumptions of the Theorem* 3.9 *we have*

$$f(t, x) - f(t, y) \leq M(x - y),$$

$v_0 \leq y \leq x \leq w_0$, $M \geq 0$ and $L \geq 0$. Then $\rho = r = x$ is the unique solution of the IVP of CFDE (3.1).

Proof. To prove this, set $p(t) = r(t) - \rho(t)$. Then we have,

$$\begin{aligned} {}^cD^q p(t) &= {}^cD^q r(t) - {}^cD^q \rho(t) = f(t, r(t)) - f(t, \rho(t)) \\ &\leq M(r(t) - \rho(t)) = Mp(t), \quad t \in [t_0, T] \end{aligned}$$

and $p(t_0) = 0$. Thus we have

$$\begin{cases} {}^cD^q p(t) \leq Mp(t), \quad t \in [t_0, T] \\ p(t_0) = 0. \end{cases}$$

Now by using the Corollary 3.7 we can have $p(t) \leq 0$, which implies that $r(t) \leq \rho(t)$, $t \in [t_0, T]$. Thus we conclude that $\rho(t) = r(t) = x(t)$ for $t \in [t_0, T]$, which proves the uniqueness of solution. Thus the proof is complete. □

4. MONOTONE ITERATIVE TECHNIQUE FOR THE CASE $\lambda = -1$

In this section we develop the monotone iterative technique for Caputo fractional differential equations (CFDE) involving integral boundary conditions for $\lambda = -1$, with the weakened hypothesis of C^q-continuity.

Consider the Caputo fractional differential equation involving integral boundary conditions for $\lambda = -1$ given by

$$\begin{cases} {}^cD^q x = f(t, x), \quad t \in [t_0, T], \\ x(t_0) = -\int_{t_0}^T x(s)ds + d, \quad d \geq 0 \end{cases} \tag{4.1}$$

for $0 < q < 1$, $f \in C[[t_0, T] \times \mathbb{R}, \mathbb{R}]$.

Definition 4.1. $v, w \in C^q[[t_0, T], \mathbb{R}]$ are said to be weakly coupled lower and upper solutions of IVP of CFDE (4.1), if and only if, they satisfy the following inequalities

$$\begin{cases} {}^cD^q v(t) \ \leq \ f(t, v(t)), \quad t \in [t_0, T], \\ v(t_0) \ \leq -\int_{t_0}^T w(s)ds + d, \quad d \geq 0 \end{cases} \tag{4.2}$$

and

$$\begin{cases} {}^cD^q w(t) \geq f(t, w(t)), \quad t \in [t_0, T], \\ w(t_0) \geq -\int_{t_0}^T v(s)ds + d, \quad d \geq 0 \end{cases} \tag{4.3}$$

respectively.

As earlier we state the fundamental results relative to strict and non strict fractional differential inequalities in Caputo fractional derivative set up, with a weakend hypothesis of C^q-continuity, for $\lambda = -1$. As the result for strict inequalities is very similar to that of Theorem 2.4 of [11], we omit.

Theorem 4.2. *Let $v, w \in C^q[[t_0, T], \mathbb{R}]$, $f \in C[[t_0, T] \times \mathbb{R}, \mathbb{R}]$ and*

$$(i) \quad {}^cD^q v(t) \leq f(t, v(t))$$

and

$$(ii) \quad {}^cD^q w(t) \geq f(t, w(t)),$$

$t_0 < t \le T$, with one of the inequalities (i) or (ii) being strict. Then $v(t_0) < w(t_0)$, where $v(t_0) \le -\int_{t_0}^{T} w_0(s)ds + d$, $d \ge 0$ and $w(t_0) \ge -\int_{t_0}^{T} v_0(s)ds + d$, $d \ge 0$ implies that

$$v(t) < w(t), \quad t_0 \le t \le T. \tag{4.4}$$

The next result deals with the inequality theorem for non strict inequalities.

Theorem 4.3. *Let $v, w \in C^q[[t_0, T], \mathbb{R}]$, $f \in C[[t_0, T] \times \mathbb{R}, \mathbb{R}]$ and*

$$(i) \quad {}^cD^q v(t) \le f(t, v(t))$$

and

$$(ii) \quad {}^cD^q w(t) \ge f(t, w(t)),$$

$t_0 < t \le T$, with non strict inequalities (i) and (ii). Further assume that f satisfies the Lipschitz condition

$$f(t,x) - f(t,y) \le \ -L(x-y), \quad x \ge y, \ L > 0. \tag{4.5}$$

Then, $v(t_0) \le w(t_0)$, where $v(t_0) \le -\int_{t_0}^{T} w(s)ds + d$, $d \ge 0$ and $w(t_0) \ge -\int_{t_0}^{T} v(s)ds + d$, $d \ge 0$ implies $v(t) \le w(t)$, $t \in [t_0, T]$.

Proof. We set

$$w_\epsilon(t) = w(t) + \epsilon\lambda(t),$$

where ϵ is a small positive number and $\lambda(t) = E_{q,q}(-2L(t-t_0)^q)$ now $w_\epsilon(t_0) = w(t_0) + \epsilon\lambda(t_0)$

$$\begin{aligned}
w_\epsilon(t_0) &= w(t_0) + \epsilon[-\int_{t_0}^{T} \lambda(s)ds + d] \\
&= w(t_0) + \epsilon\left[-\int_{t_0}^{T} \sum_{k=0}^{\infty} \frac{-2L(t-t_0)^{qk}}{\Gamma(qk+q)} ds + d\right] \\
&= w(t_0) + \epsilon d - \epsilon \sum_{k=0}^{\infty} \int_{t_0}^{T} \frac{-2L(s-t_0)^{qk}}{\Gamma(qk+q)} ds \\
&= w(t_0) + \epsilon d - \epsilon \sum_{k=0}^{\infty} \frac{-2L(s-t_0)^{qk+1}}{\Gamma(qk+q)(qk+1)}\Big|_{t_0}^{T} \\
&= w(t_0) + \epsilon d + \epsilon \sum_{k=0}^{\infty} \frac{(qk+1)(T-t_0)^{qk+1}}{\Gamma(qk+q)(qk+1)(T-t_0)}
\end{aligned}$$

by choosing

$$\begin{aligned}
2L &= \frac{(qk+1)}{(T-t_0)} \\
&= w(t_0) + \epsilon d + \epsilon \sum_{k=0}^{\infty} \frac{(T-t_0)^{qk}}{\Gamma(qk+q)} \\
&= w(t_0) + \epsilon d + \epsilon E_{q,q}((T-t_0)^q)
\end{aligned}$$

$$> w(t_0) \geq v(t_0)$$

and $w_\epsilon(t) > w(t)$. Employing the Lipschitz condition (4.5), we see that

$$\begin{aligned} {}^cD^q w_\epsilon(t) &= {}^cD^q w(t) + {}^cD^q \epsilon\lambda(t) \\ &\geq f(t, w(t)) - f(t, w_\epsilon(t)) + f(t, w_\epsilon(t)) + \epsilon[2L\lambda(t)] \\ &\geq L(-\epsilon\lambda(t)) + f(t, w_\epsilon(t)) + \epsilon(2L\lambda(t)) \\ &> f(t, w_\epsilon(t)) \end{aligned}$$

Here we employed the fact that $\lambda(t)$ is the solution of linear Caputo Fractional Differential Equation

$$\begin{cases} {}^cD^q\lambda(t) = 2L\lambda(t), \quad t_0 \leq t \leq T, \\ \lambda(t_0) = -\int_{t_0}^T \lambda(s)ds + d > 0, \quad d \geq 0. \end{cases}$$

Now applying the Theorem 4.2 to $v(t), w_\epsilon(t)$ to get $v(t) < w_\epsilon(t)$, $t_0 \leq t \leq T$. As $\epsilon \to 0$, we arrive at the desired conclusion $v(t) \leq w(t)$, $t_0 \leq t \leq T$. □

Parallel to Theorem 2.4.3 in [4], we state the comparison theorem for the Caputo fractional differential equation with integral boundary conditions for $\lambda = -1$ using the same weaker hypothesis. As the proof is similar to that of Theorem 2.4.3 in [4], we omit it.

Theorem 4.4. *Assume that $m \in C^q[[t_0, T], \mathbb{R}]$ and*

$${}^cD^q m(t) \leq g(t, m(t)), \quad t_0 \leq t \leq T,$$

where $g \in C[[t_0, T] \times \mathbb{R}, \mathbb{R}]$. Let $r(t)$ be the maximal solution of the IVP

$$\begin{cases} {}^cD^q u = g(t, u), \\ u(t_0) = -\int_{t_0}^T u(s)ds + d \geq 0, \quad d \geq 0, \end{cases} \tag{4.6}$$

existing on $[t_0, T]$ such that $m(t_0) = -\int_{t_0}^T m(s)ds + d \leq u(t_0)$. Then we have $m(t) \leq r(t)$, $t_0 \leq t \leq T$.

Before we proceed further, we need to present the following results to linear Caputo fractional differential equations in a suitable form for $\lambda = -1$.

Lemma 4.5. *The linear nonhomogeneous Caputo Fractional Differential Equation (CFDE)*

$$\begin{cases} {}^cD^q x(t) = f(t, y) - M(x - y), \quad t \in [t_0, T], \\ x(t_0) = -\int_{t_0}^T x(s)ds + d, \quad d \geq 0 \end{cases} \tag{4.7}$$

has a unique solution [2, 4] in the interval $[t_0, T]$, is given by

$$x(t) = \left(-\int_{t_0}^T x(s)ds + d \right) E_q[-M(t-t_0)^q] + \int_{t_0}^t (t-s)^{q-1} E_{q,q}(-M(t-s)^q) f(s, y(s))ds.$$

As the proof is similar to given in [2,4] hence we omit.

Lemma 4.6. *The linear homogeneous Caputo fractional differential equation (CFDE)*

$$\begin{cases} {}^cD^q p(t) = -Mp(t), \quad t \in [t_0, T], \\ p(t_0) = -\int_{t_0}^{T} p(s)ds + d, \quad d \geq 0 \end{cases} \tag{4.8}$$

has a unique solution $[2, 4]$ *in the interval* $[t_0, T]$*, is given by* $p(t) = (-\int_{t_0}^{T} p(s)ds + d)E_q[-M(t-t_0)^q]$, $t \in [t_0, T]$

Corollary 4.7. *From the above lemma we can also have*

$$\begin{cases} {}^cD^q p(t) \leq -Mp(t), \quad t \in [t_0, T], \\ p(t_0) = -\int_{t_0}^{T} p(s)ds + d, \quad d \geq 0 \end{cases} \tag{4.9}$$

has a unique solution in the interval $[t_0, T]$*, is given by*

$$p(t) \leq \left(-\int_{t_0}^{T} p(s)ds + d \right) E_q[-M(t-t_0)^q], \quad t \in [t_0, T].$$

Lemma 4.8. *Suppose that*

(i) $v_0(t), w_0(t)$ *be the weakly coupled lower and upper solutions of the IVP of CFDE* (4.1) *and* $v_0(t) \leq w_0(t)$, $t \in [t_0, T]$,

(ii) $v_1(t), w_1(t)$ *be the unique solutions of the linear nonhomogeneous Caputo fractional differential equations*

$$\begin{cases} {}^cD^q v_1 = f(t, v_0) - M(v_1 - v_0), \quad t \in [t_0, T], \\ v_1(t_0) = -\int_{t_0}^{T} w_0(s)ds + d, \quad d \geq 0 \end{cases} \tag{4.10}$$

$$\begin{cases} {}^cD^q w_1 = f(t, w_0) - M(w_1 - w_0), \quad t \in [t_0, T], \\ w_1(t_0) = -\int_{t_0}^{T} v_0(s)ds + d, \quad d \geq 0 \end{cases} \tag{4.11}$$

respectively,

(iii) $f \in C[[t_0, T] \times \mathbb{R}, \mathbb{R}]$ *and* $f(t, x) - f(t, y) \leq -M(x - y)$ *for* $v_0 \leq x \leq y \leq w_0$, $M > 0$. *Then* $v_0(t) \leq v_1(t) \leq w_1(t) \leq w_0(t)$, $t \in [t_0, T]$.

Proof. Suppose that $v_0(t)$ be weakly coupled lower solution of IVP of CFDE (4.1) and $v_1(t)$ be the unique solution of (4.10). Now, set $p(t) = v_0(t) - v_1(t)$, where $t \in [t_0, T]$. Then we have

$${}^cD^q p(t) = {}^cD^q v_0(t) - {}^cD^q v_1(t) \leq f(t, v_0) - [f(t, v_0) - M(v_1 - v_0)] \leq -Mp(t).$$

Thus ${}^cD^q p(t) \leq -Mp(t)$ and $p(t_0) = v_0(t_0) - v_1(t_0) \leq -\int_{t_0}^{T} w_0(s)ds + d + \int_{t_0}^{T} w_0(s)ds - d = 0$. Then from the Corollary 4.7 we have $p(t) \leq p(t_0)E_q(-M(t-t_0)^q)$. Which yields $p(t) \leq 0$, $t \in [t_0, T]$ since $p(t_0) \leq 0$. Thus we have $v_0(t) \leq v_1(t)$, $t \in [t_0, T]$. Similarly suppose that $w_0(t)$ be weakly coupled upper solution of IVP of CFDE (4.1) and $w_1(t)$ be the unique solution of (4.11). Now, set $p(t) = w_1(t) - w_0(t)$, where $t \in [t_0, T]$. Then we have

$${}^cD^q p(t) = {}^cD^q w_1(t) - {}^cD^q w_0(t) \leq [f(t, w_0) - M(w_1 - w_0)] - f(t, w_0) \leq -Mp(t).$$

Thus ${}^cD^q p(t) \leq -Mp(t)$ and $p(t_0) = w_1(t_0) - w_0(t_0) \leq -\int_{t_0}^{T} v_0(s)ds + d + \int_{t_0}^{T} v_0(s)ds - d = 0$. Then from the Corollary 4.7 we have $p(t) \leq p(t_0)E_q(-M(t-t_0)^q)$. Which yields $p(t) \leq 0$, $t \in [t_0, T]$ since $p(t_0) \leq 0$. Thus we have $w_1(t) \leq w_0(t)$, $t \in [t_0, T]$. To prove $v_1(t) \leq w_1(t)$, set $p(t) = v_1(t) - w_1(t)$, where $t \in [t_0, T]$. Then we have

$$\begin{aligned}
{}^cD^q p(t) &= {}^cD^q v_1(t) - {}^cD^q w_1(t) \\
&= f(t, v_0(t)) - M[v_1(t) - v_0(t))] - f(t, w_0(t)) + M[w_1(t) - w_0(t)] \\
&\leq -M[v_0(t) - w_0(t)] - M[v_1(t) - v_0(t))] + M[w_1(t) - w_0(t)] \leq -Mp(t).
\end{aligned}$$

Thus ${}^cD^q p(t) \leq -Mp(t)$ and

$$p(t_0) = v_1(t_0) - w_1(t_0) = -\int_{t_0}^{T} w_0(s)ds + d + \int_{t_0}^{T} v_0(s)ds - d \leq 0$$

since $v_o(t) \leq w_0(t)$, $t \in [t_0, T]$. Then from the Corollary 4.7 we have $p(t) \leq p(t_0)E_q(-M(t-t_0)^q)$. Which yields $p(t) \leq 0$, $t \in [t_0, T]$ since $p(t_0) \leq 0$. Thus we have $v_1(t) \leq w_1(t)$, $t \in [t_0, T]$, which completes the proof. □

We now state and prove the main result.

Theorem 4.9. *Assume that*

i) v_0, w_0 *be the weakly coupled lower and upper solutions of IVP of CFDE* (4.1) *such that* $v_0(t) \leq w_0(t)$ *for* $t \in [t_0, T]$,

ii) $f \in C[[t_0, T] \times \mathbb{R}, \mathbb{R}]$ *and* $f(t, x) - f(t, y) \leq -M(x - y)$ *for* $v_0 \leq x \leq y \leq w_0$, $M > 0$.

Then there exist monotone sequences $\{v_n\}, \{w_n\}$ *such that* $v_n \to \rho^*$, $w_n \to r^*$ *as* $n \to \infty$ *uniformly and monotonically on* $[t_0, T]$ *and* (ρ^*, r^*) *are minimal and maximal solutions of the IVP of CFDE* (4.1) *respectively.*

Proof. Consider

$$\begin{cases} {}^cD^q v_{i+1}(t) = f(t, v_i(t)) - M(v_{i+1}(t) - v_i(t)), & t \in [t_0, T], \\ v_{i+1}(t_0) \ = -\int_{t_0}^{T} w_i(s)ds + d, & d \geq 0 \end{cases} \tag{4.12}$$

and

$$\begin{cases} {}^cD^q w_{i+1}(t) = f(t, w_i(t)) - M(w_{i+1}(t) - w_i(t)), & t \in [t_0, T], \\ w_{i+1}(t_0) \ = -\int_{t_0}^{T} v_i(s)ds + d, & d \geq 0 \end{cases} \tag{4.13}$$

For $i = 0$, Lemma 4.8 shows that

$$v_0(t) \leq v_1(t) \leq w_1(t) \leq w_0(t), \quad t \in [t_0, T] \tag{4.14}$$

Next for $i > 1$, we assume that,

$$v_{i-1}(t) \le v_i(t) \le w_i(t) \le w_{i-1}(t), \quad t \in [t_0, T] \tag{4.15}$$

and prove that

$$v_i(t) \le v_{i+1}(t) \le w_{i+1}(t) \le w_i(t), \quad t \in [t_0, T]. \tag{4.16}$$

Since, $v_i(t)$ is a weakly coupled lower solution of IVP of CFDE and (4.1) and $v_{i+1}(t)$ is the unique solution of linear nonhomogeneous CFDE (4.12) by applying Lemma 4.8 we obtain that $v_i(t) \le v_{i+1}(t)$ on $t \in [t_0, T]$. Similarly $w_i(t)$ is weakly coupled upper solution of IVP of CFDE (4.1) and $w_{i+1}(t)$ is the unique solution of linear nonhomogeneous CFDE (4.13). Application of Lemma 4.8 yields $w_{i+1}(t) \le w_i(t)$, $t \in [t_0, T]$. To prove $v_{i+1}(t) \le w_{i+1}(t)$, $t \in [t_0, T]$. Set $p(t) = v_{i+1}(t) - w_{i+1}(t)$, on $t \in [t_0, T]$. Then

$$\begin{aligned} {}^cD^q p(t) &= {}^cD^q v_{i+1}(t) - {}^cD^q w_{i+1}(t) \\ &= [f(t, v_i(t)) - M(v_{i+1}(t) - v_i(t))] - [f(t, w_i(t)) - M(w_{i+1}(t) - w_i(t))] \\ &\le -Mp(t), \end{aligned}$$

and $p(t_0) = v_{i+1}(t_0) - w_{i+1}(t_0) = -\int_{t_0}^{T} w_i(s)ds + d + \int_{t_0}^{T} v_i(s)ds - d \le 0$, since $v_i(t) \le w_i(t)$ for all $t \in [t_0, T]$. Thus we have

$$\begin{cases} {}^cD^q p(t) \le -Mp(t), \\ p(t_0) \le 0. \end{cases}$$

From the Corollary 4.7 we get

$$p(t) \le p(t_0) E_q(-M(t - t_0)^q), \quad t \in [t_0, T].$$

This yields $p(t) \le 0$, $t \in [t_0, T]$, since $p(t_0) \le 0$. Thus we have $v_{i+1}(t) \le w_{i+1}(t)$, $t \in [t_0, T]$. Thus (4.16) is proved and therefore by the principle of mathematical induction, we see that

$$v_0 \le v_1 \le v_2 \le \cdots \le v_n \le w_n \le \cdots \le w_2 \le w_1 \le w_0, \quad t \in [t_0, T].$$

Thus the sequence of functions $\{v_n\}$,$\{w_n\}$ are continuous and uniformly bounded on $[t_0, T]$, also we can conclude that the sequence of functions $\{v_n\}$ and $\{w_n\}$ are equicontinuous by using Lemma 2.3.2 in [4] and the relation between the solution of Caputo fractional differential equations and the solution of R-L fractional differential equations [1]. Thus by using Arzela-Ascoli's Theorem, we obtain that the entire sequence $\{v_n\}$ converges uniformly and monotonically to $\rho^*(t)$ on $[t_0, T]$, and $\{w_n\}$ converges uniformly and monotonically to $r^*(t)$ on $[t_0, T]$, i.e

$$\lim_{n\to\infty} v_n = \rho^* \text{ and } \lim_{n\to\infty} w_n = r^*, \quad t \in [t_0, T].$$

To show that ρ^* and r^* are the solutions of the IVP of CFDE (4.1), we use the corresponding Volterra's integral equations

$$v_{n+1}(t) = -\int_{t_0}^{T} w_n(s)ds + d + \frac{1}{\Gamma(q)}\int_{t_0}^{t}(t-s)^{q-1}h(s, v_n(s))ds, \quad d \geq 0 \tag{4.17}$$

and

$$w_{n+1}(t) = -\int_{t_0}^{T} v_n(s)ds + d + \frac{1}{\Gamma(q)}\int_{t_0}^{t}(t-s)^{q-1}h(s, w_n(s))ds, \quad d \geq 0 \tag{4.18}$$

where $h(s, v_n(s)) = f(t, v_n(s)) - M(v_{n+1}(s) - v_n(s))$ and $h(s, w_n(s)) = f(t, w_n(s)) - M(w_{n+1}(s) - w_n(s))$. Now by taking the limits as $n \to \infty$, and using the uniform continuity of f and the uniform convergence of the sequences $\{v_n\}$ and $\{w_n\}$, $t \in [t_0, T]$, we get

$$\begin{cases} {}^cD^q\rho^*(t) = f(t, \rho^*(t)), \quad t \in [t_0, T], \\ \rho^*(t_0) = -\int_{t_0}^{T} x(s)ds + d, \quad d \geq 0 \end{cases} \tag{4.19}$$

and

$$\begin{cases} {}^cD^q r^*(t) = f(t, r^*(t)), \quad t \in [t_0, T], \\ r^*(t_0) = -\int_{t_0}^{T} x(s)ds + d, \quad d \geq 0 \end{cases} \tag{4.20}$$

for $t \in [t_0, T]$. Further,

$$v_0(t) \leq \rho^*(t) \leq r^*(t) \leq w_0(t)$$

$t \in [t_0, T]$. To prove $\rho^*(t)$, $r^*(t)$ are respectively minimal and maximal solutions of IVP of CFDE (4.1), we have to show that if $x(t)$ is any solution of IVP of CFDE (4.1) such that $v_0(t) \leq x(t) \leq w_0(t)$, $t \in [t_0, T]$ then

$$v_0(t) \leq \rho^*(t) \leq x(t) \leq r^*(t) \leq w_0(t)$$

$t \in [t_0, T]$. To prove this, since $v_n(t)$, $w_n(t)$ satisfy

$${}^cD^q v_n(t) = f(t, v_{n-1}(t)) - M(v_n(t) - v_{n-1}(t)), v_n(t_0) = -\int_{t_0}^{T} w_{n-1}(s)ds + d, \quad d \geq 0$$

and

$${}^cD^q w_n(t) = f(t, w_{n-1}(t)) - M(w_n(t) - w_{n-1}(t)), w_n(t_0) = -\int_{t_0}^{T} v_{n-1}(s)ds + d, \quad d \geq 0$$

respectively, suppose that for some n, $v_n(t) \leq x(t) \leq w_n(t)$, $t \in [t_0, T]$ and set $p(t) = v_{n+1}(t) - x(t)$ so that,

$$\begin{aligned} {}^cD^q p(t) &= {}^cD^q v_{n+1}(t) - {}^cD^q x(t) = [f(t, v_n(t)) - M(v_{n+1}(t) - v_n(t)] - f(t, x) \\ &\leq -M(v_{n+1}(t) - v_n(t)) - M(v_n(t) - x(t)) \\ &= -Mp(t), \quad t \in [t_0, T]. \end{aligned}$$

Also $p(t_0) = -\int_{t_0}^{T} w_n(s)ds + d + \int_{t_0}^{T} x(s)ds - d \le 0$, $d \ge 0$, since $x(t) \le w_n(t)$, $t \in [t_0, T]$. Thus

$$\begin{cases} {}^cD^q p(t) \le -Mp(t), & t \in [t_0, T], \\ p(t_0) \le 0 \end{cases}$$

Now by applying the Corollary 4.7 we get $v_{n+1}(t) \le x(t)$, $t \in [t_0, T]$. Similar process yields that $x(t) \le w_{n+1}(t)$, $t \in [t_0, T]$. Thus we have $v_{n+1}(t) \le x(t) \le w_{n+1}(t)$ on $[t_0, T]$, whenever $v_n(t) \le x(t) \le w_n(t)$ on $[t_0, T]$. Now taking the limits, we obtain that $\rho^*(t) \le x(t) \le r^*(t)$ for $t \in [t_0, T]$. □

Corollary 4.10. *If in addition to the assumptions of the Theorem* 4.9 *we have*

$$f(t, x) - f(t, y) \le M(x - y),$$

$v_0 \le y \le x \le w_0$,$M \ge 0$ *and* $L \ge 0$. *Then* $\rho^* = r^* = x$ *is the unique solution of the IVP of CFDE* (4.1).

Proof. To prove this, set $p(t) = r^*(t) - \rho^*(t)$. Then we have,

$$\begin{aligned} {}^cD^q p(t) = {}^cD^q r^*(t) - {}^cD^q \rho^*(t) &= f(t, r^*(t)) - f(t, \rho^*(t)) \\ &\le M(r^*(t) - \rho^*(t)) = Mp(t), \quad t \in [t_0, T] \end{aligned}$$

and $p(t_0) = 0$. Thus we have

$$\begin{cases} {}^cD^q p(t) \le Mp(t), & t \in [t_0, T] \\ p(t_0) = 0. \end{cases}$$

Now by Corollary 4.7 we have $p(t) \le 0$, which implies that $r^*(t) \le \rho^*(t)$, $t \in [t_0, T]$. Thus we conclude that $\rho^*(t) = r^*(t) = x(t)$ for $t \in [t_0, T]$, which proves the uniqueness of solution. Thus the proof is complete. □

Corollary 4.11. *By putting* $\lambda = 0$ *in* (2.5) *we can see the result of MIT for Caputo Fractional Differential Equations from the Section* 3 *or Section* 4 *of the above.*

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