Communications in Applied Analysis **12** (2008), no. 2, 113–136

COMPETITION FOR MARKET IN THE LAST STAGE OF PRODUCT LIFE-CYCLE: A DIFFERENTIAL GAMES APPROACH

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ABSTRACT. After stable sale in the second stage of product life cycle, the sale will decrease gradually. This paper deals with competition in the last stage of product life cycle. A differential game model will be set up for competition in this stage. We base the dynamics on Lanchester type competition and natural decrease of sales. Because the length of product life will be affected by the advertising, the **n** competitors' controls/advertising policies will be coupled in the coefficients of natural decrease. Infinite time horizon will be involved. Besides open-loop and closed-loop controls, we will discuss another type of competition, in which the roles of competitors are not symmetric: some competitors adopt open-loop controls and others adopt closed-loop controls. From Pontryagin optimal condition we derive Two-Point-Boundary-Value Problems(TPBVP). We will solve the TPBVP by an algorithm based on Newton's method. In the algorithm we will use random perturbation technique to generate Jacobian matrix and prevent the Jacobian matrix from being singular. From numerical results we draw general rules about competition policies in the final stage. From the numerical results we will also see that in some markets, small companies following the control/advertising policies of bigger companies is a good strategy.

Keywords. differential games, marketing, competitive strategy

AMS (MOS) Subject Classifications. 91A23, 91A80, 91B50, 91B60

1. INTRODUCTION

In this paper we deal with optimal competition advertising strategy in the final stage of a product life cycle. Erickson (2007) [6] deals with competition in the final stage of product life cycle using dynamics programming technique to solve a general n-person differential game model. However, in his model, the dynamics and objective function have special form. It is hard to apply to general problems. In this paper we give a general algorithm to solve more general differential game models. In this algorithm, we use random perturbation method to generate a Jacobian matrix in each iteration. So this algorithm can be said to be random quasi-Newton method.

This paper first introduces "nonsymmetric" differential game. This motivation is based on the fact that, in real market competition, every company/competitor cannot afford to current information on state variables. Bigger companies can pay

Received March 7, 2008

to get current sales information while smaller ones cannot. In this paper deal with optimality conditions taking this fact in consideration, and compare results from open- or closed-loop controls to get some deeper insight into competition and draw some useful practical guidelines.

The results strengthen those from Medhin and Wan (2007) [9], that closed-loop controls are better than open-loop controls, where one can use smaller control to get better objective value. Further, in "nonsymmetric" case, the result is mixed. Useful practical general guideline is given based on comparison of the differing differential game models. In reality, company may choose specific type of (open or closed) optimal strategy to compete with others.

The results obtained are useful for real marketing decision.

Before we discuss competition in the last stage of a product life cycle, we review an important technique drawn from dynamic programming to solve differential game problems in an infinite time horizon environment. The general **n**-player differential game is the following:

$$\begin{array}{ll} \min_{u_i} & \int_{t_0}^{\infty} e^{-rt} f_i(X, u_1, \ldots, u_n) dt \\ s.t. & \\ \dot{X} = & g(X, u_1, \ldots, u_n) \\ X(t_0) = & X_0 \\ & i = 1, \ldots, n \end{array}$$

Let

$$V_i(X_0) = \min \int_{t_0}^{\infty} e^{-r(t-t_0)} f_i(X, u_1, \ldots, u_n) dt$$

Then, an optimality condition drawn from dynamic programming is

$$rV_i(X) = \min_{u_i} [f_i(X, u_1, \ldots, u_n) + \nabla_X V(X)^T \cdot g(X, u_1, \ldots, u_n)]$$

which is called *Hamilton-Jacobi-Bellman* equation(HJB).

Let us mention a most recent paper from Erickson (2007) [6] which uses the above technique to solve a differential game problem which deals with competition at the final stage of product life cycle. The usual sales trajectories in the last stage of product life cycle are shown in *Figure 1.1.* There, we can see the decrease of sales in this stage. The dynamics in this paper (Erickson (2007)) [6] is the following:

$$\max_{a_i} V_i = \int_0^{\infty} e^{-r_i t} (q_i s_i - a_i^2) dt$$

subject to

$$\dot{s_i} = -\rho_i s_i + \beta_i a_i \sqrt{N - \sum_{j=i}^{n} s_j}$$

where
$N =$ maximum sales potential for the market.

$s_i =$ i-th company's sales at time t.
$a_i =$ i-th company's advertising at time t.
$\beta_i =$ the effectiveness of i-th company's advertising.
$\rho_i =$ the decay rate of i-th company's sales.
$q_i =$ i-th company contribution.
$r_i =$ i-th company discount rate.
Then, the Hamilton-Jacobi-Bellman equations for company $i = 1, \ldots, n$ are:

$$r_i V_i = \max_{a_i} \left[q_i s_i - a_i^2 + \sum_{j=1}^{n} \frac{\partial V_i}{\partial s_j} \left(\beta_j a_j \sqrt{N - \sum_{k=1}^{n} s_k} - \rho_j s_j \right) \right]$$

From the first order condition of maximization, we get:

$$a_i = \frac{\beta_i}{2} \sqrt{N - \sum_{k=1}^{n} s_k} \frac{\partial V_i}{\partial s_i}$$

Putting a_i back into HJB equation, we get the following nonlinear partial differential equation (PDE):

$$r_i V_i = q_i s_i - \frac{\beta_i^2}{4} \left(N - \sum_{k=1}^{n} s_k \right) \left(\frac{\partial V_i}{\partial s_i} \right)^2 + \sum_{j=1}^{n} \left(\frac{\beta_j^2}{2} \left(N - \sum_{k=1}^{n} s_k \right) \frac{\partial V_i}{\partial s_j} \frac{\partial V_j}{\partial s_j} - \rho_j s_j \frac{\partial V_i}{\partial s_j} \right)$$

To solve the system of nonlinear partial differential equations Erickson guessed that V_i has the form:

$$V_i = A_i + \sum_{j=1}^{n} B_{ij} s_j$$

Putting V_i back into the PDE system, he obtains a set of linear equations. Then, by solving for A_i, B_{ij}, one can get closed-loop controls. This result comes from the special form of the dynamics. Using HJB equation and 'guessing' the form of V_i is one of the typical ways to solve differential game problems in the literature. Of course, it has limitations in practical use. Further, the above dynamics is based on Vidale-Wolfe model, so the competitors cannot attract sales directly from other competitors, but attract the loss of others. Other previous works are due to Kenneth R. Deal (1979) [4], Roberto Cellini(2001) [2], etc.

2. MODEL DEVELOPMENT

Consider a dynamics where **n**-companies and **one** product are involved. The market managers use controls/advertising to minimize cost. Because the sale will decrease to zero in the long run, managers will not consider market share as their objectives. We use the index $i = 1, 2, \ldots, n$ to represent these **n** companies. The main notations are as follows:

$\mathbf{x}_i(t)$ Market share of company i at time t.

$\mathbf{u}_i(t)$ Control/Advertising of company i at time t.

ρ Natural sale decrease factor without advertising.

κ Effectiveness of advertising on the sales decrease rate.

$\mathbf{a}_i$ Effectiveness of control/advertising of company i.

δ_i Advertising cost parameter for company i.

$\mathbf{p}$ Price of the product.

To describe the dynamics in the final stage of the product life cycle, we consider two effects: one is natural decrease effect; the other is competition. The dynamics is the following:

$$\dot{x}_i = -f(u_1, \dots, u_n)x_i + a_i u_i(1 - x_i) - x_i \sum_{k=1, k\neq i}^{n} a_k u_k, \quad i = 1, \dots, n$$

The first term describes the natural decrease of products. It should satisfy $f(u_1, \dots, u_n) \geq 0$ and $f(0, \dots, 0) = \rho$, which means, if there is no advertising, then the sale will keep its natural decrease rate ρ. Further, we impose the rational assumption: the bigger the advertising, the smaller the decrease rate. Here, in this one-product market, the combination of all advertising effects will affect the decrease rate, so we suppose:

$$f(u_1, \dots, u_n) = \rho \left(1 - \kappa \sum_{i=1}^{n} u_i \right).$$

The second term is based on Lanchester [5] competition model. The change of $\mathbf{x}_i$ is determined by two factors: one is positive effect, which is from the effect of company **i**'s control $\mathbf{u}_i$ on the others' market, the other is negative effect, which is from the effect of competitors' controls $\sum_{\mathbf{j}\neq\mathbf{i}} \mathbf{u}_\mathbf{j}$ on **i**'s market. That is:

$$a_i u_i(1 - x_i) - x_i \sum_{k=1, k\neq i}^{n} a_k u_k$$

Let $G(u_1, u_2, \dots, u_n) = \sum_{i=1}^{n} a_i u_i$, then we get the following dynamics:

$$\begin{bmatrix} \dot{x}_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} a_1 u_1 \\ \vdots \\ a_n u_n \end{bmatrix} - \begin{bmatrix} f+G & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & f+G \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

For performance, manager's performance criteria is minimizing cost and maximizing profit, so the objective functions are of the form:

$$\min_{u_i} J_i = \int_{t_0}^{\infty} e^{-rt} [\frac{\delta_i}{2} u_i^2(t) - p x_i(t)] dt, \quad i = 1, 2, \dots, n$$

where the term $\frac{\delta_i}{2} u_i^2(t) - p x_i(t)$ is net cost at time t, which is obtained by the cost of control/advertising minus income from sale. The term e^{-rt} is discount factor.

3. OPTIMALITY CONDITIONS

3.1. **Competition in Open-loop control case.** First we consider open-loop controls. An open-loop control/advertising $\mathbf{u}(t)$ is just a function of time, and a player or competitor sets up his control/advertising policy at the beginning of the game. From theorem **1**.1, we get following optimal conditions for each competitor.

The Hamiltonian for player **i** is:

$$H_i = e^{-rt}\left[\frac{\delta_i}{2}u_i^2(t) - px_i(t)\right] + \sum_{k=1}^{n} m_{ik}(a_k u_k - (f+G)x_k)$$

Minimizing H_i with respect to u_i:

$$\frac{\partial H_i}{\partial u_i} = e^{-rt}\delta_i u_i + a_i m_{ii} - (a_i - \rho k)\sum_{k=1}^{n} m_{ik}x_k$$

Solving for u_i explicitly:

$$u_i = \frac{e^{rt}}{\delta_i}\left[(a_i - \rho k)\sum_{k=1}^{n} m_{ik}x_k - a_i m_{ii}\right]$$

The costate system for open-loop control comes from

$$\dot{m}_{ij} = -\frac{\partial H_i}{\partial x_j}, \qquad j \neq i$$

$$\dot{m}_{ii} = -\frac{\partial H_i}{\partial x_i}$$

which is

$$\dot{\begin{bmatrix} m_{i1} \\ \vdots \\ m_{ii} \\ \vdots \\ m_{in} \end{bmatrix}} = \begin{bmatrix} f+G & \cdots & 0 \\ 0 & \ddots & 0 \\ \vdots & f+G & \vdots \\ 0 & \ddots & 0 \\ 0 & \cdots & f+G \end{bmatrix}\begin{bmatrix} m_{i1} \\ \vdots \\ m_{ii} \\ \vdots \\ m_{in} \end{bmatrix} + \begin{bmatrix} 0 \\ \vdots \\ pe^{-rt} \\ \vdots \\ 0 \end{bmatrix}$$

The boundary condition for the costate system is:

$$\lim_{T\to\infty}\begin{bmatrix} m_{i1}(T) \\ \vdots \\ m_{ii}(T) \\ \vdots \\ m_{in}(T) \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

3.2. **Competition in Closed-loop control case.** A closed-loop control/advertising $\mathbf{u}_i(t,x)$ is function of state variables. Thus, we have different co-state system in the optimality conditions.

The Hamiltonian for player **i** is:

$$H_i = e^{-rt}\left[\frac{\delta_i}{2}u_i^2(t) - px_i(t)\right] + \sum_{k=1}^{n} m_{ik}(a_k u_k - (f+G)x_k)$$

Minimizing H_i with respect to u_i:

$$\frac{\partial H_i}{\partial u_i} = e^{-rt}\delta_i u_i + a_i m_{ii} - (a_i - \rho k)\sum_{k=1}^{n} m_{ik}x_k$$

Solving for u_i explicitly:

$$u_i = \frac{e^{rt}}{\delta_i}\left[(a_i - \rho k)\sum_{k=1}^{n} m_{ik}x_k - a_i m_{ii}\right]$$

The costate system for closed-loop control comes from:

$$\begin{cases} \dot{m}_{ij} = & -\frac{\partial H_i}{\partial x_j} - \sum_{k=1,k\neq i}^{n} \frac{\partial H_i}{\partial u_k}\frac{\partial u_k}{\partial x_j} \quad \text{for} \quad j=1,\dots,n, \quad j\neq i \\ \dot{m}_{ii} = & -\frac{\partial H_i}{\partial x_i} - \sum_{k=1,k\neq i}^{n} \frac{\partial H_i}{\partial u_k}\frac{\partial u_k}{\partial x_i} \end{cases}$$

where

$$\begin{cases} \frac{\partial H_i}{\partial x_j} = & -m_{ij}(f+G) & \text{for} \quad j=1,\dots,n, \quad j\neq i \\ \frac{\partial H_i}{\partial x_i} = & -m_{ii}(f+G) + pe^{-rt} & \\ \frac{\partial H_i}{\partial u_k} = & m_{ik}a_k - (a_k - \rho\kappa)\sum_{l=1}^{n} m_{il}x_l & \text{for} \quad k=1,\dots,n, \quad k\neq i \\ \frac{\partial u_i}{\partial x_j} = & \frac{e^{rt}}{\delta_i}[(a_i - \rho\kappa)m_{ij}] & \text{for all} \quad i,j \end{cases}$$

The boundary condition for the costate system is:

$$\lim_{T\to\infty}\begin{bmatrix} m_{i1}(T) \\ \vdots \\ m_{ii}(T) \\ \vdots \\ m_{in}(T) \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

3.3. **Nonsymmetric competition.** In any market, the role of competitors cannot be same, and a feedback control is not always more beneficial for a competitor. Thus, in some markets, it is not unusual that different companies adopt different control types. This motivates us to propose nonsymmetric competitions, which means that in a competition, some competitors adopt open-loop controls and others adopt closed-loop controls. In reality, in a market where big and small companies co-exist, small companies always comply with the price set up by big companies. In other words, small companies always 'give' the right of adjusting prices to the bigger companies.

We will set up the differential game model from the same dynamics as the above to explain the reason for this phenomenon.

In the following **n**-player game, we suppose company **l** adopts closed-loop control $\mathbf{u}_l(t, x(t))$ and others adopt open-loop controls $\mathbf{u}_i(t), i \neq l$. This means that at the beginning of the game, player $i, i = 1, \dots, i \neq l$ will choose their control policies. However player l will adjust his control according to the state.

The Hamiltonian for player **i** is:

$$H_i = e^{-rt}\left[\frac{\delta_i}{2}u_i^2(t) - px_i(t)\right] + \sum_{k=1}^{n} m_{ik}(a_k u_k - (f+G)x_k)$$

Minimizing H_i with respect to u_i:

$$\frac{\partial H_i}{\partial u_i} = e^{-rt}\delta_i u_i + a_i m_{ii} - (a_i - \rho k)\sum_{k=1}^{n} m_{ik}x_k$$

Solving for u_i explicitly:

$$u_i = \frac{e^{rt}}{\delta_i}\left[(a_i - \rho k)\sum_{k=1}^{n} m_{ik}x_k - a_i m_{ii}\right]$$

For player $i \neq l$, the costate system is:

$$\begin{cases} \dot{m}_{ij} = & -\frac{\partial H_i}{\partial x_j} - \frac{\partial H_i}{\partial u_l}\frac{\partial u_l}{\partial x_j} \quad \text{for} \quad j = 1, \dots, n, \quad j \neq i \\ \dot{m}_{ii} = & -\frac{\partial H_i}{\partial x_i} - \frac{\partial H_i}{\partial u_l}\frac{\partial u_l}{\partial x_i} \end{cases}$$

For player l, the costate system is:

$$\begin{cases} \dot{m}_{lj} = & -\frac{\partial H_l}{\partial x_j}, \quad j \neq l \\ \dot{m}_{ll} = & -\frac{\partial H_l}{\partial x_l} \end{cases}$$

where

$$\begin{cases} \frac{\partial H_i}{\partial x_j} = & -m_{ij}(f+G) & \text{for} \quad j = 1, \dots, n, \quad j \neq i \\ \frac{\partial H_i}{\partial x_i} = & -m_{ii}(f+G) + pe^{-rt} & \\ \frac{\partial H_i}{\partial u_k} = & m_{ik}a_k - (a_k - \rho\kappa)\sum_{l=1}^{n} m_{il}x_l & \text{for} \quad k = 1, \dots, n, \quad k \neq i \\ \frac{\partial u_i}{\partial x_j} = & \frac{e^{rt}}{\delta_i}[(a_i - \rho\kappa)m_{ij}] & \text{for} \quad all \quad i, j \end{cases}$$

The boundary condition for the costate system is:

$$\lim_{T\to\infty}\begin{bmatrix} m_{i1}(T) \\ \vdots \\ m_{ii}(T) \\ \vdots \\ m_{in}(T) \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

In order to solve our differential game models, we should solve a boundary value problem (BVP) problem, which is composed of $\mathbf{n}$ state equations with initial conditions and $\mathbf{n}^2$ costate equations with terminal conditions. In our experiment, we suppose there are **3** companies selling the same product, that is, $\mathbf{n} = \mathbf{3}$. Thus, we will pick a terminal time $\mathbf{T}$ and increase it gradually.

4. EXISTENCE OF SOLUTION

After solving for control u_i explicitly and substituting back into the state and costate equations, we get the reduced differential equations:

$$\begin{cases} \dot{x}_i(t) & = a(x_i(t), m_{ij}(t), t) \\ \dot{m}_{ij}(t) & = d(x_i(t), m_{ij}(t), t) \end{cases}$$

The boundary condition is:

$$\begin{cases} x_i(0) & = x_{i0} \\ m_{ij}(T) & = 0 \end{cases}$$

The initial value problem (IVP) associated with this BVP is

$$\begin{cases} \dot{x}_i(t) & = a(x_i(t), m_{ij}(t), t) \\ \dot{m}_{ij}(t) & = d(x_i(t), m_{ij}(t), t) \end{cases}$$

The terminal condition is

$$\begin{cases} x_i(T) & = s \\ m_{ij}(T) & = 0 \end{cases}$$

where $\mathbf{s}$ is a parameter column.

The following theorem can guarantee the existence and uniqueness of solution for the above nonlinear initial value problem(IVP).

<u>Theorem 3.1</u> : Suppose that $a(x, m, t), d(x, m, t)$ are continuous on $D = \{(x, m, t) : a \leq t \leq b, |x - \alpha_1| \leq \rho_1, |m - \alpha_2| \leq \rho_2\}$, for some ρ_1, ρ_2, and suppose that $a(x, m, t)$, $d(x, m, t)$ is Lipschitz continuous with respect to x, m respectively, that is

$$\begin{aligned} |a(x, m, t) - a(x', m, t)| & \leq L|x - x'| \\ |d(x, m, t) - d(x', m, t)| & \leq L|m - m'| \end{aligned}$$

for some constant L and any $(x, m, t), (x', m', t)$ in D. If $|a(x, m, t)| \leq M, |d(x, m, t)| \leq M$ on D, and $c = \min\{b - a, \frac{\rho_1}{M}, \frac{\rho_2}{M}\}$, then the above IVP has a unique solution for $a \leq x \leq a + c$. $\square$

The following is explicit formula for state and costate equations in open-loop differential game.

$$\begin{cases} \dot{x}_i = & -x_i^2 e^{rt} \sum_{k=1}^{n} \frac{(a_k-\rho\kappa)^2}{\delta_k} m_{ki} \\ & +x_i \Big[-\rho + e^{rt} \frac{a_i(a_i-\rho\kappa)}{\delta_i} m_{ii} + e^{rt} \sum_{k=1}^{n} \frac{a_k(a_k-\rho\kappa)}{\delta_k} m_{kk} \\ & \quad -e^{rt} \sum_{j=1, j\neq i}^{n} x_j \sum_{k=1}^{n} \frac{(a_k-\rho\kappa)^2}{\delta_k} m_{kj} \Big] \\ & +\frac{e^{rt} a_i}{\delta_i} \Big[(a_i-\rho\kappa) \sum_{j=1, j\neq i}^{n} m_{ij} x_j - m_{ii} - a_i \Big] \\ \\ \dot{m}_{ij} = & m_{ij}^2 e^{rt} \frac{(a_i-\rho\kappa)^2}{\delta_i} x_j \\ & +m_{ij} \Big[\rho + e^{rt} \sum_{k=1}^{n} \sum_{l=1, (k\neq i \ and \ l\neq j)}^{n} \frac{(a_k-\rho\kappa)^2}{\delta_k} m_{kl} x_l \\ & \quad -e^{rt} \sum_{k=1}^{n} \frac{a_k(a_k-\rho\kappa)}{\delta_k} m_{kk} \Big] \\ \\ \dot{m}_{ii} = & m_{ii}^2 e^{rt} \frac{(a_i-\rho\kappa)}{\delta_i} [x_i(a_i-\rho\kappa) - a_i] \\ & +m_{ii} \Big[\rho + e^{rt} \sum_{k=1}^{n} \sum_{l=1, (k\neq i \ and \ l\neq i)}^{n} \frac{(a_k-\rho\kappa)^2}{\delta_k} m_{kl} x_l \\ & \quad -e^{rt} \sum_{k=1, k\neq i}^{n} \frac{a_k(a_k-\rho\kappa)}{\delta_k} m_{kk} \Big] + pe^{-rt} \end{cases}$$

Then, in the open-loop case, $a(x,m,t), d(x,m,t)$ are continuous in $[0,T]$, then uniformly continuous, which means they satisfy Lipschitz continuous condition, and are bounded, so we can guarantee the existence of local solution for the open-loop case. And we can draw same conclusion in the other two cases.

So, under the condition of *Theorem 3.1*, for each $s \in R^n$ there is a unique solution of the above IVP, which is denoted by $x_i(t;s)$, $m_{ij}(t;s)$, then there is a unique $x_i(0;s)$. So we can say there exist a functional relationship between $x_i(T)$ and $x_i(0)$, that is, $x_i(0) = f_i(x_i(T))$, for which we cannot, in general, find analytic formula. Now suppose $x_i(t;s^*)$, $m_{ij}(t;s^*)$ are solved from IVP by $s = s^*$, then $x_i(t;s^*)$, $m_{ij}(t;s^*)$ satisfy the ordinary differential equation (ODE) in the above boundary value problem(BVP). So if s^* is such that the boundary conditions are satisfied, that is $x_i(0;s^*) = x_{i0}$ or $f_i(x_i(T;s^*)) - x_{i0} = 0$, then $x_i(t;s^*)$ is the solution of the BVP. We state this as the following theorem.

Theorem 3.2 : Suppose that $a(x,m,t)$, $d(x,m,t)$ are continuous on $D = \{(x,m,t) : a \le t \le b, |x| < \infty, |m| < \infty\}$ and satisfy uniform Lipschitz condition in x, m, then the above BVP has as many solutions as the number of distinct roots s^* of $f_i(x_i(T;s^*)) - x_{i0} = 0$, $i = 1, \ldots, n$. And a solution of the BVP is given by

$$\begin{aligned} x_i(t) &= x_i(t;s^*) \\ m_{ij}(t) &= m_{ij}(t;s^*) \end{aligned}$$

□

So, if we can find s^* satisfying $f_i(x_i(T;s^*)) - x_{i0} = 0, i = 1, \ldots, n$, then, we can solve the above BVP, then obtain u_i and get the solution for the differential game model. The next section will give an algorithm to find the solution to our differential game model based on the analysis of existence.

5. ALGORITHM

In order to get optimal controls, we should find solution for the following differential equations.

$$\begin{cases} \dot{x}_i(t) & = a(x_i(t), m_{ij}(t), t) \\ \dot{m}_{ij}(t) & = d(x_i(t), m_{ij}(t), t) \end{cases}$$

The boundary condition is:

$$\begin{cases} x_i(0) & = x_{i0} \\ m_{ij}(T) & = 0 \end{cases}$$

From the analysis in the previous subsection, we guess a value $x_i(T)^{(0)}$, then we use $(x_i(T)^{(0)}, m_{ij}(T))$ to solve the above differential equations backward in time and get $x_i(0)^{(0)}$. Now, we suppose $x_i(0)$ and $x_i(T)$ have the relationship:

$$\begin{cases} x_1(0) & = f_1(x_1(T), \ldots, x_n(T)) \\ \cdots & \cdots \\ x_i(0) & = f_i(x_1(T), \ldots, x_n(T)) \\ \cdots & \cdots \\ x_n(0) & = f_n(x_1(T), \ldots, x_n(T)) \end{cases}$$

Unfortunately, we cannot find an analytical expression for these functions. However we can use our algorithm to adjust systematically $x_i(T)$ according to the observed $x_i(0)$ to find $x_i(T)$ to satisfy the following equations:

$$\begin{cases} F_1(x_1(T), \ldots, x_n(T)) = & f_1(x_1(T), \ldots, x_n(T)) - x_1(0) = 0 \\ \cdots & \cdots \\ F_i(x_1(T), \ldots, x_n(T)) = & f_i(x_1(T), \ldots, x_n(T)) - x_i(0) = 0 \\ \cdots & \cdots \\ F_n(x_1(T), \ldots, x_n(T)) = & f_n(x_1(T), \ldots, x_n(T)) - x_n(0) = 0 \end{cases}$$

Our algorithm is based on Newton's method for solving nonlinear equations. As to the above functions F_i, using Taylor series expansion up to first-order about some estimate point $x_1(T)^{(k)}, \ldots, x_n(T)^{(k)}$, we have the system of equations:

$$\begin{bmatrix} F_1(x_1(T), \ldots, x_n(T)) \\ \cdots \\ F_i(x_1(T), \ldots, x_n(T)) \\ \cdots \\ F_n(x_1(T), \ldots, x_n(T)) \end{bmatrix} = \begin{bmatrix} F_1(x_1(T)^{(k)}, \ldots, x_n(T)^{(k)}) \\ \cdots \\ F_i(x_1(T)^{(k)}, \ldots, x_n(T)^{(k)}) \\ \cdots \\ F_n(x_1(T)^{(k)}, \ldots, x_n(T)^{(k)}) \end{bmatrix}$$

$$+\begin{bmatrix} \frac{\partial F_1}{\partial x_1(T)} & \cdots & \frac{\partial F_1}{\partial x_n(T)} \\ \cdots & \cdots & \cdots \\ \frac{\partial F_i}{\partial x_1(T)} & \cdots & \frac{\partial F_i}{\partial x_n(T)} \\ \cdots & \cdots & \cdots \\ \frac{\partial F_n}{\partial x_1(T)} & \cdots & \frac{\partial F_n}{\partial x_n(T)} \end{bmatrix}_{|(x_1(T)^{(k)}),\ldots,x_n(T)^{(k)})} * \begin{bmatrix} x_1(T) - x_1(T)^{(k)} \\ \cdots \\ x_i(T) - x_i(T)^{(k)} \\ \cdots \\ x_n(T) - x_n(T)^{(k)} \end{bmatrix} = \begin{bmatrix} 0 \\ \cdots \\ 0 \\ \cdots \\ 0 \end{bmatrix},$$

which we solve for $(x_1(T), \ldots, x_n(T))$ to get the updated vector estimate:

$$\begin{bmatrix} x_1(T)^{(k+1)} \\ \cdots \\ x_i(T)^{(k+1)} \\ \cdots \\ x_n(T)^{(k+1)}) \end{bmatrix} = \begin{bmatrix} x_1(T)^{(k)} \\ \cdots \\ x_i(T)^{(k)} \\ \cdots \\ x_n(T)^{(k)}) \end{bmatrix}$$

$$-\begin{bmatrix} \frac{\partial F_1}{\partial x_1(T)} & \cdots & \frac{\partial F_1}{\partial x_n(T)} \\ \cdots & \cdots & \cdots \\ \frac{\partial F_i}{\partial x_1(T)} & \cdots & \frac{\partial F_i}{\partial x_n(T)} \\ \cdots & \cdots & \cdots \\ \frac{\partial F_n}{\partial x_1(T)} & \cdots & \frac{\partial F_n}{\partial x_n(T)} \end{bmatrix}_{|(x_1(T)^{(k)}),\ldots,x_n(T)^{(k)})} * \begin{bmatrix} F_1(x_1(T)^{(k)}, \ldots, x_n(T)^{(k)}) \\ \cdots \\ F_i(x_1(T)^{(k)}, \ldots, x_n(T)^{(k)}) \\ \cdots \\ F_n(x_1(T)^{(k)}, \ldots, x_n(T)^{(k)}) \end{bmatrix},$$

that is,

$$\mathbf{X}^{(k+1)}(T) = X^{(k)}(T) - J_k^{-1} F(X^{(k)}(T))$$

In using Newton's method, we have the following issues:

1. How do we get initial guess $x_i(T)^{(0)}$? Only when $|x_i(T)^{(0)} - x_i^*(T)|$ is small enough, is there quadratic convergence.
2. How to get the *Jacobian* matrix of F. Although we know that the Jacobian matrix exists, we do not have analytic form for it. This is the case if the system is complex.

For the initial guess, a general way is to use steepest descent algorithm (*Algorithm* 1.1) to solve the differential game first. Since steepest descent is a first order convergence algorithm, it will converge slowly. However, it will give us a good hint for the initial guess of $x_i(T)$. In our differential game model of the final stage of product life cycle, we can expect that sales will decrease further and further to zero. Thus, we can use some positive small values as the initial guesses.

To obtain the *Jacobian Matrix*, the following recipe is one way to approximate it in each iteration:

$$J_k = \begin{bmatrix} \frac{\partial F_1}{\partial x_1} & \cdots & \frac{\partial F_1}{\partial x_n} \\ \cdots & \cdots & \cdots \\ \frac{\partial F_n}{\partial x_1} & \cdots & \frac{\partial F_n}{\partial x_n} \end{bmatrix},$$

where $\frac{\partial F_i}{\partial x_j} \simeq \frac{F_i(\mathbf{x(k)}) - F_i(\mathbf{x(k-1)})}{x_j(k) - x_j(k-1)}$. However, this approach does not work in solving our model's optimality condition because of singularity or near singularity of the

Jacobian. In order to prevent singular situation, at each iteration we calculate each $\frac{\partial F_i}{\partial x_j}$ in the following way. First, randomly generate a $\Delta x_j(T)$. Second, solve the ODE backward using the randomly perturbed $x_j(T)^{(k)}+\Delta x_j(T)$ and keeping $x_i(T)^{(k)}, i \neq j$ unchanged. Third, calculate $\Delta F_i, i = 1, \ldots, n$. Finally, calculate $\frac{\Delta F_i}{\Delta x_j(T)}, i = 1, \ldots, n$, which is our approximation of $\frac{\partial F_i}{\partial x_j}$. The following algorithm will be set up using this process.

Algorithm :

1. Guessing $x_i(T)^{(0)}, i = 1, \ldots, n$, and using the result from steepest descent algorithm.
2. Solving ODE backward using $RK4$ by $x_i(T)^{(0)}, m_{ij}(T)$
3. For $i = 1, \ldots, n$
   generate $\Delta x_i(T)$ randomly from Uniform Distribution in $(-\epsilon, +\epsilon)$.
   Use $[x_1(T)^{(0)}, \ldots, x_i(T)^{(0)}+\Delta x_i(T), \ldots, x_n(T)^{(0)}, m_{ij}(T)]$ to solve the ODE backward.
   Calculate ΔF_j, and $\frac{\Delta F_j}{\Delta x_i(T)}$, for $j = 1, \ldots, n$
   End. Then, proceed to get J_0
4. Solve $J_0 \cdot y_0 = F(X^0(T))$ for y_0.
5. Update $x_i(T)^{(0)}$ by $X^1(T) = X^0(T) - y_0$, and let $k = 1$
6. While $\| F_j \| > \varepsilon$ for $j = 1, \ldots, n$ and $k \leq MaxIteration$, proceed as follows
   For $i = 1, \ldots, n$
   Generating $\Delta x_i(T)$ randomly from Uniform Distribution in $(-\epsilon, +\epsilon)$.
   Using $[x_1(T)^{(k)}, \ldots, x_i(T)^{(k)} + \Delta x_i(T), \ldots, x_n(T)^{(k)}, m_{ij}(T)]$ to solve ODE backward.
   Calculate ΔF_j, and $\frac{\Delta F_j}{\Delta x_i(T)}$, for $j = 1, \ldots, n$
   End. Then, get J_k.
   Solve $J_k \cdot y_k = F(X^k(T))$ for y_k
   Update $x_i(T)^{(k)}$ by $X^{k+1}(T) = X^k(T) - y_k$
   Using $[x_1(T)^{(k+1)}, \ldots, x_i(T)^{(k+1)}, \ldots, x_n(T)^{(k+1)}, m_{ij}(T)]$ to solve ODE backward.
   Calculate $F_j(X(T)^{(k+1)})$
   End of While statement.
7. Calculate $u_i(t)$ and J_i

6. NUMERICAL RESULTS

In our experiment, we take $n = 3$, which means that there are three companies. The values of the coefficients are as follows:

$$\begin{array}{lll} \delta_1 = 20, & \delta_2 = 18, & \delta_3 = 21 \\ a_1 = 0.008, & a_2 = 0.010, & \boldsymbol{a}_3 = 0.007 \\ \kappa = 0.01, & \rho = 0.6 & p = 5 \\ r = 0.1 & & \\ x_1(0) = 0.3, & x_2(0) = 0.5, & x_3(0) = 0.2 \end{array}$$

This data set means that company 2 is relatively bigger than company 1, and company 1 is relatively bigger than company 3. The qualifier 'bigger' here implies that a bigger company's advertising cost is lower: $\delta_2 < \delta_1 < \delta_3$; and a bigger company's effectiveness of advertising is larger: $a_2 > a_1 > a_3$, etc.

In our model, the objective function is $\min_{u_i} J_i = \int_{t_0}^{\infty} e^{-rt}[\frac{\delta_i}{2}u_i^2(t) - px_i(t)]dt$. So this is infinite time horizon problem. In our experiment, we will take the time interval as $[0, T]$. From the results of the experiment, we can see the convergence of controls, state, and objective value as T goes to infinity. Thus we can infer the policies from finite time interval.

6.1. Finite time interval.

In our experiment we will take $[0, T]$ as the time interval. Then, we will discuss the results considering three different cases: open-loop, closed-loop, and nonsymmetric differential game. In the nonsymmetric differential game, we will suppose that company 2 adopts closed-loop control, and the other two companies adopt open-loop controls.

Open-loop case:

	$T = 5$	$T = 10$	$T = 20$	$T = 25$
$J1$	−2.07726113374539	−2.13914327540115	−2.13921481085360	−2.13822037010982
$J2$	−3.46211263644663	−3.56526554476421	−3.56540121159257	−3.56374755448441
$J3$	−1.38486455061329	−1.42613692152929	−1.42619951910137	−1.42553988421163

TABLE 1. Open-loop

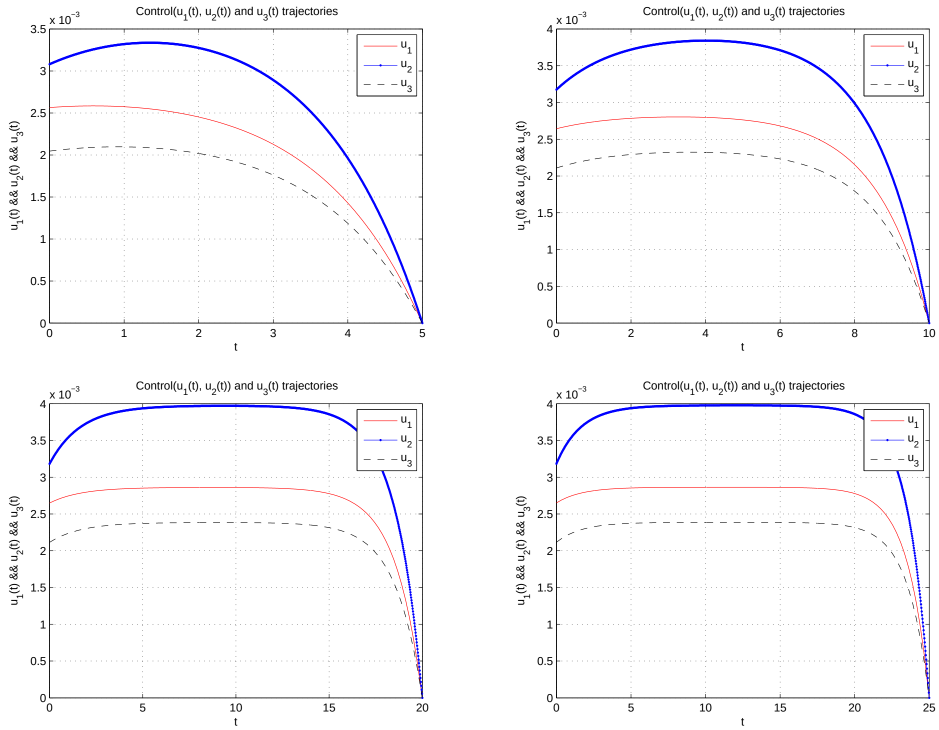

FIGURE 1. Open-loop: Control trajectories

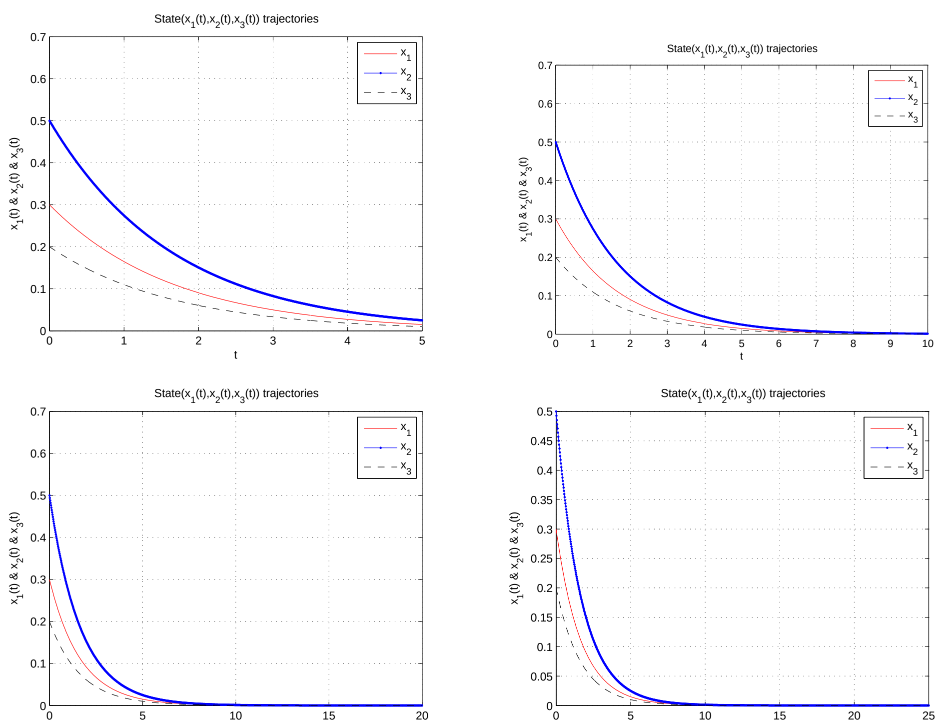

FIGURE 2. Open-loop: State trajectories

Closed-loop case:

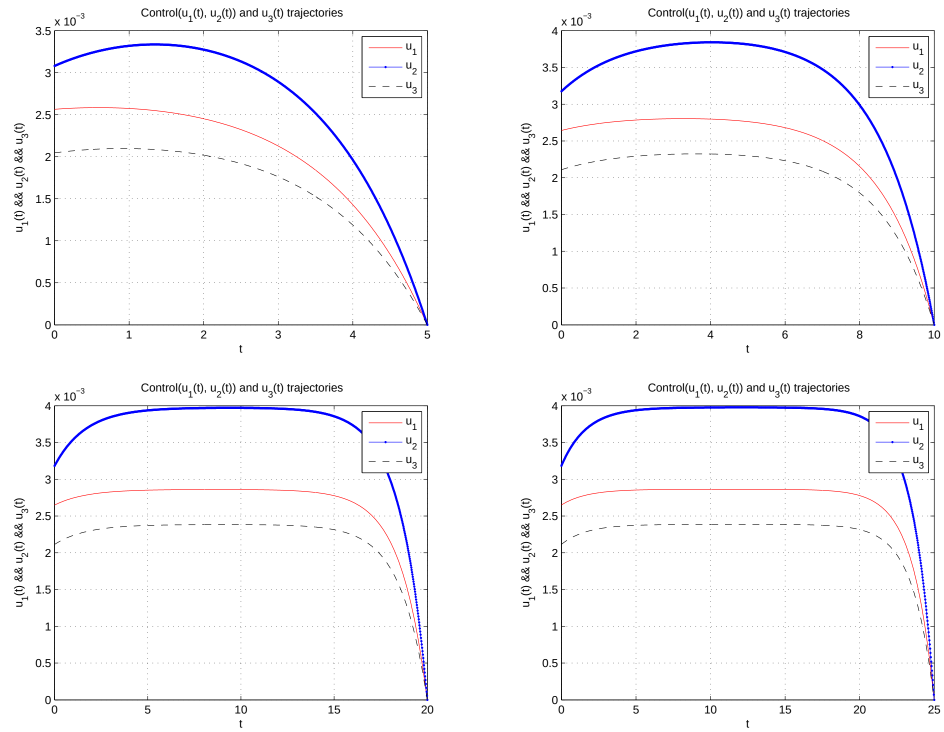

FIGURE 3. Closed-loop: Control trajectories

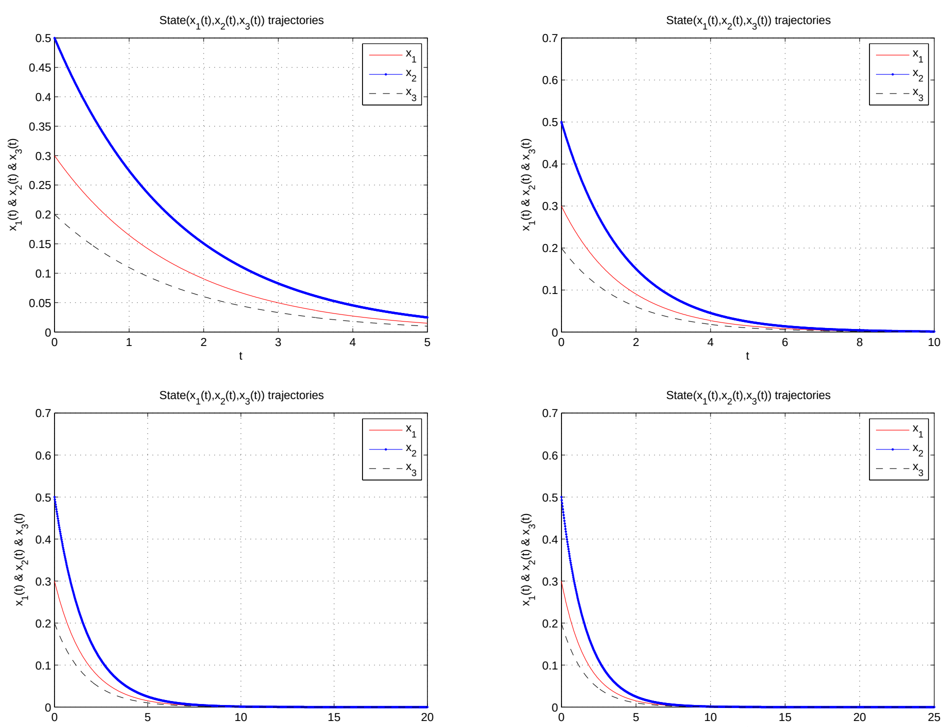

FIGURE 4. Closed-loop: State trajectories

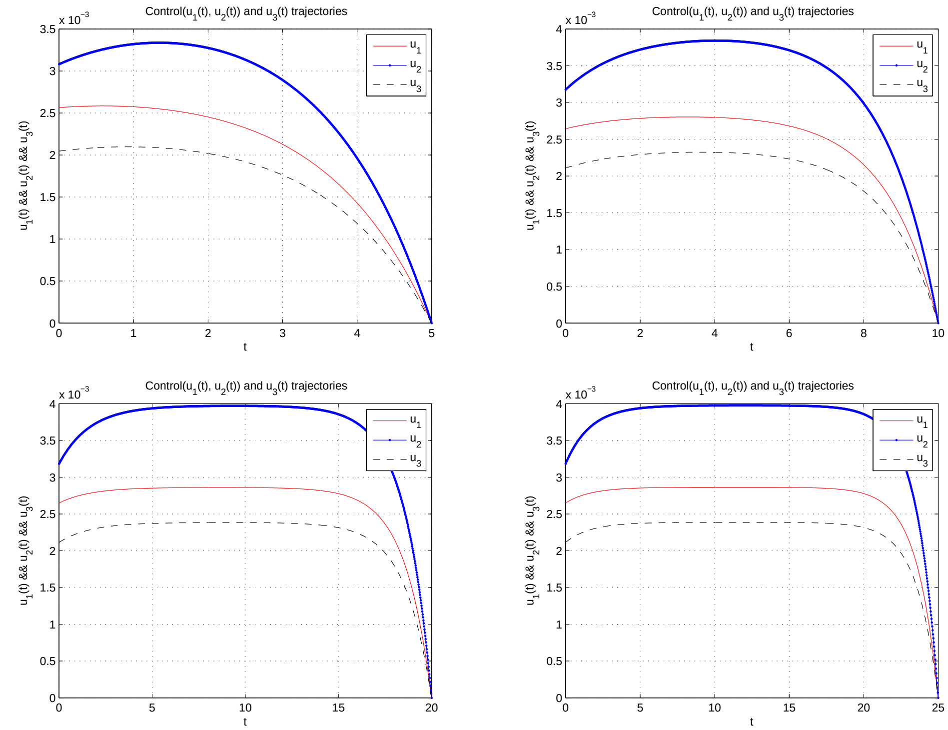

FIGURE 5. Nonsymmetry: Control trajectories

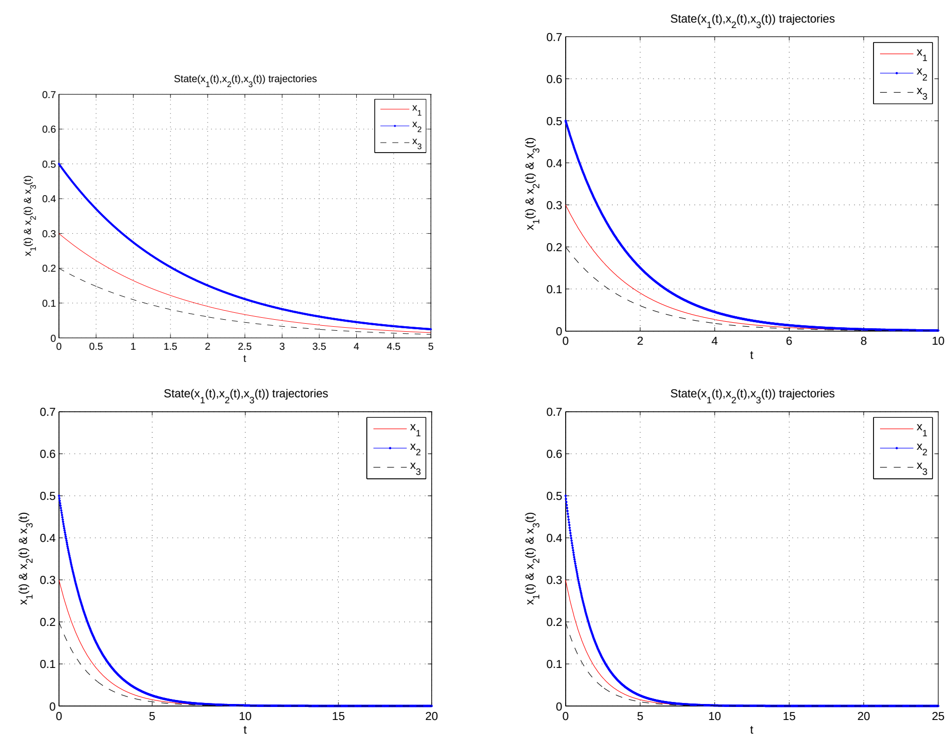

FIGURE 6. Nonsymmetry: State trajectories

	$T=5$	$T=10$	$T=20$	$T=25$
$J1$	-2.07726113374539	-2.13914327540240	-2.13921481085491	-2.13822037011187
$J2$	-3.46211263644812	-3.56526554476421	-3.56540121159449	-3.56374755448648
$J3$	-1.38486455062579	-1.42613692154521	-1.42619951911743	-1.42553988422778

Table 2. Closed-loop

	$T=5$	$T=10$	$T=20$	$T=25$
$J1$	-2.07726113374539	-2.13914327540132	-2.13921481085382	-2.13822037011061
$J2$	-3.46211263644809	-3.56526554476421	-3.56540121159438	-3.56374755448636
$J3$	-1.38486455062385	-1.42613692154275	-1.42619951911494	-1.42553988422526

Table 3. Nonsymmetric

Nonsymmetric case:

In the open-loop case, we can see that

$$x_1(T) = 0$$
$$x_2(T) = 0$$
$$x_3(T) = 0$$

when $T = 25$. From *Figure 1, Table 1* and numerical results, we can see that most of the time (excepting for the beginning and the end), control for each company is a constant:

$$u_1(t) = 0.0029$$
$$u_2(t) = 0.0040$$
$$u_3(t) = 0.0024$$

The objective function value is

$$J_1 = -2.13720767706292$$
$$J_2 = -3.56206198522349$$
$$J_3 = -1.42486673927475$$

Thus, we can conclude that in the open-loop differential game model, the optimal adverting policy for each company is increasing advertising at the beginning, which can prevent product sales from decreasing fast, then keeping at some constant level, and when the product is clear out of the market, decreasing advertising to zero at that instant.

In the closed-loop and nonsymmetric control cases, controls and state haves the same limit as that of open-loop case, and objective function values have almost the same values as those of the open-loop case. We will discuss the difference among these three cases in the following subsection. Here, in all three cases, these three companies will 'cooperate'. This cooperation is automatic and not negotiated at the beginning of the competition. That is, the companies advertise to make the sale decreasing speed slower. And then the companies will keep their controls at some constant level according to the controls' effectiveness. It is not hard to imagine that people will migrate to company 1 from others, and the sales of company 1 is the last one to reach zero.

6.2. **Comparison of Controls.** In Medhin and Wan [9], we have shown that for any competitors, closed-loop control in differential game is always smaller than open-loop control. Now, in the differential game model of this paper, we get the same conclusion about the relation of closed- and open-loop controls. We can see this result from *Figure 7*. We can also observe that for a bigger company the difference between closed-loop and open-loop controls is smaller. This means when all competitors have opportunity to exploit the state information, the weaker competitor should pay more for it. Optimal objective function value for closed-loop control setting is smaller than the open-loop for any competitor, which means, closed-loop control is better than open-loop control. We can see this result from *Tables 4, 5, 6*. And we also observe that the bigger company's value gain more with the closed-loop control.

Regarding the relation between closed-loop and nonsymmetric control, from *Figure 8*, we can see that closed-loop control is always smaller than nonsymmetric control, which holds true for all competitors. From *Tables 4, 5, 6*, we can see that the optimal value of closed-loop control is also better than nonsymmetric control. However, compared with closed-loop setting, company 2 uses much smaller control than the other two companies, and the change of optimal objective value is also smaller than the others. This is because in our nonsymmetric differential game model, company 2 uses closed-loop control, and the others use open-loop controls. Thus the status of company 2 is superior to the others.

Regarding the relation between nonsymmetric and open-loop control, from *Tables 4, 5, 6* we can see that for every competitor, the optimal value in the nonsymmetric case is better than that in the open-loop case. From *Figure 9* we can see that, in the case of companies 1 and 2, which adopt open-loop controls in nonsymmetric case, their controls in nonsymmetric case are smaller than that of open-loop case. While company 2 adopts closed-loop control at the beginning of the game, he uses a bigger control than in the case of open-loop to exploit the state information, making people migrate to his product. However, when he finds that the total sale of the product is approaching to zero, he decreases his control sharply, resulting in a much smaller control than that initial the case of open-loop control. A company's behavior says that he can completely use state information to adjust his control. Further, we find that company 1 and 3 will get more than what they get in the open-loop game, and they will not be hurt by the closed-loop type control of company 2 in nonsymmetric case, which means if company 2 wants to adopt closed-loop control to hurt others in an open-loop type game, he cannot achieve his goal. This is because, thinking from reality, the other companies can 'follow' company 2's control to adjust their own controls and get more than in the open-loop case. Thus, this result gives us hint in the n-player differential game: in some situation where the state information is hard to get or costly, the 'weaker' competitors can follow the steps of 'stronger' competitors, which can afford to pay for the state information and adopt closed-loop controls.

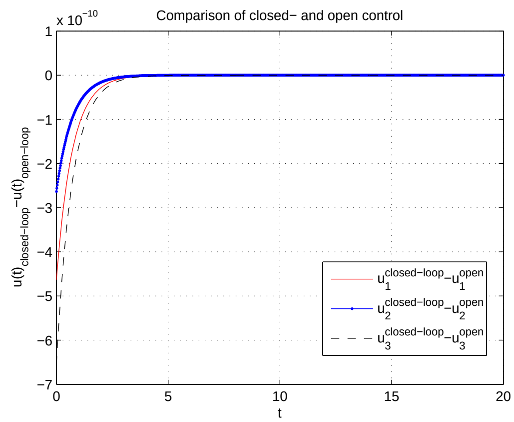

FIGURE 7. Comparison of Closed- and Open-loop control

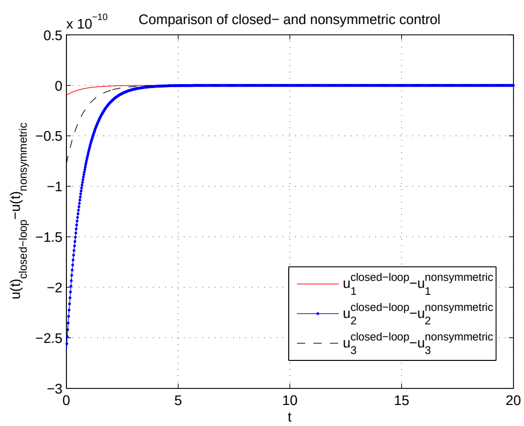

FIGURE 8. Comparison of Closed- and Nonsymmetric control

	$T = 5$	$T = 10$	$T = 20$	$T = 25$
$J_1^{Closed-loop} - J_1^{Nonsymmetry}$	$-8.8018e^{-013}$	$-1.0796e^{-012}$	$-1.0898e^{-012}$	$-1.2599e^{-012}$
$J_1^{Nonsymmetry} - J_1^{Open-loop}$	$-1.3989e^{-013}$	$-1.7009e^{-013}$	$-2.1982e^{-013}$	$-7.9003e^{-013}$

TABLE 4. Difference of J_1

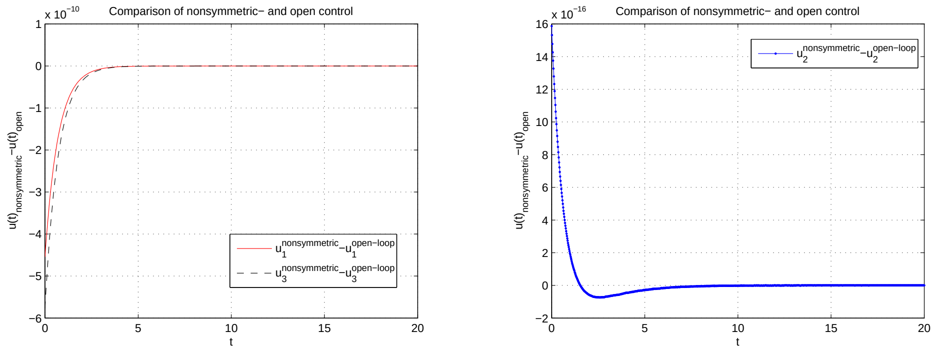

FIGURE 9. Comparison of Nonsymmetric and Open-loop control

	$T = 5$	$T = 10$	$T = 20$	$T = 25$
$J_2^{Closed-loop} - J_2^{Nonsymmetry}$	$-2.9754e^{-014}$	$-8.0380e^{-014}$	$-1.1013e^{-013}$	$-1.1990e^{-013}$
$J_2^{Nonsymmetry} - J_2^{Open-loop}$	$-1.4602e^{-012}$	$-1.7399e^{-012}$	$-1.8097e^{-012}$	$-1.9500e^{-012}$

TABLE 5. Difference of J_2

	$T = 5$	$T = 10$	$T = 20$	$T = 25$
$J_3^{Closed-loop} - J_3^{Nonsymmetry}$	$-1.9400e^{-012}$	$-2.4600e^{-012}$	$-2.4900e - 012$	$-2.5200e^{-012}$
$J_3^{Nonsymmetry} - J_3^{Open-loop}$	$-1.0560e^{-011}$	$-1.3460e^{-011}$	$-1.3570e - 011$	$-1.3630e^{-011}$

TABLE 6. Difference of J_3

6.3. **Effect of change of coefficients.** In this subsection, we will discuss the effect of change of coefficients on the optimal value. The coefficients of interest are

$$\begin{array}{lll} \delta_1, & \delta_2, & \delta_3 \\ a_1, & a_2, & \boldsymbol{a}_3 \\ \rho, & p, & r \end{array}$$

We want to figure out their effects on the optimal value, that is: $\frac{\partial J_i}{\partial r}, \frac{\partial J_i}{\partial p}, \frac{\partial J_i}{\partial \rho}, \frac{\partial J_i}{\partial \boldsymbol{a}_j}, \frac{\partial J_i}{\partial \delta_j}$. By keeping other coefficients constant, we calculate $\frac{\Delta J_i}{\Delta r}$, etc, and get the results summarized in *Tables 7, 8, 9* for open-, closed- and nonsymmetric cases respectively.

1. r: interest rate. The increase of interest rate will increase the optimal cost of each company. It has the biggest effect on the bigger company 2. Because company 2's sale is largest, the current income will decrease most as r increases.
2. ρ: natural decrease of sale. The positive relation of J_i and ρ is reasonable, because the more rapid the decrease rate, the less the income from sale. And company 2 has biggest change for the same reason as above.
3. p: the price of product. The higher the price is, the more income the companies will get, so J_i has negative relation to p. Company 2 has biggest $\frac{\partial J_i}{\partial p}$ because of the same reason as in the case of r.

4. a_i: the effectiveness of control/advertising. When company i's effectiveness of advertising increases, which means that same amount of advertising will bring more customers in, thus $\frac{\partial J_i}{\partial a_i}$ is negative, and others are positive. The absolute of $\frac{\partial J_2}{\partial a_2}$ is largest because its status in the competition will bring in more customers when the effectiveness of the company's control increases. Status here means that the company already has more customers, and lower control cost.
5. δ_i: the cost of control/advertsing. It is natural that the higher cost of advertising, the higher optimal objective value. We can see that the absolute value of $\frac{\partial J_2}{\partial \delta_2}$ is largest. In these three companies, company 2 has advantages, one of which is lower advertising cost, so if his advertising cost increases, then he will lose his advantage, which will hurt him more than the other companies. On the other hand, if his advertising cost decreases, which means that his advantage has been strengthened, thus his cost will decrease more than the others.

Other observations are:

1. Interest rate r has the biggest effect on the objective value J_i than the other coefficients. The companies cannot affect the value of r, but they should know this factor.
2. Following r, price p is the second important coefficient. If price does not affect the demand of the product, companies will 'cooperate' to increase the price to benefit them all.
3. Although the values of coefficients will affect the objective values, these values cannot affect the quality of optimal controls, which mean that the optimal policy is still increasing control at the beginning of competition, and keeping control at some level and decrease to zero at the instant the sale becomes zero.

	$\frac{\Delta J_i}{\Delta r}$	$\frac{\Delta J_i}{\Delta \rho}$	$\frac{\Delta J_i}{\Delta p}$	$\frac{\Delta J_i}{\Delta a_1}$	$\frac{\Delta J_i}{\Delta a_2}$	$\frac{\Delta J_i}{\Delta a_3}$	$\frac{\Delta J_i}{\Delta \delta_1}$	$\frac{\Delta J_i}{\Delta \delta_2}$	$\frac{\Delta J_i}{\Delta \delta_3}$
Company 1	3.19	2.85	-0.427	-0.158	0.014	0.008	$3.2e^{-005}$	$-2.3e^{-006}$	$-3.3e^{-007}$
Company 2	5.32	4.75	-0.713	0.017	-0.208	0.013	$-1.3e^{-006}$	$6.0e^{-005}$	$-5.6e^{-007}$
Company 3	2.13	1.90	-0.285	0.006	0.005	-0.133	$-5.5e^{-007}$	$-1.5e^{-006}$	$2.3e^{-005}$

TABLE 7. Change of coefficients(Open-loop case)

	$\frac{\Delta J_i}{\Delta r}$	$\frac{\Delta J_i}{\Delta \rho}$	$\frac{\Delta J_i}{\Delta p}$	$\frac{\Delta J_i}{\Delta a_1}$	$\frac{\Delta J_i}{\Delta a_2}$	$\frac{\Delta J_i}{\Delta a_3}$	$\frac{\Delta J_i}{\Delta \delta_1}$	$\frac{\Delta J_i}{\Delta \delta_2}$	$\frac{\Delta J_i}{\Delta \delta_3}$
Company 1	3.19	3.01	-0.427	-0.158	0.014	0.008	$3.2e^{-005}$	$-2.3e^{-006}$	$-3.3e^{-007}$
Company 2	5.32	5.02	-0.713	0.017	-0.208	0.013	$-1.3e^{-006}$	$6.0e^{-005}$	$-5.6e^{-007}$
Company 3	2.13	2.00	-0.285	0.006	0.005	-0.133	$-5.5e^{-007}$	$-1.5e^{-006}$	$2.3e^{-005}$

TABLE 8. Change of coefficients(Closed-loop case)

	$\frac{\Delta J_i}{\Delta r}$	$\frac{\Delta J_i}{\Delta \rho}$	$\frac{\Delta J_i}{\Delta p}$	$\frac{\Delta J_i}{\Delta a_1}$	$\frac{\Delta J_i}{\Delta a_2}$	$\frac{\Delta J_i}{\Delta a_3}$	$\frac{\Delta J_i}{\Delta \delta_1}$	$\frac{\Delta J_i}{\Delta \delta_2}$	$\frac{\Delta J_i}{\Delta \delta_3}$
Company 1	3.19	3.01	-0.427	-0.158	0.014	0.008	$3.2e^{-005}$	$-2.3e^{-006}$	$-3.3e^{-007}$
Company 2	5.32	5.02	-0.713	0.017	-0.208	0.013	$-1.3e^{-006}$	$6.0e^{-005}$	$-5.6e^{-007}$
Company 3	2.13	2.00	-0.285	0.006	0.005	-0.133	$-5.5e^{-007}$	$-1.5e^{-006}$	$2.3e^{-005}$

TABLE 9. Change of coefficients(Nonsymmetric-loop case)

7. CONCLUSION

In this paper we set up general n-person differential game model for competition in the last stage of product life cycle. Besides open- and closed-loop differential games, we introduced another type of game, nonsymmetric differential game, in which some competitors adopt open-loop controls and others adopt closed-loop controls. We set up Pontryagin optimal conditions for our differential game model, which is a Two Point Boundary Value Problem(TPBVP). In our algorithm, we used random perturbation technique to avoid singular *Jacobian* matrix. This technique is useful to solve a nonlinear BVP. From our numerical results we find that the closed-loop controls are better than open-loop controls, as has been stated in Medhin and Wan [9]. In closed-loop differential game, all competitors have better objective values, and smaller controls. Thus, if possible, closed-loop controls are a better choice for the competitors. However, in reality, the status of competitors are not symmetric. Only bigger companies can pay for the state information, which means that some competitors cannot get the value of the state variables. By comparing open-loop and nonsymmetric differential games, we find that all competitors' objective value will be better in nonsymmetric control games than in the open-loop games. In this case adopting an open-loop control is not harmful. In reality, when a company cannot get instant value of the state variable, following the steps of bigger company is a good choice. In another paper we will set up 'Leader-follower' type differential game model to get deeper understanding of nonsymmetric status in the marketing competition.

To deal with the type of problem we treated here, Erickson guesses the form of objective function. In a general differential game model, it is hard to guess the form of the objective function. Thus adopting more general approach based on Pontryagin optimal conditions, and the random perturbation technique proposed her is a strong tool.

REFERENCES

[1] Frank M. Bass, Anand Krishnamoorthy, Ashutosh Prasad, and Suresh P. Sethi *Generic and Brand Advertising Strategies in a Dynamic Duopoly*, Marketing Science, 2005, Vol. 24, No. 4,Fall, pp. 556–568.

[2] Roberto Cellini, Luca Lambertini, *Advertising in a Differential Oligopoly Game*, 2001.

[3] James W. Demmel, *Applied Numerical Linear Algebra*, SIAM, 1997.

[4] Kenneth R. Deal, *Optimization Advertising Expenditures in a Dyanmic Duopoly*, Operations Research, Vol. 27, No. 4, July-Aug. 1979, pp. 682–692.

[5] Gary M. Erickson, *Dynamic Models of Advertising Competition*, 2003, Second Edtion, Kluwer Academic Publishers, Boston/Dordrecht/London.

[6] Gary M. Erickson, *An Oligopoly of Dynamic Advertising Competition*, 2007, To be published.

[7] E. J. Hinch, *Perturbation Methods*, Cambridge University Press , 1991.

[8] Mark H. Holmes , *Introduction to Perturbation Methods* , Springer, 1998.

[9] Medhin, N. G., Wan, Wei, *Marketing Competition in the Middle of Product-Life Cycle: A Differential Game Analysis*, 2007.

[10] Ali H. Nayfeh, *Perturbation Methods*, Wiley-Interscience , 2000