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EIGENVALUE COMPARISONS FOR BOUNDARY VALUE PROBLEMS OF THE DISCRETE ELLIPTIC EQUATION

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ABSTRACT. In this paper we study a boundary value problem for a discrete elliptic equation. The focus will be on the structure of the spectrum of this problem and the existence of a positive eigenvector corresponding to the smallest eigenvalue. Comparison results for the eigenvalues are also established as the coefficients of the problem changes.

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1. INTRODUCTION

We consider the Dirichlet boundary value problem for the elliptic differential equation in the rectangle $[0, m+1] \times [0, n+1]$

$$\begin{aligned} u_{xx} + u_{yy} + \lambda a(x,y)u(x,y) &= 0, \quad 0 < x < m+1, 0 < y < n+1, && (1.1)\\ u(x,0) = u(x,n+1) &= 0, \quad 0 < x < m+1, && (1.2)\\ u(0,y) = u(m+1,y) &= 0, \quad 0 < y < n+1, && (1.3) \end{aligned}$$

where $m, n \geq 1$ are fixed integers. Define

$$\begin{aligned} u_{ij} &= u(i,j), \quad a_{ij} = a(i,j), \quad 0 \leq i \leq m+1, \quad 0 \leq j \leq n+1,\\ u &= (u_{11}, \cdots, u_{m1}, u_{12}, \cdots, u_{m2}, \cdots, u_{1n}, \cdots, u_{mn})^T,\\ A &= \operatorname{diag}(a_{11}, \cdots, a_{m1}, a_{12}, \cdots, a_{m2}, \cdots, a_{1n}, \cdots, a_{mn}). \end{aligned}$$

Then the system (1.1) with the boundary conditions (1.2)-(1.3) is discretized as

$$Du = \lambda A u, \tag{1.4}$$

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where D is an $mn \times mn$ matrix given by

$$D = \begin{pmatrix} L & -I_m & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ -I_m & L & -I_m & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & -I_m & L & -I_m & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & -I_m & L & \cdots & 0 & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & L & -I_m & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & -I_m & L & -I_m & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & -I_m & L & -I_m \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & -I_m & L \end{pmatrix},$$

I_m is the identity matrix of order m, and L is an $m \times m$ matrix given by

$$L = \begin{pmatrix} 4 & -1 & 0 & \cdots & 0 & 0 & 0 \\ -1 & 4 & -1 & \cdots & 0 & 0 & 0 \\ 0 & -1 & 4 & \cdots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 4 & -1 & 0 \\ 0 & 0 & 0 & \cdots & -1 & 4 & -1 \\ 0 & 0 & 0 & \cdots & 0 & -1 & 4 \end{pmatrix}.$$

We also consider the Dirichlet boundary value problem for the elliptic differential equation in the rectangle $[0, m+1] \times [0, n+1]$

$$\begin{aligned} u_{xx} + u_{yy} + \lambda b(x,y)u(x,y) &= 0, \quad 0 < x < m+1, 0 < y < n+1, \\ u(x,0) = u(x,n+1) &= 0, \quad 0 < x < m+1, \\ u(0,y) = u(m+1,y) &= 0, \quad 0 < y < n+1, \end{aligned}$$

whose discretization is

$$Du = \lambda Bu, \tag{1.5}$$

where

$$B = \operatorname{diag}(b_{11}, \cdots, b_{m1}, b_{12}, \cdots, b_{m2}, \cdots, b_{1n}, \cdots, b_{mn}).$$

Throughout the paper, we assume that m and n are fixed integers and

(H) a_{ij} and b_{ij} are non-negative for $1 \le i \le m, 1 \le j \le n$ with $\sum_{i,j} a_{ij} > 0$ and $\sum_{i,j} b_{ij} > 0$.

If λ is a number (maybe complex) such that the problem (1.4) has a nontrivial solution $\{y_i\}_{i=1}^{mn}$, then λ is said to be an eigenvalue of the problem (1.4), and the corresponding nontrivial solution $\{y_i\}_{i=1}^{mn}$ is called an eigenvector of the problem (1.4) corresponding to λ. Similarly, if μ is a number such that the problem (1.5) has a nontrivial solution $\{y_i\}_{i=1}^{mn}$, then μ is said to be an eigenvalue of the problem (1.5), and the corresponding nontrivial solution $\{y_i\}_{i=1}^{mn}$ is called an eigenvector of the problem (1.5) corresponding to μ.

The research on comparison of eigenvalues has been very active recently since the earlier work of Travis [14]. A representative set of references for these works would be Davis, Eloe, and Henderson [2], Diaz and Peterson [3], Hankerson and Henderson [4], Hankerson and Peterson [5, 6, 7], Henderson and Prasad [8], and Travis [14]. However, in all the aforementioned papers, the focus has been on the smallest eigenvalue.

Recently, a new approach was introduced in [9] for the eigenvalue comparisons of second-order discrete Sturm-Liouville problem. With this approach, we were able to compare all eigenvalues of a larger class of problems which has never been studied in the literature (see, for example, Atkinson [1], Jirari [10], Shi and Chen [12, 13]). Along the same lines, in this paper we will establish the comparison theorems for all the eigenvalues of the problems (1.4) and (1.5). We will also prove the existence of positive eigenvector corresponding to the smallest eigenvalue of the problem (1.4).

2. EIGENVALUE COMPARISONS

In this section, we denote by x^* the conjugate transpose of a vector x. A hermitian matrix C is said to be positive semidefinite if $x^*Cx \geq 0$ for any x. It is said to be positive definite if $x^*Cx > 0$ for any nonzero x. In what follows, we will write $X \succeq Y$ if X and Y are hermitian matrices of the same order and $X - Y$ is positive semidefinite.

First, we establish a few technical results.

Lemma 2.1. *D is positive definite.*

Proof. Obviously, both L and D are real symmetric. For any $y = (y_1, \ldots, y_m)^T \in R^m$,

$$y^T L y = 4\sum_{i=1}^{m} y_i^2 - 2\sum_{i=1}^{m-1} y_i y_{i+1} = y_1^2 + 2\sum_{i=1}^{m} y_i^2 + y_m^2 + \sum_{i=1}^{m-1}(y_i - y_{i+1})^2 \geq 2y^T y \ . \tag{2.1}$$

Let x be a vector in R^{mn}, being partitioned according to the block matrix D, i.e.,

$$x = (x_1^T, x_2^T, \ldots, x_n^T)^T \quad \text{and} \quad x_i \in R^m, \quad 1 \leq i \leq n \ .$$

In view of (2.1), we have

$$\begin{aligned} x^T D x &= \sum_{i=1}^{n} x_i^T L x_i - 2\sum_{i=1}^{n-1} x_i^T x_{i+1} \geq 2\sum_{i=1}^{n} x_i^T x_i - 2\sum_{i=1}^{n-1} x_i^T x_{i+1} \\ &= x_1^T x_1 + \sum_{i=1}^{n-1}(x_i - x_{i+1})^T (x_i - x_{i+1}) + x_n^T x_n \geq 0 \ . \end{aligned} \tag{2.2}$$

Whenever $x^T D x = 0$, the equation (2.2) indicates that $x_1 = 0, x_i - x_{i+1} = 0, 1 \leq i \leq n-1$, and $x_n = 0$, i.e., $x = 0$. Thus, we have $x^T D x > 0$ for $x \neq 0$. The proof is complete. □

Next, we will focus on the study of the elements of the inverse matrix of D. To this end, in what follows we will write $X = (x_{ij}) \geq Y = (y_{ij})$ if $x_{ij} \geq y_{ij}$ for all i, j, and write $X = (x_{ij}) > Y = (y_{ij})$ if $x_{ij} > y_{ij}$ for all i, j. A matrix is said to be positive if each element of the matrix is positive. We also need to employ the properties of the Kronecker product $A \otimes B = (a_{ij}B) \in R^{pm\times qn}$ of two matrices $A = (a_{ij}) \in R^{p\times q}$ and $B \in R^{m\times n}$. Let us first collect a few properties of $A \otimes B$.

Lemma 2.2. *There hold the following statements:*

(1) $I_m \otimes I_n = I_{mn}$.
(2) *If* $A \geq 0$ *and* $B \geq C$, *then* $A \otimes B \geq A \otimes C$.
(3) *If* AC *and* BD *exist, then* $(A \otimes B)(C \otimes D) = AC \otimes BD$.
(4) *If* A *and* B *are nonsingular, then* $A \otimes B$ *is also nonsingular and* $(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}$.
(5) *Let* A *be an* $m \times m$ *matrix with eigenvalues* $\lambda_1, \lambda_2, \ldots, \lambda_m$ *and let* B *be a* $p \times p$ *matrix with eigenvalues* $\mu_1, \mu_2, \ldots, \mu_p$. *Then the* mp *eigenvalues of* $A \otimes B$ *are* $\lambda_i\mu_j$, $1 \leq i \leq m$, $1 \leq j \leq p$.

The first four results of Lemma 2.2 can be derived immediately from the definition of Kronecker product. The proof of the last part of the lemma and many other interesting properties of the Kronecker product can be found in [11, page 28].

Denote by J_m the $m \times m$ matrix of the form

$$J_m = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 0 & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 \end{pmatrix}.$$

It is easy to see that

$$I_m + \left(J_m + J_m^T\right) + \left(J_m^2 + (J_m^T)^2\right) + \cdots + \left(J_m^{m-1} + (J_m^T)^{m-1}\right) = e_m e_m^T\ , \tag{2.3}$$

where $e_m = (1, 1, \ldots, 1)^T \in R^m$, a vector of all ones. We note that $L = 4(I_m - F)$ where $F = \frac{1}{4}(J_m + J_m^T)$ and that the spectral radius $\rho(F)$ satisfies $\rho(F) \leq \|F\|_\infty = \frac{1}{2}$. Thus, we have

$$L^{-1} = \frac{1}{4}(I_m - F)^{-1} = \frac{1}{4}\sum_{i=0}^{\infty} F^i\ . \tag{2.4}$$

It is easily seen from $J_m \geq 0$ that

$$F^i = \frac{1}{4^i}(J_m + J_m^T)^i \geq \frac{1}{4^i}\left(J_m^i + (J_m^T)^i\right), \quad \text{for } i \geq 1\ . \tag{2.5}$$

Combining (2.3), (2.4), and (2.5), we have

$$\begin{aligned} L^{-1} &\geq \frac{1}{4}\left(I_m + \sum_{i=1}^{\infty} \frac{1}{4^i}\left(J_m^i + (J_m^T)^i\right)\right) \geq \frac{1}{4^m}\left(I_m + \sum_{i=1}^{m-1}\left(J_m^i + (J_m^T)^i\right)\right) \\ &= \frac{1}{4^m} e_m e_m^T > 0 \; . \end{aligned} \tag{2.6}$$

In view of (2.6), we have

$$(L^{-1})^i \geq \frac{m^{i-1}}{4^{i\,m}} e_m e_m^T \equiv \tau_i e_m e_m^T > 0, \quad \text{for } i \geq 1 \; , \tag{2.7}$$

where $\tau_i \equiv m^{i-1}/4^{mi}$. It is seen from (2.1) that $y^T L y > 2y^T y$ for $y \neq 0$. Therefore, we have $\min \lambda_i(L) = \min\{y^T L y / y^T y : y \neq 0\} > 2$ which implies $\rho(L^{-1}) < 1/2$. Define $G \equiv (J_n + J_n^T) \otimes L^{-1}$. It is obvious from (2.6) that $G \geq 0$. Also, in view of part (5) of Lemma 2.2, together with $|\lambda_i(J_n + J_n^T)| \leq 2$ and $\rho(L^{-1}) < 1/2$, we have $\rho(G) < 1$.

Observe that the matrix D can be written as

$$D = I_n \otimes L - (J_n + J_n^T) \otimes I_m = (I_n \otimes L)(I_{mn} - G) \; .$$

Therefore, together with (2.6) and the fact that $G \geq 0$ and $I_n \otimes L^{-1} \geq 0$, we have

$$\begin{aligned} D^{-1} &= (I_{mn} - G)^{-1}(I_n \otimes L)^{-1} = (\sum_{i=0}^{\infty} G^i)(I_n \otimes L^{-1}) \\ &\geq (\sum_{i=0}^{n-1} G^i)(I_n \otimes L^{-1}) = \left(\sum_{i=0}^{n-1}(J_n + J_n^T)^i \otimes (L^{-1})^i\right)(I_n \otimes L^{-1}) \\ &= \left(\sum_{i=0}^{n-1}(J_n + J_n^T)^i \otimes (L^{-1})^{i+1}\right) \; . \end{aligned} \tag{2.8}$$

Lemma 2.3. *D^{-1} is a positive matrix.*

Proof. Define $\tau = \min_{1 \leq i \leq n} \tau_i = m^{n-1}/4^{nm}$. It is seen from (2.7) that

$$(L^{-1})^{i+1} \geq \tau e_m e_m^T, \quad 0 \leq i \leq n-1 \; . \tag{2.9}$$

Combining (2.3), (2.8), and (2.9), with the help of part (2) of Lemma 2.2, we have

$$\begin{aligned} D^{-1} &\geq \left(\sum_{i=0}^{n-1}(J_n + J_n^T)^i \otimes \tau e_m e_m^T\right) \geq \tau\left(I_n + \sum_{i=1}^{n-1}\left((J_n)^i + (J_n^T)^i\right)\right) \otimes e_m e_m^T \\ &= \tau e_n e_n^T \otimes e_m e_m^T = \tau e_{mn} e_{mn}^T > 0 \; . \end{aligned} \tag{2.10}$$

Note that the identity given in (2.3) is used here for J_n instead of J_m. □

Lemma 2.4. *If λ is an eigenvalue of the problem* (1.4) *and y is a corresponding eigenvector, then*

(a) *$y^* A y > 0$.*
(b) *λ is real and positive.*

(c) *If ρ is an eigenvalue of the problem* (1.4) *which is different from λ and x is a corresponding eigenvector, then we have $x^T Ay = 0$.*

Proof. (a) The assumption (H) indicates that $y^*Ay \geq 0$. Assume to the contrary that $y^*Ay = 0$. Obviously, we have $\sqrt{A}\,y = 0$ where

$$\sqrt{A} = \operatorname{diag}(\sqrt{a_{11}}, \cdots, \sqrt{a_{m1}}, \sqrt{a_{12}}, \cdots, \sqrt{a_{m2}}, \cdots, \sqrt{a_{1n}}, \cdots, \sqrt{a_{mn}}).$$

Then, we have $Dy = \lambda Ay = \lambda\sqrt{A}\,\sqrt{A}\,y = 0$, which, together with Lemma 2.1, implies $y = 0$. This is a contradiction.

(b) We can write

$$\lambda y^*Ay = y^*(\lambda Ay) = y^*Dy = y^*D^*y = (Dy)^*y = (\lambda Ay)^*y = \bar{\lambda}y^*A^*y = \bar{\lambda}y^*Ay,$$

which, together with (a), implies that $\lambda = \bar{\lambda}$, i.e., λ is real. Finally, the relations above indicate that $\lambda = y^*Dy/(y^*Ay) > 0$ thanks to Lemma 2.1 and the first part of this lemma.

Part (c) follows from

$$(\lambda - \rho)x^T Ay = \lambda x^T Ay - \rho x^T Ay = x^T(\lambda Ay) - (\rho Ax)^T y = x^T Dy - (Dx)^T y = 0.$$

The proof is complete. □

Lemma 2.5. *The eigenvalues of the problem* (1.4) *are related to those of the matrix $D^{-\frac{1}{2}}AD^{-\frac{1}{2}}$ as follows.*

(a) *If λ is an eigenvalue of the problem* (1.4)*, then $1/\lambda$ is an eigenvalue of the matrix $D^{-\frac{1}{2}}AD^{-\frac{1}{2}}$.*
(b) *If α is a positive eigenvalue of $D^{-\frac{1}{2}}AD^{-\frac{1}{2}}$, then $1/\alpha$ is an eigenvalue of the problem* (1.4).

Proof. (a) If λ is an eigenvalue of the problem (1.4), and y is a corresponding eigenvector, then we have $\lambda Ay = Dy$ with $\lambda > 0$ due to Lemma 2.4. Thus, we have

$$\lambda Ay = D^{\frac{1}{2}}D^{\frac{1}{2}}y, \quad \text{and} \quad D^{-\frac{1}{2}}AD^{-\frac{1}{2}}(D^{\frac{1}{2}}y) = \frac{1}{\lambda}(D^{\frac{1}{2}}y).$$

The result in (b) can be proved similarly. The proof is complete. □

Next, we state the well-known Perron-Frobenius Theorem [15, page 30].

Theorem 2.6 (Perron-Frobenius)**.** *Let C be a real square matrix. If C is also a non-negative irreducible matrix, then the spectral radius $\rho(C)$ of C is a simple eigenvalue of C, associated with a positive eigenvector. Moreover, $\rho(C) > 0$.*

Theorem 2.7. *If $\lambda_1 > 0$ is the smallest eigenvalue of the problem* (1.4)*, then λ_1 is a simple eigenvalue, and there exists a positive eigenvector $y > 0$ corresponding to λ_1.*

Proof. We note that $D^{-1}Ay = (1/\lambda_1)y$. Thus, $1/\lambda_1$ is the maximum eigenvalue of $D^{-1}A$ and y is an eigenvector corresponding to $1/\lambda_1$.

In the case where $a_{ij} > 0$ for all $1 \le i \le m$ and $1 \le j \le n$, we obtain that the matrix $D^{-1}A$ is positive (therefore irreducible) in view of Lemma 2.3. Therefore, the result follows immediately from Theorem 2.6.

In the case where some of the a_{ij}'s are zero, there exists a permutation matrix P such that

$$P^T AP = \begin{pmatrix} Z & 0 \\ 0 & 0 \end{pmatrix},$$

where $Z = \mathrm{diag}(\bar{a}_1, \ldots, \bar{a}_t)$, and $\bar{a}_1, \ldots, \bar{a}_t$ are the positive elements in the set

$$\{a_{11}, \cdots, a_{m1}, a_{12}, \cdots, a_{m2}, \cdots, a_{1n}, \cdots, a_{mn}\}.$$

Then, (1.4) becomes

$$\frac{1}{\lambda} P^T u = P^T D^{-1} P \begin{pmatrix} Z & 0 \\ 0 & 0 \end{pmatrix} P^T u. \tag{2.11}$$

Note that $P^T D^{-1} P \begin{pmatrix} Z & 0 \\ 0 & 0 \end{pmatrix}$ is in the form $\begin{pmatrix} W & 0 \\ V & 0 \end{pmatrix}$, where W is nonsingular and both W and V are positive matrices in view of Lemma 2.3. If λ_1 is the smallest eigenvalue of (1.4), then $1/\lambda_1$ is the largest eigenvalue of W. Therefore, λ_1 is simple. Lemma 2.6 indicates that there exists a positive eigenvector u_1 of W corresponding to $1/\lambda_1$. Finally, we see that

$$y \equiv P \begin{pmatrix} u_1 \\ \lambda_1 V u_1 \end{pmatrix} > 0$$

is a positive eigenvector of (1.4) corresponding to λ_1. This completes the proof. □

Lemma 2.8. *Let $N \ge 1$ be the number of positive elements in the set*

$$\{a_{11}, \cdots, a_{m1}, a_{12}, \cdots, a_{m2}, \cdots, a_{1n}, \cdots, a_{mn}\}.$$

Then there are N eigenvalues λ_i $(i = 1, 2, \ldots, N)$ of the problem (1.4) *and $\alpha_i = 1/\lambda_i$ $(i = 1, 2 \ldots, N)$ are the only positive eigenvalues of $D^{-\frac{1}{2}}AD^{-\frac{1}{2}}$.*

Proof. The assumption (H) implies $N \ge 1$. Suppose that $\alpha_1 \ge \alpha_2 \ge \cdots \ge \alpha_{mn} \ge 0$ are all eigenvalues of $D^{-\frac{1}{2}}AD^{-\frac{1}{2}}$. The fact that $D^{-\frac{1}{2}}AD^{-\frac{1}{2}}$ is real and symmetric indicates that there exists an orthogonal matrix Q such that

$$Q^T D^{-\frac{1}{2}} AD^{-\frac{1}{2}} Q = \mathrm{diag}(\alpha_1, \alpha_2, \ldots, \alpha_{mn}). \tag{2.12}$$

Therefore, we have

$$\mathrm{rank}(A) = \mathrm{rank}\left(Q^T D^{-\frac{1}{2}} AD^{-\frac{1}{2}} Q\right) = \mathrm{rank}\left(\mathrm{diag}(\alpha_1, \alpha_2, \ldots, \alpha_{mn})\right),$$

indicating that the number of positive α_i is the same as that of positive a_{ij} in A, which is equal to N.

Thus, in view of Lemma 2.5, we see that $\{\lambda_i = 1/\alpha_i : i = 1, 2 \dots, N\}$ gives the complete set of eigenvalues of the problem (1.4). The proof is complete. $\square$

Theorem 2.9. *Assume that hypothesis (H) holds. Let N be the number of positive elements in the set*

$$\{a_{11}, \cdots, a_{m1}, a_{12}, \cdots, a_{m2}, \cdots, a_{1n}, \cdots, a_{mn}\}$$

and M be the number of positive elements in the set

$$\{b_{11}, \cdots, b_{m1}, b_{12}, \cdots, b_{m2}, \cdots, b_{1n}, \cdots, b_{mn}\}.$$

Let $\{\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_N\}$ be the set of all eigenvalues of the problem (1.4) *and $\{\mu_1 \leq \mu_2 \leq \cdots \leq \mu_M\}$ be the set of all eigenvalues of the problem* (1.5)*. If $a_{ij} \geq b_{ij}$ for $1 \leq i \leq m$ and $1 \leq j \leq n$, then $\lambda_i \leq \mu_i$ for $1 \leq i \leq M$.*

Proof. In view of Lemma 2.8, it is easily seen that

$$\alpha_1 = \frac{1}{\lambda_1} \geq \alpha_2 = \frac{1}{\lambda_2} \geq \cdots \geq \alpha_N = \frac{1}{\lambda_N} > 0, \quad \text{and} \quad \alpha_{N+1} = \cdots = \alpha_{mn} = 0 \tag{2.13}$$

and

$$\beta_1 = \frac{1}{\mu_1} \geq \beta_2 = \frac{1}{\mu_2} \geq \cdots \geq \beta_M = \frac{1}{\mu_M} > 0, \quad \text{and} \qquad \beta_{M+1} = \cdots = \beta_{mn} = 0 \tag{2.14}$$

are the eigenvalues of $D^{-\frac{1}{2}}AD^{-\frac{1}{2}}$ and $D^{-\frac{1}{2}}BD^{-\frac{1}{2}}$, respectively. If $a_{ij} \geq b_{ij}$ for all $1 \leq i \leq m$, $1 \leq j \leq n$, then $A \succeq B$, implying

$$D^{-\frac{1}{2}}AD^{-\frac{1}{2}} \succeq D^{-\frac{1}{2}}BD^{-\frac{1}{2}}. \tag{2.15}$$

By Weyl's inequality and (2.15), we have

$$\alpha_i \geq \beta_i \geq 0, \quad 1 \leq i \leq mn. \tag{2.16}$$

The desired result follows immediately from (2.13), (2.14), and (2.16). $\square$

Finally, we remark that the smallest eigenvalue of the problem (1.4) is simple as indicated in Theorem 2.7. But the other eigenvalues of the problem, in general, may not be simple as is illustrated by the following example.

Example 2.10. Consider the simple case where $m = n = 2$ and $a(x, y) \equiv 1$ in (1.1). Then we have $Du = \lambda Au$ where $u = (u_{11}, u_{21}, u_{12}, u_{22})^T$, $A = \mathrm{diag}(1, 1, 1, 1)$, and

$$D = \begin{pmatrix} 4 & -1 & -1 & 0 \\ -1 & 4 & 0 & -1 \\ -1 & 0 & 4 & -1 \\ 0 & -1 & -1 & 4 \end{pmatrix}.$$

A simple calculation leads to $\lambda_1 = 2, \lambda_2 = \lambda_3 = 4, \lambda_4 = 6$. In this example, λ_1 is simple, and $\lambda_2 = \lambda_3$ has multiplicity 2.

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