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STABILITY OF CAPUTO FRACTIONAL DIFFERENTIAL EQUATIONS WITH NON-INSTANTANEOUS IMPULSES

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ABSTRACT. The stability of the zero solution of a nonlinear Caputo fractional differential equation with noninstantaneous impulses is studied using Lyapunov like functions. The theory is based on the derivative of a Lyapunov like function along the given noninstantaneous impulsive fractional differential equations. Two types of fractional derivatives of Lyapunov functions are introduced and their applications are discussed. Several sufficient conditions for uniform stability and asymptotic uniform stability of the zero solution, based on the definition of the derivative of Lyapunov functions are established. Some examples are given to illustrate the results.

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1. INTRODUCTION

In the real world life there are many processes and phenomena that are characterized by rapid changes in their state. In the literature there are two popular types of impulses:

- *instantaneous impulses*- the duration of these changes is relatively short compared to the overall duration of the whole process. The model is given by impulsive differential equations (see, for example, the monographs [17], [20] and the cited references therein);
- *noninstantaneous impulses* - an impulsive action, which starts at an arbitrary fixed point and remains active on a finite time interval. E. Hernandez and D. O'Regan ([16]) introduced a new class of abstract differential equations for which the impulses are not instantaneous and they investigated the existence of

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mild and classical solutions. For recent work we refer the reader to [12], [27], [29], [30], [31], [34], [37], [38], [39].

Results on stability of fractional differential equations in the literature via Lyapunov functions could be divided into two main groups:

- continuously differentiable Lyapunov functions with their Caputo fractional derivatives (see, for example, the papers [4], [18], [23], [26]).
- continuous Lyapunov functions with their fractional Dini derivatives (see, for example, the papers [10], [21], [22]).

Introducing impulses into fractional differential equations leads to complications on the stability properties of the solutions. In the literature there are several papers which examine existence and qualitative properties of fractional differential equations with instantaneous impulses (see, for example, [2], [5], [7], [8], [36]). There are also results concerning fractional differential equations with noninstantaneous impulses; see for example [1], [13], [14], [15], [19], [24], [28], [41], [42]. In the above cited papers the impulses start abruptly at some points and their action continues on a finite interval. As a motivation for the study of such kind of systems we consider the following simplified situation concerning the hemodynamical equilibrium of a person. In the case of a decompensation (for example, high or low levels of glucose) one can prescribe some intravenous drugs (insulin). Since the introduction of the drugs in the bloodstream and the consequent absorption for the body are gradual and continuous processes, we can interpret the situation as an impulsive action which starts abruptly and stays active on a finite time interval.

In this paper the stability of the zero solution of the noninstantaneous impulsive nonlinear Caputo fractional differential equations is studied. We define in an appropriate way two types of fractional derivatives of Lyapunov functions: Fractional Dini derivative as well as Caputo fractional Dini derivative. Their applications is discussed. Several sufficient conditions for uniform stability and asymptotic uniform stability are obtained. Some examples illustrating the obtained results are given.

2. NOTES ON FRACTIONAL CALCULUS

Fractional calculus generalizes the derivative and the integral of a function to a non-integer order [9, 11, 21, 32, 33] and there are several definitions of fractional derivatives and fractional integrals. In engineering, the fractional order q is often less than 1, so we restrict our attention to $q \in (0, 1)$.

1: The Riemann–Liouville (RL) fractional derivative of order $q \in (0,1)$ of $m(t)$ is given by (see, for example, 1.4.1.1 [9], or [32])

$$ {}_{t_0}^{RL}D^q m(t) = \frac{1}{\Gamma(1-q)} \frac{d}{dt} \int_{t_0}^{t} (t-s)^{-q} m(s) ds, \quad t \geq t_0. $$

where $\Gamma(\cdot)$ denotes the Gamma function.

2: The Caputo fractional derivative of order $q \in (0,1)$ is defined by (see, for example, 1.4.1.3 [9])

$$ {}_{t_0}^{c}D^q m(t) = \frac{1}{\Gamma(1-q)} \int_{t_0}^{t} (t-s)^{-q} m'(s) ds, \quad t \geq t_0. \tag{2.1} $$

The Caputo and Riemann-Liouville formulations coincide when $m(t_0) = 0$. The properties of the Caputo derivative are quite similar to those of ordinary derivatives. Also, the initial conditions of fractional differential equations with the Caputo derivative has a clear physical meaning and as a result the Caputo derivative is usually used in real applications.

3: The Grunwald-Letnikov fractional derivative is given by (see, for example, 1.4.1.2 [9])

$$ {}_{t_0}^{GL}D^q m(t) = \lim_{h \to 0} \frac{1}{h^q} \sum_{r=0}^{[\frac{t-t_0}{h}]} (-1)^r (qCr) m(t-rh), \quad t \geq t_0, $$

and the Grunwald-Letnikov fractional Dini derivative by

$$ {}_{t_0}^{GL}D_+^q m(t) = \limsup_{h \to 0+} \frac{1}{h^q} \sum_{r=0}^{[\frac{t-t_0}{h}]} (-1)^r (qCr) m(t-rh), \quad t \geq t_0, \tag{2.2} $$

where qCr are the Binomial coefficients and $[\frac{t-t_0}{h}]$ denotes the integer part of the fraction $\frac{t-t_0}{h}$.

The relations between the three types of fractional derivatives are given by ${}_{t_0}^{c}D^q m(t) = {}_{t_0}^{RL}D^q [m(t) - m(t_0)]$

Proposition 1 (Theorem 2.25 [11]). *Let $m \in C^1[t_0, b]$. Then, for $t \in (t_0, b]$,* ${}_{t_0}^{GL}D^q m(t) = {}_{t_0}^{RL}D^q m(t)$.

Also, according to Lemma 3.4 ([11]), ${}_{t_0}^{c}D_t^q m(t) = {}_{t_0}^{RL} D_t^q m(t) - m(t_0) \frac{(t-t_0)^{-q}}{\Gamma(1-q)}$ holds.

From the relation between the Caputo fractional derivative and the Grunwald-Letnikov fractional derivative using (2.2) we define the Caputo fractional Dini derivative as

$$ {}_{t_0}^{c}D_+^q m(t) = {}_{t_0}^{GL}D_+^q [m(t) - m(t_0)], $$

i.e.

$$ {}_{t_0}^{\ c}D_+^q m(t) = \limsup_{h\to 0+} \frac{1}{h^q}\Big[m(t)-m(t_0)-\sum_{r=1}^{[\frac{t-t_0}{h}]}(-1)^{r+1}(qCr)\big(m(t-rh)-m(t_0)\big)\Big]. \quad (2.3) $$

Definition 1 ([10]). We say $m \in C^q([t_0,T],\mathbb{R}^n)$ if $m(t)$ is differentiable (i.e. $m'(t)$ exists), the Caputo derivative ${}_{t_0}^{\ c}D^q m(t)$ exists and satisfies (2.1) for $t \in [t_0,T]$.

Remark 1. Definition 1 coud be extended to any interval $I \subset \mathbb{R}_+$.

Remark 2. If $m \in C^q([t_0,T],\mathbb{R}^n)$ then ${}_{t_0}^{\ c}D_+^q m(t) = {}_{t_0}^{\ c}D^q m(t)$.

3. NONINSTANTANEOUS IMPULSES IN FRACTIONAL DIFFERENTIAL EQUATIONS

In this paper we will assume two increasing sequences of points $\{t_i\}_{i=1}^\infty$ and $\{s_i\}_{i=0}^\infty$ are given such that $s_0 = 0 < t_i \le s_i < t_{i+1}$, $i = 1,2,\dots$, and $\lim_{k\to\infty} t_k = \infty$.

Let $t_0 \in \cup_{k=0}^\infty [s_k, t_{k+1})$ be a given arbitrary point. Without loss of generality we will assume that $t_0 \in [s_0,t_1)$, i.e. $0 \le t_0 < t_1$. Define the sequence of points $\{\tau_i\}_{i=0}^\infty$ by

$$ \tau_k = \begin{cases} t_0 & \text{for } k = 0, \\ s_k & \text{for } k \ge 1. \end{cases} $$

Consider the initial value problem (IVP) for the system of *noninstantaneous impulsive fractional differential equations* (NIFrDE) with a Caputo derivative for $0 < q < 1$,

$$ \begin{aligned} &{}_{\tau_k}^{\ c}D^q x = f(t,x) \text{ for } t \in (\tau_k, t_{k+1}],\ k = 0,1,2,\dots, \\ &x(t) = \phi_i(t, x(t_i - 0)) \text{ for } t \in (t_i, s_i],\ k = 1,2,\dots, \\ &x(t_0) = x_0 \end{aligned} \quad (3.1) $$

where $x, x_0 \in \mathbb{R}^n$, $f : \cup_{k=0}^\infty [\tau_k, t_{k+1}] \times \mathbb{R}^n \to \mathbb{R}^n$, $\phi_i : [t_i, s_i] \times \mathbb{R}^n \to \mathbb{R}^n$, $(i = 1,2,3,\dots)$.

We suppose that the function $f(t,x)$ is smooth enough, such that for any initial value $\tilde{x}_0 \in \mathbb{R}^n$ the IVP for the system of Caputo *fractional differential equations* (FrDE)

$$ {}_{\tau_k}^{\ c}D^q x = f(t,x) \quad \text{for } t \in [\tau_k, t_{k+1}] \text{ with } x(\tau_k) = \tilde{x}_0 \quad (3.2) $$

has a solution $x(t) = x(t;\tau_k,\tilde{x}_0) \in C^q([\tau_k,t_k],\mathbb{R}^n)$. Some sufficient conditions for global existence of solutions of (3.2) are given in [6], [21].

The IVP for FrDE (3.2) is equivalent to the integral equation

$$ x(t) = \tilde{x}_0 + \frac{1}{\Gamma(q)}\int_{\tau_k}^t (t-s)^{q-1} f(s,x(s))ds \quad \text{for } t \in [\tau_k, t_{k+1}]. \quad (3.3) $$

Remark 3. The intervals $(t_k, s_k]$, $k = 1, 2, \dots$ are called intervals of noninstantaneous impulses and the functions $\phi_k(t, x)$, $k = 1, 2, \dots,$ are called noninstantaneous impulsive functions.

Now t_0 is the initial time. We will assume throughout this paper that the initial time t_0 is not in an interval of noninstantaneous impulses, i.e. we will assume $t_0 \in \cup_{k=0}^{\infty}[s_k, t_{k+1})$.

We will give a brief description of the solution of IVP for NIFrDE (3.1). The solution $x(t; t_0, x_0)$, $t \geq t_0$ of (3.1) is given by

$$x(t;t_0,x_0) = \begin{cases} X_k(t) & \text{for } t \in (s_k, t_{k+1}],\ k = 0,1,2,\dots, \\ \phi_k(t, X_{k-1}(t_k - 0)) & \text{for } t \in (t_k, s_k],\ \ k = 1,2,\dots, \end{cases}$$

where

- $X_0(t)$ is the solution of IVP for FrDE (3.2) for $k = 0$, $t \in [t_0, t_1]$, $\tilde{x}_0 = x_0$ and $X_0(t)$ satisfies (3.3) on $[t_0, t_1]$;
- $X_1(t)$ is the solution of IVP for FrDE (3.2) for $k = 1$, $t \in [s_1, t_2]$, $\tilde{x}_0 = \phi_1(s_1, X_0(t_1 - 0))$, and $X_1(t)$ satisfies (3.3) on $[s_1, t_2]$;
- $X_2(t)$ is the solution of IVP for FrDE (3.2) for $k = 2$, $t \in [s_2, t_3]$, $\tilde{x}_0 = \phi_2(s_2, X_1(t_2 - 0))$, and $X_2(t)$ satisfies (3.3) on $[s_2, t_3]$;

and so on.

Also, the solution $x(t) = x(t; t_0, x_0)$, $t \geq t_0$ of (3.1) is given by

$$x(t) = \begin{cases} x_0 + \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t-s)^{q-1} f(s, x(s)) ds & \text{for } t \in [t_0, t_1], \\ \phi_1(t, x(t_1 - 0)) & \text{for } t \in (t_1, s_1], \\ \phi_1(s_1, x(t_1 - 0)) + \frac{1}{\Gamma(q)} \int_{s_1}^{t} (t-s)^{q-1} f(s, x(s)) ds & \text{for } t \in [s_1, t_2], \\ \phi_2(t, x(t_2 - 0)) & \text{for } t \in (t_2, s_2], \\ \phi_2(s_2, x(t_2 - 0)) + \frac{1}{\Gamma(q)} \int_{s_2}^{t} (t-s)^{q-1} f(s, x(s)) ds & \text{for } t \in [s_2, t_3], \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots & \\ \phi_k(t, x(t_k - 0)) & \text{for } t \in (t_k, s_k], \\ \phi_k(s_k, x(t_k - 0)) + \frac{1}{\Gamma(q)} \int_{s_k}^{t} (t-s)^{q-1} f(s, x(s)) ds & \text{for } t \in [s_k, t_{k+1}], \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots & \end{cases}$$

Remark 4. If $t_k = s_k$, $k = 1, 2, \dots$ then the IVP for NIFrDE (3.1) reduces to an IVP for impulsive fractional differential equations studied in [2], [5], [7], [8]. In this case at any point of instantaneous impulse t_k the amount of jump of the solution $x(t)$ is given by $\Delta x(t_k) = x(t_k + 0) - x(t_k - 0) = \Phi_k(x(t_k - 0)) = \phi_k(t_k, x(t_k - 0)) - x(t_k - 0)$.

Remark 5. In the case $q = 1$ the IVP for NIFrDE (3.1) reduces to an IVP for noninstantaneous impulsive differential equations studied in [24], [34], [37], [38].

Remark 6. In the case $q=1$, $t_k=s_k$, $k=1,2,\dots$ the IVP for NIFrDE (3.1) reduces to an IVP for impulsive differential equations (see for example the books [17], [20] and the cited references therein).

Let $J\subset\mathbb{R}_+$ be a given interval. We introduce the following classes of functions

$$\begin{aligned}
PC^q(J) &= \{u\in C^q\big(J\cap\big(\cup_{k=0}^{\infty}[s_k,t_{k+1})\big),\mathbb{R}^n\big)\bigcup C\big(J\cap\big(\cup_{k=1}^{\infty}(t_k,s_k]\big),\mathbb{R}^n\big):\\
&\qquad u(t_k)=u(t_k-0)=\lim_{t\uparrow t_k}u(t)<\infty,\ u(t_k+0)=\lim_{t\downarrow t_k}u(t)<\infty\\
&\qquad\qquad \text{for } k:\ t_k\in J,\\
&\qquad u(s_k)=u(s_k-0)=\lim_{t\uparrow s_k}u(t)=u(s_k+0)=\lim_{t\downarrow s_k}u(t)\ \text{ for } k:\ s_k\in J\},\\
PC(J) &= \{u\in C\big(J\cap\big(\cup_{k=0}^{\infty}(t_k,t_{k+1})\big),\mathbb{R}^n\big):\\
&\qquad u(t_k)=u(t_k-0)=\lim_{t\uparrow t_k}u(t)<\infty \text{ and } u(t_k+0)=\lim_{t\downarrow t_k}u(t)<\infty\\
&\qquad\qquad \text{for } k:\ t_k\in J\}.
\end{aligned}$$

Remark 7. According to the above description any solution of (3.1) is from the class $PC^q([t_0,b))$, $b\le\infty$, i.e. any solution could have a discontinuity at points t_k, $k=1,2,\dots$.

Example 1. Consider the IVP for the scalar NIFrDE

$$\begin{aligned}
&{}_{\tau_k}^{c}D^q x = Ax \text{ for } t\in(\tau_k,t_{k+1}],\ k=0,1,2,\dots,\\
&x(t)=\Psi_k(t,x(t_k-0)) \text{ for } t\in(t_k,s_k],\ k=1,2,\dots,\\
&x(t_0)=x_0,
\end{aligned}\tag{3.4}$$

where $x,x_0\in\mathbb{R}$, A is a constant.

The solution of (3.4) is given by

$$x(t;0,x_0)=\begin{cases}
x_0E_q(A(t-t_0)^q) & \text{for } t\in[t_0,t_1]\\
\Psi_k(t,x(t_k-0)) & \text{for } t\in(t_k,s_k],\quad k=1,2,\dots\\
\Psi_k\big(s_k,x(t_k-0)\big)E_q\big(A(t-s_k)^q\big) & \text{for } t\in[s_k,t_{k+1}],\ k=1,2,3,\dots
\end{cases}$$

where the Mittag-Leffler function (with one parameter) is defined by $E_q(z)=\sum\limits_{k=0}^{\infty}\frac{z^k}{\Gamma(qk+1)}$.

If $\Psi_k(t,x)=a_k(t)x$, $a_k:[t_k,s_k]\to\mathbb{R}$, $k=1,2,3,\dots$, then the solution of NIFrDE (3.4) is given by

$$x(t;0,x_0)=\begin{cases}
x_0E_q(A(t-t_0)^q) \qquad \text{for } t\in[0,t_1],\\
x_0E_q(AC^q)\big(\prod_{i=1}^{k-1}a_i(s_i)E_q(AC_i^q)\big)a_k(t)\\
\qquad\qquad \text{for } t\in(t_k,s_k],\ k=1,2,\dots,\\
x_0E_q(AC^q)a_k(s_k)\big(\prod_{i=1}^{k-1}a_i(s_i)E_q(AC_i^q)\big)E_q(A(t-s_k)^q)\\
\qquad\qquad \text{for } t\in(s_k,t_{k+1}],\ k=1,2,\dots.
\end{cases}$$

where $C = t_1 - t_0$ and $C_k = s_k - t_k \geq 0$, $k = 1, 2, \ldots$.

If $A = 0$ and $\Psi_k(t, x) = a_k(t)x$, $a_k : [t_k, s_k] \to \mathbb{R}$, $k = 1, 2, 3, \ldots$ the solution of NIFrDE (3.4) is given by

$$x(t; 0, x_0) = \begin{cases} x_0 & \text{for } t \in [t_0, t_1], \\ x_0\big(\prod_{i=1}^{k-1} a_i(s_i)\big) a_k(t) & \text{for } t \in (t_k, s_k],\ k = 1, 2, \ldots \\ x_0\big(\prod_{i=1}^{k} a_i(s_i)\big) & \text{for } t \in (s_k, t_{k+1}],\ k = 1, 2, \ldots \end{cases}$$

□

4. DEFINITIONS CONCERNING STABILITY AND LYAPUNOV FUNCTIONS

The goal of the paper is to study the stability properties of the system NIFrDEs (3.1). In the definition below we denote by $x(t; t_0, x_0) \in PC^q([t_0, \infty), \mathbb{R}^n)$ any solution of (3.1).

Definition 2. The zero solution of the IVP for NIFrDE (3.1) is said to be

- *stable* if for every $\epsilon > 0$ and $t_0 \in \cup_{k=0}^{\infty}[s_k, t_{k+1})$ there exist $\delta = \delta(\epsilon, t_0) > 0$ such that for any $x_0 \in \mathbb{R}^n$ the inequality $\|x_0\| < \delta$ implies $\|x(t; t_0, x_0)\| < \epsilon$ for $t \geq t_0$;
- *uniformly stable* if for every $\epsilon > 0$ there exist $\delta = \delta(\epsilon) > 0$ such that for any initial point $t_0 \in \cup_{k=0}^{\infty}[s_k, t_{k+1})$ and any inital value $x_0 \in \mathbb{R}^n$ with $\|x_0\| < \delta$ the inequality $\|x(t; t_0, x_0)\| < \epsilon$ holds for $t \geq t_0$;
- *uniformly attractive* if for $\beta > 0$: for every $\epsilon > 0$ there exist $T = T(\epsilon) > 0$ such that for any initial pont $t_0 \in \cup_{k=0}^{\infty}[s_k, t_{k+1})$ and any initial value $x_0 \in \mathbb{R}^n$ with $\|x_0\| < \beta$ the inequality $\|x(t; t_0, x_0)\| < \epsilon$ holds for $t \geq t_0 + T$;
- *uniformly asymptotically stable* if the zero solution is uniformly stable and uniformly attractive.

Example 2. Consider the scalar NIFrDE (3.4) where $A \leq 0$ and $\Psi_k(t, x) = a_k(t)x$ and $a_k :\ [t_k, s_k] \to \mathbb{R}$, $k = 1, 2, 3, \ldots$ are such that $\sup_{t\in[t_k,s_k]} |a_k(t)| \leq M_k$, $\prod_{i=1}^{\infty} M_i < \infty$ where $M_k > 0$ are constants. According to Example 1 and the inequality $0 < E_q(A(T - \tau)^q) \leq 1$ for $T \geq \tau$ there exists a constant $M > 0$ such that

$$|x(t; t_0, x_0)| \leq M\, |x_0| \quad \text{for } t \geq t_0. \tag{4.1}$$

Inequality (4.1) guarantees that the zero solution of (3.4) is uniformly stable. □

In this paper we will use the followings sets:

$$\begin{aligned} \mathcal{K} &= \{a \in C[\mathbb{R}_+, \mathbb{R}_+] : a \text{ is strictly increasing and } a(0) = 0\}, \\ S(A) &= \{x \in \mathbb{R}^n : \|x\| \leq A\}, \quad A > 0. \end{aligned}$$

We now introduce the class Λ of Lyapunov-like functions which will be used to investigate the stability of the zero solution of the system NIFrDE (3.1).

Definition 3. Let $J \in \mathbb{R}_+$ be a given interval, and $\Delta \subset \mathbb{R}^n$, $0 \in \Delta$ be a given set. We will say that the function $V(t,x) : J \times \Delta \to \mathbb{R}_+$, $V(t,0) \equiv 0$ belongs to the class $\Lambda(J,\Delta)$ if

1. The function $V(t,x)$ is continuous on $J/\{t_k \in J\} \times \Delta$ and it is locally Lipschitzian with respect to its second argument;
2. For each $t_k \in Int(J)$ and $x \in \Delta$ there exist finite limits

$$V(t_k - 0, x) = \lim_{t \uparrow t_k} V(t,x) < \infty, \quad \text{and} \quad V(t_k + 0, x) = \lim_{t \downarrow t_k} V(t,x) < \infty$$

and the following equalities are valid

$$V(t_k - 0, x) = V(t_k, x).$$

Remark 8. In the case when the Lyapunov function does not depend on the time t, i.e. $V(t,x) = V(x) \in C[\Delta, \mathbb{R}_+]$, $V(0) = 0$, $\Delta \subset \mathbb{R}^n$, $0 \in \Delta$, and the function $V(x)$ is locally Lipschitzian then we will say $V(x) \in \Lambda^C(\Delta)$.

Lyapunov-like functions used to discuss stability for differential equations require an appropriate definition of their derivatives along the studied differential equations. For fractional differential equations some authors (see, for example, [23], [26]) used the so called Caputo fractional derivative of Lyapunov function ${}_{t_0}^{c}D^q V(t, x(t))$ where $x(t)$ is the unknown solution of the studied fractional differential equation. This approach requires the function to be smooth enough (at least continuously differentiable) and also some conditions involved are quite restrictive. Other authors used the so called Dini fractional derivative of Lyapunov function ([21], [22]). This is based on the Dini derivative of the Lyapunov function $V(t,x)$ among the ordinary differential equation $x' = f(t,x)$ given by

$$DV(t,x) = \limsup_{h \to 0} \frac{1}{h}\big[V(t,x) - V(t-h, x - hf(t,x))\big]. \tag{4.2}$$

The authors generalized (4.2) to the *Dini fractional derivative* along the FrDE ${}_{t_0}^{c}D^q x = f(t,x),\ t \geq t_0$ by

$${}^{c}D_+^q V(t,x) = \limsup_{h \to 0} \frac{1}{h^q}\big[V(t,x) - V(t-h, x - h^q f(t,x))\big]. \tag{4.3}$$

This definition requires only the continuity of the Lyapunov function.

In this paper we will use piecewise continuous Lyapunov functions from the above introduced class $\Lambda([t_0,T),\Delta)$. We will introduce the derivative of Lyapunov function in two different ways and we will discuss their applications.

We now define the *generalized Caputo fractional Dini derivative* of the Lyapunov-like function $V(t,x) \in \Lambda([t_0,T),\Delta)$ along trajectories of solutions of IVP for the system NIFrDE (3.1). It is based on the Caputo fractional Dini derivative of a function $m(t)$ given by (2.3). The Caputo fractional Dini derivative along trajectories of solutions of IVP for the system NIFrDE (3.1) is given by:

$$\begin{aligned} {}^{c}_{(3.1)}D^q_+ V(t,x;t_0,x_0) &= \limsup_{h\to 0^+} \frac{1}{h^q}\Big\{ V(t,x) - V(t_0,x_0) \\ &- \sum_{r=1}^{[\frac{t-t_0}{h}]} (-1)^{r+1} qCr \Big[V(t-rh, x-h^q f(t,x)) - V(t_0,x_0)\Big]\Big\} \\ &\text{for } t\in (s_k,t_{k+1})\cap(t_0,T),\ k=0,1,2,\dots, \end{aligned} \tag{4.4}$$

where $x, x_0 \in \Delta$, and for any $t \in (s_k,t_{k+1})\cap(t_0,T)$ there exists $h_t > 0$ such that $t-h \in (s_k,t_{k+1})\cap(t_0,T)$, $x - h^q f(t,x) \in \Delta$ for $0 < h \le h_t$.

Remark 9. The generalized Caputo fractional Dini derivative of the Lyapunov function was introduced and used for studying stability properties of the zero solution of Caputo fractional differential equations in [3].

The formula (4.4) could be reduced to

$$\begin{aligned} &{}^{c}_{(3.1)}D^q_+ V(t,x;t_0,x_0) \\ &= \limsup_{h\to 0^+} \frac{1}{h^q}\Big\{ V(t,x) - \sum_{r=1}^{[\frac{t-t_0}{h}]} (-1)^{r+1} qCr V(t-rh, x-h^q f(t,x))\Big\} \\ &- V(t_0,x_0)\frac{(t-t_0)^{-q}}{\Gamma(1-q)} \quad \text{for } t\in (s_k,t_{k+1})\cap(t_0,T),\ k=0,1,2,\dots, \end{aligned} \tag{4.5}$$

Now, based on the Dini fractional derivative of a continuous Lyapunov function defined by (4.3), we will define the *generalized Dini fractional derivative* of the Lyapunov-like function $V(t,x) \in \Lambda([t_0,T),\Delta)$ along trajectories of solutions of IVP for the system NIFrDE (3.1) by

$$\begin{aligned} {}^{c}_{(3.1)}\mathcal{D}^q_+ V(t,x) &= \lim_{h\to 0^+} \sup \frac{1}{h^q}\big[V(t,x) - V(t-h, x-h^q f(t,x))\big] \\ qquad &\quad \text{for } t\in (s_k,t_{k+1})\cap(t_0,T),\ k=0,1,2,\dots, \end{aligned} \tag{4.6}$$

where $x \in \Delta$, and for any $t \in (s_k,t_{k+1})\cap(t_0,T)$ there exists $h_t > 0$ such that $t-h \in (s_k,t_{k+1})\cap(t_0,T)$, $x - h^q f(t,x) \in \Delta$ for $0 < h \le h_t$.

Example 3. Let $V \in \Lambda(\mathbb{R}_+,\mathbb{R})$ be given by $V(t,x) = m(t)g(x)$ where the function $m \in C^1(\cup_{i=0}^{\infty}(s_k,t_{k+1}),\mathbb{R}_+)$, $g:\mathbb{R}^n \to \mathbb{R}_+$ is a locally Lipshitz function such that the limit

$${}^{Fr}D_q g(x) = \lim_{h\to 0^+} \sup \frac{g(x) - g(x-h^q f(t,x))}{h^q}$$

exists for $x \in \mathbb{R}^n$.

First we apply the formula (4.6) to obtain the generalized Dini fractional derivative of the considered Lyapunov function. We obtain

$$\begin{aligned}
&{}_{(3.1)}^{\;c}\mathcal{D}_+^q V(t,x) = \lim_{h\to 0^+}\sup \frac{1}{h^q}\big[m(t)g(x) - m(t-h)g\big(x - h^q f(t,x)\big)\big] \\
&= m(t)\lim_{h\to 0^+}\sup \frac{g(x) - g(x-h^q f(t,x))}{h^q} \\
&\qquad + \Big(\lim_{h\to 0^+}\sup \frac{m(t)-m(t-h)}{h}\Big)\Big(\lim_{h\to 0^+}\sup h^{1-q} g\big(x - h^q f(t,x)\big)\Big) \\
&= m(t)\ {}^{Fr}D_q g(x) \quad \text{for } t\in(s_k,t_{k+1}),\ k=0,1,2,\dots.
\end{aligned} \tag{4.7}$$

Next we use (4.5) to obtain the generalized Caputo fractional Dini derivative of the function V. Let $t\in(s_k,t_{k+1}),\ k=0,1,2,\dots$. Apply equalities (2.2), ${}_{t_0}^{RL}D^q 1 = \frac{(t-t_0)^{-q}}{\Gamma(1-q)}$, and $\lim_{h\to 0^+}\sup \frac{1}{h^q}\sum_{r=0}^{[\frac{t-t_0}{h}]}(-1)^r\, qCr\ m(t-rh) = {}_{t_0}^{RL}D^q\big(m(t)\big)$ to (4.5) and obtain

$$\begin{aligned}
&{}_{(3.1)}^{\;c}D_+^q V(t,x;t_0,x_0) \\
&= \lim_{h\to 0^+}\sup \frac{1}{h^q}\Big[m(t)g(x) + g(x-h^q f(t,x))\sum_{r=1}^{[\frac{t-t_0}{h}]}(-1)^r\, qCr\ m(t-rh)\Big] \\
&\qquad - m(t_0)g(x_0)\frac{(t-t_0)^{-q}}{\Gamma(1-q)} \\
&= m(t)\lim_{h\to 0^+}\sup \frac{g(x)-g(x-h^q f(t,x))}{h^q} \\
&\qquad + \lim_{h\to 0} g(x-h^q f(t,x))\lim_{h\to 0^+}\sup \frac{1}{h^q}\sum_{r=0}^{[\frac{t-t_0}{h}]}(-1)^r\, qCr\ m(t-rh)\Big] \\
&\qquad - m(t_0)g(x_0)\frac{(t-t_0)^{-q}}{\Gamma(1-q)} \\
&= m(t)\ {}^{Fr}D_q g(x) + g(x)\ {}_{t_0}^{RL}D^q\big(m(t)\big) - m(t_0)g(x_0)\frac{(t-t_0)^{-q}}{\Gamma(1-q)}, \\
&\qquad \text{for } t\in(s_k,t_{k+1}),\ k=0,1,2,\dots.
\end{aligned} \tag{4.8}$$

or

$$\begin{aligned}
&{}_{(3.1)}^{\;c}D_+^q V(t,x;t_0,x_0) \\
&= m(t)\ {}^{Fr}D_q g(x) + g(x)\ {}_{t_0}^{C}D^q\big(m(t)\big) + \big(g(x)-g(x_0)\big)m(t_0)\frac{(t-t_0)^{-q}}{\Gamma(1-q)}, \\
&\qquad \text{for } t\in(s_k,t_{k+1}),\ k=0,1,2,\dots.
\end{aligned} \tag{4.9}$$

Example 4. Let $V\in\Lambda(\mathbb{R}_+,\mathbb{R})$ be given by the equality $V(t,x)=m(t)x^2$ where $m\in C^1(\cup_{i=0}^{\infty}(s_k,t_{k+1}),\mathbb{R}_+)$ and $x\in\mathbb{R}$.

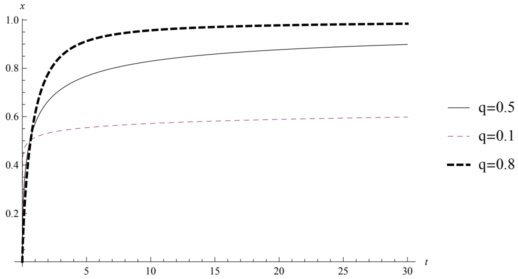

Figure 1. Graphs of solutions of ${}_0^cD^q x + x(t) = 1$ *for various* q.

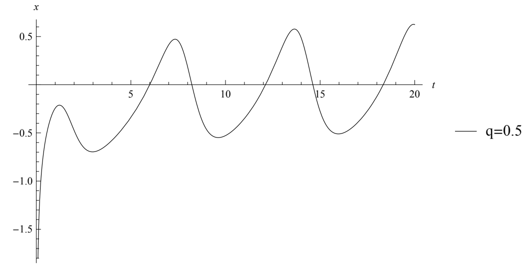

Figure 2. Example 8. Graph of $f(t)$.

In this case

$$\begin{aligned} {}^{Fr}D_q g(x) = {}^{Fr}D_q(x^2) &= \lim_{h\to 0^+} \sup \frac{x^2 - (x - h^q f(t,x))^2}{h^q} \\ &= \lim_{h\to 0} f(t,x)(2x - h^q f(t,x)) = 2x f(t,x) \end{aligned} \tag{4.10}$$

First we apply the formula (4.6) to obtain the generalized Dini fractional derivative of the considered Lyapunov function. From (4.7) and (4.10) we obtain

$${}_{(3.1)}^{c}\mathcal{D}_+^q V(t,x) = 2x\ m(t) f(t,x), \quad \text{for } t \in (s_k, t_{k+1}),\ k = 0,1,2,\dots. \tag{4.11}$$

Note ${}_{(3.1)}^{c}\mathcal{D}_+^q V(t,x)$ does not depend on the order q of the fractional differential equation. The behavior of solutions of fractional differential equations depends significantly on the order q. For example, let us consider the simple fractional differential equation ${}_0^cD^q x + x(t) = 1,\ x(0) = 0$ whose solution is given by $x(t) = t^q E_{q,1+q}(-t^q)$. From Figure 1 it can be seen $\lim_{t\to\infty} x(t) = a$ where a is different for different values of the order q of fractional differential equation.

Next we use (4.5) to obtain the generalized Caputo fractional Dini derivative of the function V. Let $t \in (s_k, t_{k+1}),\ k = 0,1,2,\dots$. From (4.8), (4.9) for $g(x) = x^2$ and (4.10) we obtain

$$\begin{aligned} &{}_{(3.1)}^{c}D_+^q V(t,x;t_0,x_0) \\ &= 2x\ m^2(t) f(t,x) + x^2\ {}_{t_0}^{RL}D^q\big(m^2(t)\big) - (x_0)^2 m^2(t_0) \frac{(t-t_0)^{-q}}{\Gamma(1-q)}, \\ &\qquad \text{for } t \in (s_k, t_{k+1}),\ k = 0,1,2,\dots. \end{aligned} \tag{4.12}$$

or

$$\begin{aligned} &{}_{(3.1)}^{c}D_+^q V(t,x;t_0,x_0) \\ &= 2x\ m^2(t) f(t,x) + x^2\ {}_{t_0}^{C}D^q\big(m^2(t)\big) + \big(x^2 - (x_0)^2\big) m^2(t_0) \frac{(t-t_0)^{-q}}{\Gamma(1-q)}, \\ &\qquad \text{for } t \in (s_k, t_{k+1}),\ k = 0,1,2,\dots. \end{aligned} \tag{4.13}$$

Note the generalized Caputo fractional Dini derivative ${}_{(3.1)}^{\;\;c}D_+^qV(t,x;t_0,x_0)$ depends significantly not only on the order q of the fractional differential equation but also on the initial data.

The derivative of the Lyapunov function in the well known case for first order impulsive differential equations ($q=1$) is

$$D_+V(t,x) = 2x\,m^2(t)f(t,x)+x^2\frac{d}{dt}\big[m^2(t)\big], \qquad t\in(s_k,t_{k+1}),\ k=0,1,2,\dots. \tag{4.14}$$

The generalized Caputo fractional Dini derivative given by formula (4.5) seems to be the natural generalization of the derivative of Lyapunov functions for ordinary differential equations (with or without impulses). □

5. COMPARISON RESULTS

Again in this section we assume $0<q<1$. Also, $t_0\in\cup_{k=0}^{\infty}[s_k,t_{k+1})$ so without loss of generality we assume $t_0\in[s_0,t_1)$. We will obtain some comparison results for NIFrDE (3.1) using the definitions (4.5) and (4.6) for a derivative of Lyapunov-like function.

5.1. **Generalized Caputo fractional Dini derivative.** In this section we will use the following result for fractional differential equations:

Lemma 1 ([3]). *Assume the following conditions are satisfied:*

1. *The function $x^*(t)=x(t;t_0,x_0)\in C^q([t_0,T],\Delta)$, is a solution of the FrDE*
$$ {}_{t_0}^{\,c}D^qx = f(t,x), \quad t\in[t_0,T] \text{ with } x(t_0)=x_0 \tag{5.1}$$
where $\Delta\subset\mathbb{R}^n$, $0\in\Delta$, $x_0\in\Delta$ and t_0, $T\in\mathbb{R}_+$, $t_0<T$ are given constants.
2. *The function $V\in\Lambda^C([t_0,T],\Delta)$ and the inequality*
$$ {}_{(5.1)}^{\;\;c}D_+^qV(t,x;t_0,x_0)\le 0 \qquad \text{for } (t,x)\in[t_0,T]\times\Delta$$
holds.

Then the inequality $V(t,x^*(t))\le V(t_0,x_0)$ holds for $t\in[t_0,T]$.

Now we will prove some comparison results for noninstantaneous impulsive Caputo fractional differential equations.

Lemma 2 (Comparison result for NIFrDE by generalized Caputo fractional Dini derivative). *Let:*

1. *The function $x^*(t)=x(t;t_0,x_0)\in PC^q([t_0,T],\Delta)$ is a solution of the NIFrDE (3.1) where $\Delta\subset\mathbb{R}^n$, $0\in\Delta$, $x_0\in\Delta$ and t_0,T are given constants such that $t_0\in[s_0,t_1)$, $T>t_0$.*
2. *The function $V\in\Lambda([t_0,T],\Delta)$ and*

(i) *the inequality*

$$ {}_{(3.1)}^{\ c}D_+^q V(t, x^*(t); t_0, x_0) \le 0 \quad \text{for } t \in (t_0, T) \bigcap \cup_{k=0}^{\infty} (s_k, t_{k+1}) \tag{5.2} $$

holds;

(ii) *the inequalities*

$$ V(t, x^*(t)) \le V(t_k - 0, x^*(t_k - 0)) \quad \text{for } t \in [t_0, T] \bigcap (t_k, s_k] \text{ for } k = 1, 2, 3, \dots $$

hold.

Then the inequality $V(t, x^*(t)) \le V(t_0, x_0)$ holds on $[t_0, T]$.

P r o o f: We use induction to prove Lemma 2.

Let $t \in [t_0, t_1] \cap [t_0, T]$. The function $x^*(t) \in C^q([t_0, t_1] \cap [t_0, T], \mathbb{R}^n)$, satisfies FrDE (5.1) and from Lemma 1 (with $T = t_1$) the inequality $V(t, x^*(t)) \le V(t_0, x_0)$ holds on $[t_0, t_1] \cap [t_0, T]$.

Let $T > t_1$ and $t \in (t_1, s_1] \cap [t_0, T]$. From condition 2(ii) and the above we get $V(t, x^*(t)) \le V(t_1 - 0, x^*(t_1 - 0)) = V(t_1, x^*(t_1)) \le V(t_0, x_0)$.

Let $T > s_1$ and $t \in (s_1, t_2] \cap [t_0, T]$. Consider the function $\overline{x}_1(t) = x^*(t)$ for $t \in (s_1, t_2]$ and $\overline{x}_1(s_1) = x^*(s_1) = \phi_1(s_1, x^*(t_1 - 0))$. The function $\overline{x}_1(t) \in C^q([s_1, t_2], \mathbb{R}^n)$ and satisfies IVP for FrDE (5.1) with $t_0 = s_1$, $x_0 = x^*(s_1)$, and $T = t_2$. Using condition 2(i), Lemma 1 for the function $\overline{x}_1(t)$ and the above we obtain $V(t, x^*(t)) = V(t, \overline{x}_1(t)) \le V(s_1, \overline{x}_1(s_1)) = V(s_1, x^*(s_1)) \le V(t_0, x_0)$.

Continue this process and an induction argument proves the claim of Lemma 2 is true for $t \in [t_0, T]$. □

Lemma 3 ([3]). *Let the following conditions be satisfied:*

1. *The function $x^*(t) = x(t; t_0, x_0) \in C^q([t_0, T], \Delta)$ is a solution of the FrDE (5.1) where $\Delta \subset \mathbb{R}^n$, $0 \in \Delta$, $x_0 \in \Delta$, t_0, $T \in \mathbb{R}_+$, $t_0 < T$ are given constants.*
2. *The function $V \in \Lambda^C([t_0, T], \Delta)$ is such that for any points $t \in [t_0, T]$, $x \in \Delta$ the inequality*

$$ {}_{(3.2)}^{\ c}D_+^q V(t, x; \tau_0, x_0) \le -c(\|x\|) $$

holds where $c \in \mathcal{K}$.

Then for $t \in [t_0, T]$ the inequality

$$ V(t, x^*(t)) \le V(t_0, x_0) - \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t - s)^{q-1} c(\|x^*(s)\|) ds \tag{5.3} $$

holds.

Lemma 4 (Comparison result for NIFrDE, negative generalized Caputo fractional Dini derivtive). *Assume the following conditions are satisfied:*

1. *The function* $x^*(t)=x(t;t_0,x_0)\in PC^q([t_0,T],\Delta)$ *is a solution of the NIFrDE* (3.1) *where* $\Delta\subset\mathbb{R}^n$, $0\in\Delta$, $x_0\in\Delta$ *and* t_0,T *are given constants such that* $t_0\in[s_0,t_1)$, $T>t_0$.
2. *The function* $V\in\Lambda([t_0,T],\Delta)$ *and*
   (i) *the inequality* ${}^{c}_{(3.1)}D^q_+V(t,x^*(t);t_0,x_0)\le -c(\|x^*(t)\|)$ *for* $t\in(t_0,T)\bigcap\cup_{k=0}^{\infty}(s_k,t_{k+1})$ *holds where* $c\in\mathcal{K}$;
   (ii) *for any* $k=1,2\dots$ *the inequalities*
$$V(t,x^*(t))\le V(t_k-0,x^*(t_k-0))\quad \textit{for } t\in[t_0,T]\bigcap(t_k,s_k]$$
   hold.

Then for $t\in[t_0,T]$ *the inequality*

$$V(t,x^*(t))\le\begin{cases}V(t_0,x_0)-\frac{1}{\Gamma(q)}\int_{t_0}^t(t-s)^{q-1}c(\|x^*(s)\|)ds,\\ \qquad t\in[t_0,t_1]\\ V(t_0,x_0)-\frac{1}{\Gamma(q)}\big(\sum_{i=0}^{k-1}\int_{\tau_i}^{t_{i+1}}(t_{i+1}-s)^{q-1}c(\|x^*(s)\|)ds\\ \qquad +\int_{s_k}^t(t-s)^{q-1}c(\|x^*(s)\|)ds\big),\\ \qquad t\in(s_k,t_{k+1}]\cap(t_0,T),\ k\ge1\\ V(t_0,x_0)-\frac{1}{\Gamma(q)}\sum_{i=0}^{k-1}\int_{\tau_i}^{t_{i+1}}(t_{i+1}-s)^{q-1}c(\|x^*(s)\|)ds,\\ \qquad t\in(t_k,s_k]\cap[t_0,T],\ k\ge1\end{cases}$$

holds where

$$\tau_k=\begin{cases}t_0 & \textit{for } k=0,\\ s_k & \textit{for } k\ge1.\end{cases}$$

P r o o f: Let $t\in[t_0,t_1]\cap[t_0,T]$. The function $x^*(t)\in C^q([t_0,t_1]\cap[t_0,T],\Delta)$ and satisfies the IVP for FrDE (5.1) for $T=t_1$. From Lemma 3 the inequality $V(t,x^*(t))\le V(t_0,x_0)-\frac{1}{\Gamma(q)}\int_{t_0}^t(t-s)^{q-1}c(\|x^*(s)\|)ds$ holds, i.e. the claim of Lemma 4 is true on $[t_0,t_1]\cap[t_0,T]$.

Let $T>t_1$ and $t\in(t_1,s_1]\cap[t_0,T]$. From condition 2(ii) and the above we get

$$\begin{aligned}V(t,x^*(t))&\le V(t_1-0,x^*(t_1-0))=V(t_1,x^*(t_1))\\&\le V(t_0,x_0)-\frac{1}{\Gamma(q)}\int_{t_0}^{t_1}(t_1-s)^{q-1}c(\|x^*(s)\|)ds.\end{aligned}$$

Let $T>s_1$ and $t\in(s_1,t_2]\cap[t_0,T]$. Consider the function $\overline{x}_1(t)=x^*(t)$ for $t\in(s_1,t_2]$ and $\overline{x}_1(s_1)=x^*(s_1)=\phi_1(s_1,x^*(t_1-0))$. The function $\overline{x}_1(t)\in C^q([s_1,t_2],\mathbb{R}^n)$ and satisfies IVP for FrDE (5.1) with $t_0=s_1$, $x_0=x^*(s_1)$ and $T=t_2$. Using condition 2(i), Lemma 3 for the function $\overline{x}_1(t)$, and the above we obtain

$$V(t,x^*(t))=V(t,\overline{x}_1(t))\le V(s_1+0,\overline{x}_1(s_1))-\frac{1}{\Gamma(q)}\int_{s_1}^t(t-s)^{q-1}c(\|x^*(s)\|)ds$$

$$= V(s_1, x^*(s_1)) - \frac{1}{\Gamma(q)} \int_{s_1}^{t} (t-s)^{q-1} c(\|x^*(s)\|)ds$$

$$\leq V(t_0, x_0) - \frac{1}{\Gamma(q)} \int_{t_0}^{t_1} (t_1 - s)^{q-1} c(\|x^*(s)\|)ds - \frac{1}{\Gamma(q)} \int_{s_1}^{t} (t-s)^{q-1} c(\|x^*(s)\|)ds.$$

Therefore, the claim of Lemma 4 is true on $(s_1, t_2] \cap [t_0, T]$.

Let $T > t_2$ and $t \in (t_2, s_2] \cap [t_0, T]$. From condition 2(ii) and the above we obtain

$$V(t, x^*(t)) \leq V(t_2 - 0, x^*(t_2 - 0)) = V(t_2, x^*(t_2))$$

$$\leq V(t_0, x_0) - \frac{1}{\Gamma(q)} \int_{t_0}^{t_1} (t_1 - s)^{q-1} c(\|x^*(s)\|)ds - \frac{1}{\Gamma(q)} \int_{s_1}^{t_2} (t_2 - s)^{q-1} c(\|x^*(s)\|)ds.$$

Continue this process and an induction argument proves the claim is true for $t \in [t_0, T]$. □

Remark 10. The results of Lemma 2 and Lemma 4 are true on the half line (recall [3] that Lemma 1 and Lemma 3 extend to the half line).

Remark 11. The results of Lemma 2 and Lemma 4 will be similar with slight changes of condition 2(ii) if the initial time t_0 is in a interval of noninstantaneous impulses, i.e. $t_0 \in \cup_{k=1}^{\infty} (t_k, s_k]$.

5.2. **Generalized Dini fractional derivative.** Now we present the analogue results of Section 5.1 using (4.3).

Lemma 5 ([22])**.** *Let $V \in C(\mathbb{R}_+ \times \mathbb{R}^n, \mathbb{R}_+)$ and $V(t,x)$ be locally Lipschitzian in x. Assume that*

$${}^{c}D_{+}^{q} V(t,x) \leq g(t, V(t,x)), \quad (t,x) \in \mathbb{R}_+ \times \mathbb{R}^n,$$

where $g \in C(\mathbb{R}_+^2, \mathbb{R})$ and ${}^{c}D_{+}^{q} V(t,x)$ is defined by (4.3). Suppose that the maximal solution $r(t; t_0, u_0)$ of IVP

$${}_{t_0}^{c}D^q u = g(t,u), \quad u(t_0) = u_0 \geq 0$$

exists on $[t_0, \infty)$. Then $V(t_0, u_0) \leq u_0$ implies $V(t, x(t)) \leq r(t)$ for $t \geq t_0$ where $x(t) = x(t; t_0, x_0)$ is any solution of IVP (5.1) existing on $[t_0, \infty)$.

Corollary 1. *Let $V \in C(\mathbb{R}_+ \times \mathbb{R}^n, \mathbb{R}_+)$ and $V(t,x)$ be locally Lipschitz in x and ${}^{c}D_{+}^{q} V(t,x) \leq 0$ for $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^n$. Then $V(t, x(t)) \leq V(t_0, x_0)$ for $t \geq t_0$ where $x(t) = x(t; t_0, x_0)$ is any solution of IVP (5.1) existing on $[t_0, \infty)$.*

Now we will give a comparison result for NIFrDE (3.1) by application of generalized Dini fractional derivative of Lyapunov function.

Lemma 6 (Comparison result for NIFrDE by generalized Dini fractional derivative)**.** *Assume the conditions of Lemma* 2 *are satisfied where the inequality* (5.2) *is replaced by*

$$ {}^{c}_{(3.1)}\mathcal{D}^{q}_{+}V(t,x^*(t)) \le 0 \quad \text{for } t\in[t_0,\infty)\bigcap\cup_{k=0}^{\infty}(s_k,t_{k+1}). \tag{5.4}$$

Then the inequality $V(t,x^*(t))\le V(t_0,x_0)$ *holds on* $[t_0,\infty)$.

P r o o f: The proof of Lemma 6 is similar to that in Lemma 2 where instead of Lemma 1 we apply Corollary 1. □

If $V(x)=x^Tx=\sum_{k=1}^{n}x_k^2\in\Lambda^C(\mathbb{R}^n)$, $x=(x_1,x_2,\dots,x_n)^T$ we obtain the following result:

Corollary 2 (Comparison result by a quadratic Lyapunov function)**.** *Let the function* $x^*(t)=x(t;t_0,x_0)\in PC^q([t_0,\infty),\mathbb{R}^n)$ *be a solution of the NIFrDE* (3.1) *such that*

(i) $\big(x^*(t)\big)^T f(t,x^*(t))\le 0$ *for* $t\in[t_0,\infty)\bigcap(s_k,t_{k+1})$, $k=0,1,2,\dots$.
(ii) *for any* $k=1,2\dots$ *the inequalities*

$$\|x^*(t)\|\le\|x^*(\tau_k-0)\| \quad \text{for } t\in[t_0,\infty)\bigcap(t_k,s_k] \text{ with } \tau_k=\max\{t_0,t_k\}$$

hold.

Then the inequality $\|x^*(t)\|\le\|x_0\|$ *holds on* $[t_0,\infty)$.

P r o o f: The proof follows from Lemma 6 applied with $V(x)=x^Tx=\sum_{k=1}^{n}x_k^2$ and note

$$\begin{aligned} {}^{c}_{(3.1)}\mathcal{D}^{q}_{+}V(x) &= \lim_{h\to 0^+}\sup\frac{1}{h^q}\Big[\sum_{i=1}^{n}x_i^2-\sum_{i=1}^{n}\big(x_i-h^q f^{(i)}(t,x)\big)^2\Big]\\ &=\sum_{i=1}^{n}2x_i f^{(i)}(t,x)=2\ x^Tf(t,x)\\ &\quad \text{for } t\in[t_0,\infty)\bigcap(s_k,t_{k+1}),\ k=0,1,2,\dots \end{aligned} \tag{5.5}$$

□

Example 5. Consider the scalar NIFrDE (3.4) where $A<0$, and $\Psi_k(t,x)=a_k(t)x$, and $a_k:[t_k,s_k]\to\mathbb{R}$, $k=1,2,3,\dots$ are such that $\sup_{t\in[t_k,s_k]}|a_k(t)|\le 1$. Its solution $x^*(t)=x(t;0,x_0)$ is given in Example 1. Using $0<E_q(A(T-\tau)^q)\le 1$ for $T\ge\tau$ it follows that the conditions of Corollary 2 are satisfied and therefore $|x^*(t)|\le|x_0|$ for $t\ge 0$. □

Remark 12. In this paper we assumed an infinite number of points $\{t_i\}_{i=1}^{\infty}$, $\{s_i\}_{i=1}^{\infty}$ with $0<t_i\le s_i<t_{i+1}$ and $\lim_{k\to\infty}t_k=\infty$. However it is worth noting that the results in Section 5 (and elsewhere) hold true if we only consider a finite of points $\{t_i\}_{i=1}^{p}$, $\{s_i\}_{i=1}^{p}$ with $0<t_i\le s_i<t_{i+1}$, $i=1,\dots,p$ with $t_{p+1}=T$.

6. MAIN RESULTS

We will obtain sufficient conditions for stability of the zero solution of nonlinear impulsive Caputo fractional differential equations. Again we assume $0 < q < 1$.

We say conditions (H) are satisfied if :

(H1) The function $f \in C(\cup_{k=0}^{\infty}[s_k, t_{k+1}], \mathbb{R}^n)$, $f(t, 0) \equiv 0$ for $t \in \cup_{k=0}^{\infty}[s_k, t_{k+1}]$ is such that for any initial point $(\tilde{t}_0, \tilde{x}_0) \in \cup_{k=0}^{\infty}[s_k, t_{k+1}) \times \mathbb{R}^n$, $s_k \leq \tilde{t}_0 < t_{p+1}$, p is a nonzero integer, the IVP for the system of FrDE (3.2) with $\tau_k = \tilde{t}_0$ has a solution $x(t; \tilde{t}_0, \tilde{x}_0) \in C^q([\tilde{t}_0, t_{p+1}], \mathbb{R}^n)$.

(H2) The functions $\phi_k \in C([t_k, s_k] \times \mathbb{R}^n, \mathbb{R}^n)$, $\phi_k(t, 0) \equiv 0$ for $t \in [t_k, s_k]$, $k = 1, 2, \ldots$.

Remark 13. Conditions (H) guarantee the existence of a solution $x(t; t_0, x_0) \in PC^q([t_0, \infty), R^n)$ of NIFrDE (3.1) for any initial data $(t_0, x_0) \in \cup_{k=0}^{\infty}[s_k, t_{k+1}) \times \mathbb{R}^n$. If (H) is satisfied then (3.1) has a zero solution.

6.1. Stability by generalized Caputo fractional Dini derivative.

Theorem 1 (Stability)**.** *Let the following conditions be satisfied:*

1. *Condition (H) is satisfied.*
2. *There exists a function $V \in \Lambda(\mathbb{R}_+, \mathbb{R}^n)$ such that $V(t, 0) = 0$ and*
   (i) *the inequality*

$$ {}_{(3.1)}^{\ c}D_+^q V(t, x; t_0, x_0) \leq 0 \quad \text{for } t \in \cup_{k=0}^{\infty}(s_k, t_{k+1}),\ x, x_0 \in \mathbb{R}^n \tag{6.1}$$

   holds;
   (ii) *for any point $x \in \mathbb{R}^n$ and any $t \in (t_k, s_k]$, $k = 1, 2, 3, \ldots$ the inequality $V(t, \phi_k(t, x)) \leq V(t_k - 0, x)$ holds;*
   (iii) *$b(\|x\|) \leq V(t, x)$ for $t \in \mathbb{R}_+$, $x \in \mathbb{R}^n$, where $b \in \mathcal{K}$.*

Then the zero solution of the NIFrDE (3.1) *is stable.*

P r o o f: Let $\epsilon > 0$ and $t_0 \in \cup_{k=0}^{\infty}[s_k, t_{k+1})$ be arbitrary given numbers. There exists a $p \in \{0, 1, \ldots\}$ with $t_0 \in [s_p, t_{p+1})$. Without loss of generality assume $p = 0$.

Since $V(t_0, 0) = 0$ there exists $\delta_1 = \delta_1(t_0, \varepsilon) > 0$ such that $V(t_0, x) < b(\varepsilon)$ for $\|x\| < \delta_1$. Let $x_0 \in \mathbb{R}^n$ with $\|x_0\| < \delta_1$. Then $V(t_0, x_0) < b(\varepsilon)$. Consider any solution $x^*(t) = x(t; t_0, x_0) \in PC^q([t_0, \infty), \mathbb{R}^n)$ of NIFrDE (3.1). From inequality (6.1) it follows that

$$ {}_{(3.1)}^{\ c}D_+^q V(t, x^*(t); t_0, x_0) \leq 0 \quad \text{for } t \in (t_0, \infty) \bigcap \cup_{k=0}^{\infty}(s_k, t_{k+1}),$$

i.e. condition 2(i) of Lemma 2 (with $T = \infty$, see Remark 10) is satisfied.

Let $t \in (t_k, s_k] \bigcap [t_0, \infty)$, $k = 1, 2, 3, \dots$. From condition 2(ii) of Theorem 1 we get

$$V(t, x^*(t)) = V(t, \phi_k(t, x^*(t_k))) \leq V(t_k - 0, x^*(t_k)) = V(t_k - 0, x^*(t_k - 0)).$$

Therefore, condition 2(ii) of Lemma 2 is fulfilled.

From Lemma 2 applied to the solution $x^*(t)$ with $T = \infty$ (see Remark 10) and condition $2(iii)$ we obtain

$$b(\|x^*(t)\|) \leq V(t, x^*(t)) \leq V(t_0, x_0) < b(\epsilon),$$

so the result follows. □

Theorem 2 (Uniform stability)**.** *Let the following conditions be satisfied:*

1. *Condition (H) is satisfied.*
2. *There exists a function $V \in \Lambda(\mathbb{R}_+, \mathbb{R}^n)$ such that*
   (i) *the inequality is satisfied*

$$ {}_{(3.1)}^{\;\;c}D_+^q V(t, x; t_0, x_0) \leq 0 \quad \textit{for } t \in \cup_{k=0}^{\infty}(s_k, t_{k+1}),\ x, x_0 \in S(\lambda) \tag{6.2}$$

   where $\lambda > 0$ is a given number;
   (ii) *for any point $t \in (t_k, s_k]$, $k = 1, 2, 3, \dots$ and any $x \in S(\lambda)$ the inequality $V(t, \phi_k(t, x)) \leq V(t_k - 0, x)$ holds;*
   (iii) *$b(\|x\|) \leq V(t, x) \leq a(\|x\|)$ for $t \in \mathbb{R}_+$, $x \in \mathbb{R}^n$, where $a, b \in \mathcal{K}$.*

Then the zero solution of NIFrDE (3.1) *is uniformly stable.*

P r o o f: Let $\epsilon \in (0, \lambda]$ and $t_0 \in \cup_{k=0}^{\infty}[s_k, t_{k+1})$ be arbitrary given numbers. There exists $p \in \{0, 1, \dots\}$ with $t_0 \in [s_p, t_{p+1})$. Without loss of generality assume $p = 0$.

Let $\delta_1 < \min\{\epsilon, b(\epsilon)\}$. From $a \in \mathcal{K}$ there exists $\delta_2 = \delta_2(\epsilon) > 0$ so if $s < \delta_2$ then $a(s) < \delta_1$. Let $\delta = \min(\epsilon, \delta_2)$. Choose the initial value $x_0 \in \mathbb{R}^n$ such that $\|x_0\| < \delta$ and let $x^*(t) = x(t; t_0, x_0) \in PC^q([t_0, \infty), \mathbb{R}^n)$ be a solution of the IVP for NIFrDE (3.1). We now prove that

$$\|x^*(t)\| < \epsilon, \quad t \geq t_0. \tag{6.3}$$

Assume inequality (6.3) is not true and let $t^* = \inf\{t > t_0 :\ \|x^*(t)\| \geq \epsilon\}$. Then

$$\|x^*(t)\| < \epsilon \quad \text{for } t \in [t_0, t^*) \text{ and } |x^*(t^*)\| = \epsilon. \tag{6.4}$$

If $t^* \neq t_k$, $k = 1, 2, \dots$ or if $t^* = t_p$ for some natural number p and $\|x^*(t_p - 0)\| = \epsilon$ then (6.4) is true. If for a natural number p we have $t^* = t_p$ and $\|x^*(t_p - 0)\| < \epsilon$, then according to Lemma 2 for $T = t_p$ and $\Delta = S(\lambda)$ we obtain $V(t, x^*(t)) \leq V(t_0, x_0)$ for $t \in [t_0, t_p]$. Then for all $t \in (t_p, s_p]$ from condition 2(iii) we get $b(\|x^*(t)\|) \leq V(t, x^*(t)) = V(t, \phi_p(t, x^*(t_p - 0))) \leq V(t_p - 0, x^*(t_p - 0)) \leq V(t_0, x_0) \leq a(\delta) < \delta_1 < b(\varepsilon)$. Thus $||x^*(t)|| < b^{-1}(\delta_1) < \epsilon$ for $t \in (t_p, s_p]$, and this contradicts the choice of t^*. Therefore, (6.4) holds.

Then, $x^*(t) \in S(\lambda)$ on $[t_0, t^*]$ and conditions 2(i) and 2(ii) of Lemma 2 are satisfied on $[t_0, t^*]$. From Lemma 2 applied to the solution $x^*(t)$ with $T = t^*$ and $\Delta = S(\lambda)$ we get $V(t, x^*(t)) \leq V(t_0, x_0)$ on $[t_0, t^*]$. Then applying condition 2 (iii) of Theorem 2 we obtain $b(\varepsilon) = b(\|x^*(t^*)\|) \leq V(t^*, x^*(t^*)) \leq V(t_0, x_0) \leq a(\delta) < \delta_1 < b(\varepsilon)$. The contradiction proves (6.3) and therefore, the zero solution of NIFrDE (3.1) is uniformly stable. □

Now we present some sufficient conditions for the uniform asymptotic stability of the zero solution of the NIFrDE.

Theorem 3 (Uniform asymptotic stability)**.** *Let the following conditions be satisfied:*

1. *Condition (H) is satisfied.*
2. *There exists a positive constant $M < \infty$ such that $\sum_{i=1}^{\infty}(s_i - t_i) \leq M$.*
3. *There exists a function $V \in \Lambda(\mathbb{R}_+, \mathbb{R}^n)$ such that*
   (i) *the inequality is satisfied*

$$ {}_{(3.1)}^{\;\;c}D_+^q V(t, x; t_0, x_0) \leq -c(\|x\|) \quad \text{for } t \in \cup_{k=0}^{\infty}(s_k, t_{k+1}),\ x, x_0 \in S(\lambda) \tag{6.5} $$

   where $\lambda > 0$ is a given number, $c \in \mathcal{K}$;
   (ii) *for any point $t \in (t_k, s_k]$, $k = 1, 2, 3, \dots$ and any $x \in S(\lambda)$ the inequality $V(t, \phi_k(t, x)) \leq V(t_k - 0, x)$ holds;*
   (iii) *$b(\|x\|) \leq V(t, x) \leq a(\|x\|)$ for $t \in \mathbb{R}_+$, $x \in \mathbb{R}^n$, where $a, b \in \mathcal{K}$.*

Then the zero solution of NIFrDE (3.1) *is uniformly asymptotically stable.*

P r o o f: From Theorem 2 the zero solution of the NIFrDE (3.1) is uniformly stable. Therefore, for the number λ there exists $\alpha = \alpha(\lambda) \in (0, \lambda)$ such that for any $\tilde{t}_0 \in \cup_{k=0}^{\infty}[s_k, t_{k+1})$ and $\tilde{x}_0 \in \mathbb{R}^n$ the inequality $\|\tilde{x}_0\| < \alpha$ implies

$$ \|x(t; \tilde{t}_0, \tilde{x}_0)\| < \lambda \quad \text{for } t \geq \tilde{t}_0 \tag{6.6} $$

where $x(t; \tilde{t}_0, \tilde{x}_0)$ is any solution of the NIFrDE (3.1) (with initial data $(\tilde{t}_0, \tilde{x}_0)$).

Now we will prove that the zero solution of the fractional differential equations (3.1) is uniformly attractive. Consider the constant $\beta \in (0, \alpha]$ such that $a(\beta) \leq b(\alpha)$. Let $\epsilon \in (0, \lambda]$ and $t_0 \in \cup_{k=0}^{\infty}[s_k, t_{k+1})$ be arbitrary given numbers. There exists a $p \in \{0, 1, \dots\}$ with $t_0 \in [s_p, t_{p+1})$. Without loss of generality assume $p = 0$.

Let the point $x_0 \in \mathbb{R}^n$, $\|x_0\| < \beta$ and $x^*(t) = x(t; t_0, x_0)$ be any solution of (3.1). Then $b(\|x_0\|) \leq a(\|x_0\|) < a(\beta) < b(\alpha)$, i.e. $\|x_0\| < \alpha$ and according to (6.6) the inequality

$$ \|x^*(t)\| < \lambda \quad \text{for } t \geq t_0 \tag{6.7} $$

holds, i.e. the solution $x^*(t) \in S(\lambda)$ on $[t_0, \infty)$.

Choose a constant $\gamma = \gamma(\epsilon) \in (0, \epsilon]$ such that $a(\gamma) < b(\epsilon)$. Let $T > \sqrt[q]{a(\alpha)\frac{q\Gamma(q)}{c(\gamma)}} + M$ and m be a natural number such that $s_m < t_0 + T \leq t_{m+1}$. Note T depends only on ε but not on t_0. We now prove that

$$\|x^*(t)\| < \epsilon \quad \text{for } t \geq t_0 + T. \tag{6.8}$$

Assume

$$\|x^*(t)\| \geq \gamma \quad \text{for every } t \in [t_0, t_0 + T]. \tag{6.9}$$

Then from Lemma 4 (applied to the interval $[t_0, t_0 + T]$ and $\Delta = S(\lambda)$), conditions 2 and 3 (ii) of Theorem 3, inequality $a^q + b^q \geq (a + b)^q$ for $a, b > 0$ and the choice of T we get

$$\begin{aligned}
&V(t_0 + T, x^*(t_0 + T)) \\
&\leq V(t_0, x_0) - \frac{1}{\Gamma(q)}\Big(\sum_{i=0}^{m-1}\int_{\tau_i}^{t_{i+1}}(t_{i+1} - s)^{q-1}c(\|x^*(s)\|)ds \\
&\qquad + \int_{s_m}^{t_0+T}(t_0 + T - s)^{q-1}c(\|x^*(s)\|)ds\Big) \\
&\leq a(\|x_0\|) - \frac{c(\gamma)}{\Gamma(q)}\Big(\sum_{i=0}^{m-1}\int_{\tau_i}^{t_{i+1}}(t_{i+1} - s)^{q-1}ds + \int_{s_m}^{t_0+T}(t_0 + T - s)^{q-1}ds\Big) \\
&< a(\alpha) - \frac{c(\gamma)}{q\Gamma(q)}\Big((t_1 - t_0)^q + \sum_{i=1}^{m-1}(t_{i+1} - s_i)^q + (T + t_0 - s_m)^q\Big) \\
&\leq a(\alpha) - \frac{c(\gamma)}{q\Gamma(q)}\Big((t_1 - t_0) + \sum_{i=1}^{m-1}(t_{i+1} - s_i) + (T + t_0 - s_m)\Big)^q \\
&= a(\alpha) - \frac{c(\gamma)}{q\Gamma(q)}\Big(-\sum_{i=1}^{m}(s_i - t_i) + T\Big)^q \leq a(\alpha) - \frac{c(\gamma)}{q\Gamma(q)}\Big(-M + T\Big)^q < 0.
\end{aligned}$$

The above contradiction proves there exists $t^* \in [t_0, t_0 + T]$ such that $\|x^*(t^*)\| < \gamma$. Let the natural number p be such that $t_{p-1} \leq t^* < t_p$.

Case 1. Let $t \in [t^*, t_p]$.

If $s_{p-1} < t^* < t_p$ then for $t \in [t^*, t_p]$ the function $x^*(t) \in C^q([t^*, t_p], \mathbb{R}^n)$ and from Lemma 3 we get $V(t, x^*(t)) \leq V(t^*, x^*(t^*)) - \frac{1}{\Gamma(q)}\int_{t^*}^{t}(t - s)^{q-1}c(\|x^*(s)\|)ds \leq V(t^*, x^*(t^*))$.

If $t_{p-1} < t^* \leq s_{p-1}$ then for $t \in [t^*, t_p]$ the function $x^*(t) \in PC^q([t^*, t_p], \mathbb{R}^n)$ and from Lemma 4 we get $V(t, x^*(t)) \leq V(t^*, x^*(t^*))$.

Case 2. For any $t > t^*$, $t \in (s_k, t_{k+1}]$, $k = p, p+1, \dots,$ from Lemma 4 for $\Delta = S(\lambda)$ we obtain

$$\begin{aligned} V(t, x^*(t)) \le V(t^*, x^*(t^*)) - \frac{1}{\Gamma(q)}\Big(&\int_{t^*}^{t_p} (t-s)^{q-1} c(\|x^*(s)\|)ds \\ &+ \sum_{i=p}^{k-1} \int_{s_i}^{t_{i+1}} (t_{i+1}-s)^{q-1} c(\|x^*(s)\|)ds \\ &+ \int_{s_k}^{t} (t-s)^{q-1} c(\|x^*(s)\|)ds\Big) \le V(t^*, x^*(t^*)). \end{aligned}$$

Case 3. For any $t > t^*$, $t \in (t_k, s_k]$, $k = p+1, p+2, \dots,$ from Lemma 4 for $\Delta = \mathbb{R}^n$ we obtain

$$\begin{aligned} V(t, x^*(t)) \le V(t^*, x^*(t^*)) - \frac{1}{\Gamma(q)}\Big(&\int_{t^*}^{t_p} (t-s)^{q-1} c(\|x^*(s)\|)ds \\ &+ \sum_{i=p}^{k-1} \int_{s_i}^{t_{i+1}} (t_{i+1}-s)^{q-1} c(\|x^*(s)\|)ds\Big) \\ \le V(t^*, x^*(t^*)). \end{aligned}$$

Therefore, for $t \ge t^*$ the following inequality is satisfied:

$$V(t, x^*(t)) \le V(t^*, x^*(t^*)). \tag{6.10}$$

Then for any $t \ge t^*$ applying (6.10), condition 3(iii) and inequality (6.7) we get the inequalities

$$b(\|x^*(t)\|) \le V(t, x^*(t)) \le V(t^*, x^*(t^*)) \le a(\|x^*(t^*)\|) \le a(\gamma) < b(\epsilon).$$

Therefore, the inequality (6.8) holds for all $t \ge t^*$ (hence for $t \ge t_0 + T$). □

Remark 14. If the initial time t_0 is in an interval of noninstantaneous impulses, i.e. $t_0 \in \cup_{k=1}^{\infty}(t_k, s_k]$ then the results of Theorem 1, Theorem 2 and Theorem 3 will be similar with slight changes in Definition 2 and condition 2(ii) (Theorems 1,2) or condition 3(ii) (Theorem 3).

6.2. **Stability by generalized Dini fractional derivative.** The proofs below are similar to those in Theorem 1 and Theorem 2 where Lemma 6 is used instead of Lemma 2.

Theorem 4 (Stability)**.** *Let the conditions of Theorem* 1 *be satisfied where the inequality* (6.1) *is replaced by*

$$ {}_{(3.1)}^{\ c}\mathcal{D}_+^q V(t,x) \le 0 \quad \textit{for } t \in (s_k, t_{k+1}],\ k = 0, 1, 2, \dots, \quad x \in \mathbb{R}^n. \tag{6.11}$$

Then the zero solution of the NIFrDE (3.1) *is stable.*

Theorem 5 (Uniform stability). *Let the conditions of Theorem 2 be satisfied where the inequality (6.2) is replaced by*

$$ {}_{(3.1)}^{\;\;c}\mathcal{D}_+^q V(t,x) \le 0 \quad \textit{for } t\in(s_k,t_{k+1}],\ k=0,1,2,\dots,\ x\in S(\lambda). \tag{6.12}$$

Then the zero solution of the IFrDE (3.1) is uniformly stable.

We will give the comparison result with the quadratic Lyapunov function $V(x) = x^T x = \sum_{k=1}^n x_k^2 \in \Lambda^C(\mathbb{R}^n)$, $x = (x_1, x_2, \dots, x_n)$.

Corollary 3 (Stability by a quadratic function). *Let condition (H) be satisfied and*

(i) $x^T f(t,x) \le 0$ *for* $t \in (s_k, t_{k+1})$, $k = 0,1,2,\dots$, $x \in S(\lambda)$;

(ii) *for any* $k = 1,2,3,\dots$ *and* $x \in S(\lambda)$ *for* $t \in [s_k, t_k]$ *the inequality* $\big(\phi_k(t,x)\big)^T \phi_k(t,x) \le x^T x$ *for* $t \in (t_k, s_k]$ *holds;*

(iii) $b(\|x\|) \le x^T x \le a(\|x\|)$ *for* $t \in \mathbb{R}_+$, $x \in \mathbb{R}^n$, *where* $a, b \in \mathcal{K}$.

Then the zero solution of (3.1) is uniformly stable.

Remark 15. If $t_k = s_k$ and $y = \phi_k(t_k, y)$, $y \in \mathbb{R}^n$ for all $k = 1,2\dots$ system (3.1) reduces to a system of fractional differential equations. For the reduced (3.1) in [40] the author defines a generalized fractional-order derivative (Dini-like derivative) in the Caputo sense based on the fractional-order Dini derivative in the Caputo sense ([21], [22]) and some sufficient conditions for the stability with initial time difference are obtained.

7. APPLICATIONS

7.1. Quadratic Lyapunov function.

We will apply the quadratic Lyaunov function and its fractional derivative.

Example 6. Consider the scalar NIFrDE (3.4) where $A \le 0$ and $\phi_k(t,x) = a_k(t)x$, $a_k : [t_k, s_k] \to \mathbb{R}$, $k = 1,2,3,\dots$ are such that $\sup_{t\in[t_k,s_k]} |a_k(t)| \le 1$. Consider the quadratic Lyapunov function $V(x) = x^2$. Then $xf(t,x) = Ax^2 \le 0$ for $t \in (s_k, t_{k+1})$, $k = 0,1,2,\dots$, $x \in S(\lambda)$, i.e. condition (i) of Corollary 3 is satisfied. Also, $\big(\phi_k(t,x)\big)^2 = \big(a_k(t)\big)^2 x^2 \le x^2$ for $t \in [t_k, s_k]$, i.e. condition (ii) is satisfied.

From Corollary 3 the zero solution of the scalar NIFrDE (3.4) is uniformly stable (this was proved directly in Example 2). □

Example 7. Let the points t_k, s_k, $k = 0,1,2,\dots$, be such that $0 \le t_k < s_k < t_{k+1}$, $\lim_{k\to\infty} t_k = \infty$. Consider the scalar noninstantaneous impulsive Caputo fractional differential equation

$$\begin{aligned} &{}_{s_k}^{c}D^q x = -a(t)x(1+x^2) \quad \text{for } t\in(s_k,t_{k+1}],\ k=0,1,2,\dots,\\ &x(t) = c_k(t)x(t_k-0) \quad \text{for } t\in(t_k,s_k],\ k=1,2,3,\dots, \\ &x(0)=x_0 \end{aligned} \tag{7.1}$$

where $x \in \mathbb{R}$, $a(t) \in C(\cup_{k=0}^{\infty}(s_k, t_{k+1}], \mathbb{R}_+)$, $c_k(t) \in C([t_k, s_k], [-1, 1])$, $k = 1, 2, \dots$.

Consider the function $V(t, x) = x^2$. Then $xf(t, x) = -a(t)x^2(1 + x^2) \leq 0$, i.e. condition (i) of Corollary 3 is satisfied. Also, $\big(c_k(t)x\big)^2 = (c_k(t)x)^2 \leq x^2$ for $t \in (s_k, t_k]$, $k = 1, 2, 3, \dots$, i.e. condition (ii) is satisfied.

From Corollary 3, the trivial solution of NIFrDE (7.1) is uniformly stable. $\square$

7.2. **General Lyapunov function.** Now we will apply a general Lyapunov function and its generalized Caputo fractional Dini derivative.

Example 8. Let points $s_k = (4k + 1)\frac{\pi}{2}$, $t_k = (4k - 1)\frac{\pi}{2}$, $k = 1, 2, \dots$, $s_0 = 0$. Consider the following initial value problem for the scalar noninstantaneous impulsive Caputo fractional differential equation

$$\begin{aligned} &{}_{s_k}^{c}D^{0.5}x(t) = xf(t), \quad t \in (s_k, t_{k+1}],\ k = 0, 1, 2, \dots, \\ &x(t) = c_k(t)x(t_k - 0), \quad t \in [t_k, s_k],\ k = 1, 2, \dots, \\ &x(0) = x_0, \end{aligned} \tag{7.2}$$

where $x \in \mathbb{R}$, $c_k \in C([t_k, s_k], [-1, 1])$, $f(t) = 0.5\frac{\frac{-2}{\sqrt{t\pi}} + \sqrt{t}E_{2,1.5}(-t^2)}{2 - \sin(t)}$, $k = 0, 1, 2, \dots$.

Let $V(t, x) = x^2$. Then $x\big(xf(t)\big) = x^2 f(t)$. Since the sign of the function $f(t)$ changes (see Figure 2) Corollary 3 and the quadratic Lyapunov function are not applicable to the fractional equation (7.2).

Let $V(t, x) = (2 - \sin(t))x^2$.

Apply generalized Dini fractional derivative given by (4.6), Example 4 and formula (4.11) and we get

$${}_{(7.2)}^{c}\mathcal{D}_{+}^{0.5}V(t, x) = 2x^2\ (2 - \sin(t))f(t).$$

The sign of the function $f(t)$ and the derivative ${}_{(7.2)}^{c}\mathcal{D}_{+}^{0.5}V(t, x)$ are changeable. Therefore, the application of fractional Dini derivative (4.6) does not give us a conclusion about stability properties of NIFrDE (7.2).

Now apply generalized Caputo fractional Dini derivative to the considered Lyapunov function. According to Example 4 for $t \in (s_k, t_{k+1})$, $k = 0, 1, 2, \dots$ and $\Gamma(0.5) = \sqrt{\pi}$ we obtain

$$\begin{aligned} {}_{(7.2)}^{c}D_{+}^{0.5}V(t, x; 0, x_0) &= 2x^2(2 - \sin(t))f(t) + x^2\ {}_{0}^{RL}D^{0.5}(2 - \sin(t)) - \frac{2(x_0)^2}{\sqrt{t\pi}} \\ &= 2x^2(2 - \sin(t))f(t) + x^2\left(\frac{2}{\sqrt{t\pi}} - \sqrt{t}E_{2,1.5}(-t^2)\right) - \frac{2(x_0)^2}{\sqrt{t\pi}} \leq 0. \end{aligned} \tag{7.3}$$

Also, for $t \in [t_k, s_k]$, $k = 1, 2, \dots$ we get $V(t, c_k(t)x) = \big(2 - \sin(t)\big)(c_k(t)x)^2 \leq \big(2 - \sin(t)\big)x^2 \leq \big(2 - \sin(t_k)\big)x^2 = \big(2 - \sin((4k - 1)\frac{\pi}{2})\big)x^2 = V(t_k - 0, x)$, i.e. condition 2(ii) of Theorem 1 is satisfied.

According to Theorem 1 the zero solution of (7.2) is stable. $\square$

8. CONCLUSSIONS

Piecewise continuous scalar Lyapunov functions are applied to study stability, uniform stability and asymptotic uniform stability of the zero solution of nonlinear Caputo fractional differential equations with not instantaneous impulses. Two types of derivatives of Lyapunov function among NIFrDE are introduced and their applications are discussed. Several sufficient conditions for stability, uniform stability and uniform asymptotic stability of the zero solution of nonlinear NIFrDE are obtained. The results are illustrated with several examples.

Note the above description and all consideration in the paper could be generalized when additionally an instantaneous jump at the points s_k are given and the right side part of the fractional equation is changing (so called variable structure), i.e. problem (3.1) can be generalized to

$$\begin{aligned}
&{}_{s_k}^{c}D^q x = f_k(t,x) \quad \text{for } t \in (s_k, t_{k+1}],\ k = 0,1,2,\dots \\
&x(t) = \phi_i(t, x(t_i - 0)) \quad \text{for } t \in (t_i, s_i],\ i = 1,2,\dots, \\
&x(s_i + 0) = G_k(x(s_i - 0)) \quad \text{for } i = 1,2,\dots, \\
&x(t_0) = x_0.
\end{aligned}$$

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