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POSITIVE SOLUTIONS AND EIGENVALUE INTERVALS FOR DISCRETE BOUNDARY VALUE PROBLEMS

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ABSTRACT. In this paper, we study the existence of positive solutions and characterize the eigenvalue intervals for discrete boundary value problems. Both second order problems and p-Laplacian problems are considered. The proof relies on a well-known fixed point theorem in cones.

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1. INTRODUCTION

The purpose of this paper is to obtain the existence of positive solutions to discrete boundary value problems. We consider the second order problem

$$\begin{cases} \Delta^2 y(k-1) + f(k, y(k)) = 0, \ k \in N, \\ y(0) = 0, \quad y(T+1) = 0, \end{cases} \tag{1.1}$$

and the one-dimensional p-Laplacian problem

$$\begin{cases} \Delta(\phi(\Delta y(k-1))) + f(k, y(k)) = 0, \ k \in N, \\ y(0) = 0, \quad y(T+1) = 0, \end{cases} \tag{1.2}$$

where T is a positive integer, N is the discrete interval $\{1, 2, \ldots, T\}$, $\phi(s) = |s|^{p-2}s, \ p > 1$ and $\Delta y(k) = y(k+1) - y(k)$ is the forward difference operator. Also we consider the second order system

$$\begin{cases} \Delta^2 y_i(k-1) + f^i(k, y(k)) = 0, \quad k \in N, \\ y(0) = 0, \quad y(T+1) = 0, \quad i = 1, 2, \ldots, n, \end{cases} \tag{1.3}$$

and the p-Laplacian system

$$\begin{cases} \Delta(\phi(\Delta y_i(k-1))) + f^i(k, y(k)) = 0, \quad k \in N, \\ y(0) = 0, \quad y(T+1) = 0, \quad i = 1, 2, \ldots, n, \end{cases} \tag{1.4}$$

where $y = (y_1, y_2, \ldots, y_n)$.

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Recently, discrete boundary value problems have been studied by many authors [1, 4, 10, 15]. In [4], positive solutions for nonpositone second order discrete problems were considered using a conical shell fixed point theorem. Multiple positive solutions of singular discrete problems were also studied in [6, 7] using variational methods. Moreover, discrete $p-$Laplacian problems have been considered in [9, 10, 15]. We refer the reader to [2] for a broad introduction to difference equations.

The main results in this paper are proved by employing a fixed point theorem (see Theorem 2.6) for compact maps on conical shells. To do this, we extend the ideas introduced by Lan in [18] to the discrete case. In the continuous case, this approach was also used to prove the existence of positive periodic solutions for systems of second order differential equations in [12]. The same problems for scalar functional differential equation were studied in [13]. The results obtained in [13] improve some results in [21]. Here we mention that the systems of integral equations or differential equations have been studied in [8, 14, 19, 20] by employing another fixed point theorem in cones (see Theorem 1.1 in [10]).

As applications, we characterize the eigenvalue intervals for problems

$$\begin{cases} \Delta^2 y(k-1) + \lambda h(k) g(y(k)) = 0, \ k \in N, \\ y(0) = 0, \quad y(T+1) = 0, \end{cases} \tag{1.5}$$

$$\begin{cases} \Delta(\phi(\Delta y(k-1))) + \lambda h(k) g(y(k)) = 0, \ k \in N, \\ y(0) = 0, \quad y(T+1) = 0, \end{cases} \tag{1.6}$$

and the $n-$dimensional systems

$$\begin{cases} \Delta^2 y_i(k-1) + \lambda h_i(k) g^i(y(k)) = 0, \quad k \in N, \\ y(0) = 0, \quad y(T+1) = 0, \quad i = 1, 2, \ldots, n, \end{cases} \tag{1.7}$$

$$\begin{cases} \Delta(\phi(\Delta y_i(k-1))) + \lambda h_i(k) g^i(y(k)) = 0, \quad k \in N, \\ y(0) = 0, \quad y(T+1) = 0, \quad i = 1, 2, \ldots, n. \end{cases} \tag{1.8}$$

Here $\lambda > 0$ is a positive parameter. We prove that the above eigenvalue problems have at least one positive solution for each λ in an explicit eigenvalue interval. Recently, several eigenvalue characterizations for different kinds of boundary value problems have appeared and we refer the reader to [3, 5, 10, 11, 20].

The notation used is as follows. $\mathbb{R}_+ = [0, \infty)$, $\mathbb{R}^n_+ = \prod_{i=1}^n \mathbb{R}_+$, $N^+ = \{0, 1, \ldots, T+1\}$. Recall that a map $f : N^+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is continuous if it is continuous as a map of the topological space $N^+ \times \mathbb{R}_+$ into the topological space $\mathbb{R}_+$. In this paper, the topology on N^+ will be the discrete topology. Let $C(N^+, \mathbb{R})$ denote the class of map y continuous on N^+ (discrete topology), with norm $|y|_0 = \max_{k \in N^+} |y(k)|$, then $C(N^+, \mathbb{R})$ is a Banach space. For any $y = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n_+$, $\|y\| = \max_{i=1,2,\ldots,n} |y_i|$.

The remaining part of the paper is organized as follows. In Section 2 some preliminary results will be given. In Section 3, we study the existence of positive solutions for problem (1.1) and characterize the eigenvalue intervals for (1.5). Positive solutions of the second order system (1.3) and eigenvalue intervals for (1.7) are considered in section 4. In section 5, we study the existence of positive solutions for problem (1.2) and characterize the eigenvalue intervals for (1.6). Finally, section 6 studies the p-Laplacian system (1.4) and (1.8). It is worth remarking here that, Lemma 3.2 and Lemma 4.2 play a fundamental role in the proof of our main results.

2. PRELIMINARIES

In this section, we present some results, which will be needed in the following four sections. First we state some well-known results.

Lemma 2.1 ([4]). *Let $u \in C(N^+, \mathbb{R})$ satisfy $u(k) \geq 0$ for $k \in N^+$. If $y \in C(N^+, \mathbb{R})$ satisfies*

$$\begin{cases} \Delta^2 y(k-1) + u(k) = 0, \ k \in N, \\ y(0) = 0, \quad y(T+1) = 0, \end{cases}$$

then

$$y(k) \geq q(k)\|y\| \text{ for } k \in N^+;$$

here

$$q(k) = \min\{\frac{T+1-k}{T+1}, \ \frac{k}{T}\}. \tag{2.1}$$

Remark 2.2. From the definition of $q(k)$, we have $q(k) \geq \frac{1}{T+1}$ for $k \in N$.

Lemma 2.3 ([15]). *If $y \in C(N^+, \mathbb{R})$ satisfies*

$$\begin{cases} \Delta(\phi(\Delta y(k-1))) \leq 0, \ k \in N, \\ y(0) = 0, \quad y(T+1) = 0, \end{cases}$$

then $y(k) \geq q(k)\|y\|$ for $k \in N^+$; here $q(k)$ is defined in Lemma 2.1.

Lemma 2.4 ([15]). *If $u, \ v \in C(N^+, \mathbb{R})$ satisfies*

$$\begin{cases} \Delta(\phi(\Delta u(k-1))) \leq \Delta(\phi(\Delta v(k-1))), \ k \in N, \\ u(0) \geq v(0), \ \ u(T+1) \geq v(T+1), \end{cases}$$

then $u(k) \geq v(k)$ for $k \in N^+$.

As indicated in the introduction, the proof of our main results is based on a fixed point theorem for compact maps on conical shells. We recall the statement of this result below, after introducing the definition of a cone.

Definition 2.5. Let X be a Banach space and K be a closed, nonempty subset of X. K is a cone if

(i) $\alpha u + \beta v \in K$ for all $u, v \in K$ and all $\alpha, \beta > 0$.
(ii) $u, -u \in K$ implies $u = 0$.

We also recall that a completely continuous operator means a continuous operator which transforms every bounded set into a relatively compact set. If D is a subset X, we write $D_K = D \cap K$ and $\partial_K D = (\partial D) \cap K$.

Theorem 2.6 ([17]). *Let X be a Banach space and K a cone in X. Assume Ω^1, Ω^2 are open bounded subsets of X with $\Omega^1_K \neq \emptyset$, $\overline{\Omega^1}_K \subset \Omega^2_K$. Let*

$$T : \overline{\Omega^2}_K \to K$$

be a continuous and compact operator such that

(i) $\|Tx\| \leq \|x\|$ *for* $x \in \partial_K \Omega^1$, *and*
(ii) *there exists* $e \in K \backslash \{0\}$ *such that* $x \neq Tx + \lambda e$ *for all* $x \in \partial_K \Omega^2$ *and all* $\lambda > 0$.

Then T has a fixed point in $\overline{\Omega^2}_K \setminus \Omega^1_K$. The same conclusion remains valid if (i) *holds on $\partial_K \Omega^2$ and* (ii) *holds on $\partial_K \Omega^1$.*

3. SECOND ORDER PROBLEMS

In this section we study the existence of positive solutions to second order discrete boundary value problem (1.1) and characterize the eigenvalue intervals for (1.5).

Define the operator

$$Ty(k) = \sum_{j=1}^{T} G(k,j) f(j, y(j));$$

here $G(k,j)$ is defined by

$$G(k,j) = \begin{cases} \dfrac{j(T+1-k)}{T+1}, & 0 \leq j \leq k-1, \\ \dfrac{k(T+1-j)}{T+1}, & k \leq j \leq T+1. \end{cases} \tag{3.1}$$

In order to apply Theorem 2.6, we take $X_1 = C(N^+, \mathbb{R})$ with norm $|y|_0 = \max_{k \in N^+} |y(k)|$. Define a cone K_1 in X_1 by

$$K_1 = \{y \in X_1 : \ y(k) \geq q(k)\|y\| \text{ for } k \in N^+\}. \tag{3.2}$$

Lemma 3.1 ([1]). *Assume that $f : N^+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is continuous. Then T is well defined and maps K_1 into K_1. Moreover, T is continuous and completely continuous.*

Following the ideas in [13, 18], we define the open sets

$$\Omega^r = \{y \in X_1 : \ \min_{k \in N} y(k) < \frac{r}{T+1}\} \text{ and } B^r = \{y \in X_1 : |y|_0 < r\}.$$

Lemma 3.2. Ω^r, B^r *defined above have the following properties:*

(a) $\Omega^r_{K_1}$ *and* $B^r_{K_1}$ *are open relative to* K_1.
(b) $B^{r/T+1}_{K_1} \subset \Omega^r_{K_1} \subset B^r_{K_1}$.

(c) $y \in \partial_{K_1}\Omega^r$ *if and only if* $y \in K_1$ *and* $\min_{k\in N} y(k) = \dfrac{r}{T+1}$.

(d) *If* $y \in \partial_{K_1}\Omega^r$, *then* $\frac{r}{T+1} \le y(k) \le r$ *for* $k \in N$ *and* $|y|_0 \le r$.

Proof. (a) holds since $\min_{k\in N} y(k)$ is continuous (discrete topology). (c) is clear. Let $y \in \partial_{K_1}\Omega^r$, so by (c), we have $\min_{k\in N} y(k) = \dfrac{r}{T+1}$ and $y(k) \ge q(k)|y|_0 \ge \frac{1}{T+1}|y|_0$ for $k \in N$. Thus $\frac{1}{T+1}|y|_0 \le \min_{k\in N} y(k) = \dfrac{r}{T+1}$, so $|y|_0 \le r$ and $\frac{r}{T+1} \le y(k) \le r$ for $k \in N$, i.e., (d) holds.

Finally we prove (b). Let $y \in B_{K_1}^{r/T+1}$, then $|y|_0 < \frac{r}{T+1}$, so $\min_{k\in N} y(k) < \dfrac{r}{T+1}$ and $y \in \Omega_{K_1}^r$. If $y \in \Omega_{K_1}^r$. Then $\min_{k\in N} y(k) < \dfrac{r}{T+1}$ and $y(k) \ge q(k)|y|_0 \ge \frac{1}{T+1}|y|_0$ for $k \in N$. This implies $|y|_0 < r$, i. e., $\Omega_{K_1}^r \subset B_{K_1}^r$. Hence (b) holds. □

It is clear that the sets Ω^r are unbounded sets for each $r > 0$, so we cannot use Theorem 2.6 with Ω^r. However we will be able to apply Theorem 2.6 taking into account that, for each $\delta > r$, the following relations hold:

$$\Omega_{K_1}^r = (\Omega^r \cap B^\delta)_{K_1} \quad \text{and} \quad \overline{\Omega^r}_{K_1} = (\overline{\Omega^r \cap B^\delta})_{K_1}.$$

The first equality follows immediately from Lemma 3.2 (b). For the second let $y \in \overline{\Omega^r}_{K_1}$, then from Lemma 3.2 (c), we have that

$$\frac{1}{T+1}|y|_0 \le \min_{k\in N} y(k) \le \frac{r}{T+1} < \frac{\delta}{T+1},$$

so $y \in (\overline{\Omega^r} \cap B^\delta) \cap K_1$. Now, since Ω^r and B^δ are open sets we have $\overline{\Omega^r} \cap B^\delta \subset \overline{\Omega^r \cap B^\delta}$. Thus $y \in (\overline{\Omega^r \cap B^\delta})_{K_1}$, and therefore $\overline{\Omega^r}_{K_1} \subseteq (\overline{\Omega^r \cap B^\delta})_{K_1}$. The reverse inclusion is trivial.

Theorem 3.3. *Assume that* $f : N^+ \times \mathbb{R}_+ \to \mathbb{R}_+$ *is continuous. Furthermore, it is assumed that the following two hypotheses hold:*

(D_1) *there exist a constant* $\alpha > 0$ *and a continuous function* $\psi : N \to (0, \infty)$ *such that*

$$f(j, y) \ge \frac{1}{T+1}\alpha\psi(j), \qquad \text{for all } j \in N,\ \frac{\alpha}{T+1} \le y \le \alpha$$

and

$$\min_{k\in N} \sum_{j=1}^{T} G(k, j)\psi(j) \ge 1;$$

(D_2) *there exist a constant* $\beta > 0$ *and a continuous function* $\chi : N \to (0, \infty)$ *such that*

$$f(j, y) \le \beta\chi(j) \qquad \text{for all } j \in N,\ 0 < y \le \beta$$

and

$$\max_{k\in N} \sum_{j=1}^{T} G(k, j)\chi(j) \le 1.$$

Then, the following results hold:

(a) *if* $\beta < \frac{\alpha}{T+1}$, *then problem* (1.1) *has at least one positive solution* y *satisfying*

$$\beta \le |y|_0 \le \alpha;$$

(b) *if* $\alpha < \beta$, *then problem* (1.1) *has at least one positive solution* y *satisfying*

$$\frac{1}{T+1}\alpha \le |y|_0 \le \beta.$$

Proof. Since $y = y(k)$ is a solution of (1.1) whenever y is a fixed point of T, we only need to prove that T has at least one positive fixed point. To do this, we show that :

(i) $|Ty|_0 \le |y|_0$ for $y \in \partial_{K_1} B^\beta$, and
(ii) there exists $e \in K_1 \backslash \{0\}$ such that $y \ne Ty + \lambda e$, for all $y \in \partial_{K_1}\Omega^\alpha$ and all $\lambda > 0$.

We start with (i). Now for any $y \in \partial_{K_1} B^\beta$, we have $|y|_0 = \beta$. Then from (D_2) we obtain, for each $k \in N$,

$$\begin{aligned} Ty(k) &= \sum_{j=1}^{T} G(k,j) f(j, y(j)) \le \beta \sum_{j=1}^{T} G(k,j)\chi(j) \\ &\le \beta \max_{k\in N} \sum_{j=1}^{T} G(k,j)\chi(j) \le \beta. \end{aligned}$$

Hence, $|Ty|_0 \le |y|_0$ for each $y \in \partial_{K_1} B^\beta$ and so (i) holds.

Next we consider part (ii). Let $e(t) \equiv 1$, so $e \in K_1 \setminus \{0\}$. Next, suppose that there exists $y \in \partial_{K_1}\Omega^\alpha$ and $\lambda > 0$ such that $y = Ty + \lambda e$. Since $y \in \partial_{K_1}\Omega^\alpha$, then from Lemma 3.2 (d), we have $\frac{1}{T+1}\alpha \le y(k) \le \alpha,\ k \in N$.

From (D_1) we have, for $k \in N$, that

$$\begin{aligned} y(k) &= Ty(k) + \lambda = \sum_{j=1}^{T} G(k,j) f(j, y(j)) + \lambda \\ &\ge \frac{1}{T+1}\alpha \sum_{j=1}^{T} G(k,j)\psi(j) + \lambda \\ &\ge \frac{1}{T+1}\alpha \min_{k\in N} \sum_{j=1}^{T} G(k,j)\psi(j) + \lambda \ge \frac{1}{T+1}\alpha + \lambda. \end{aligned}$$

Hence $\min\limits_{k\in N} y(k) \ge \dfrac{1}{T+1}\alpha + \lambda > \dfrac{1}{T+1}\alpha$, contradicting the statement of Lemma 3.2 (c). This contradiction proves part (ii) above.

Now suppose that $\beta < \frac{1}{T+1}\alpha$. Then one has from Lemma 3.2 that $\overline{B^\beta}_{K_1} \subset B_{K_1}^{\alpha/T+1} \subset \Omega_{K_1}^\alpha$ and therefore it follows from Theorem 2.6 that T has at least one fixed point $y \in \overline{\Omega^\alpha}_{K_1} \setminus B^\beta_{K_1}$. Hence $|y|_0 \geq \beta$ and $\frac{1}{T+1}\beta \leq \min_{k\in N} y(k) \leq \dfrac{1}{T+1}\alpha$. On the other hand, $\frac{1}{T+1}|y|_0 \leq \min_{k\in N} y(k) \leq \dfrac{1}{T+1}\alpha$. This implies that $|y|_0 \leq \alpha$.

Finally, if $\alpha < \beta$ one has $\overline{\Omega^\alpha}_{K_1} \subset B^\beta_{K_1}$, and then Theorem 2.6 guarantees the existence of at least one fixed point $y \in \overline{B^\beta}_{K_1} \setminus \Omega^\alpha_{K_1}$ of T. Hence we obtain the inequality $\frac{1}{T+1}\alpha \leq |y|_0 \leq \beta$. $\square$

Next we employ Theorem 3.3 to establish the existence of positive solutions to the following discrete boundary value problem

$$\begin{cases} \Delta^2 y(k-1) + h(k)g(y(k)) = 0, \ k \in N, \\ y(0) = 0, \quad y(T+1) = 0. \end{cases} \tag{3.3}$$

We assume that

(H_1) $g : \mathbb{R}_+ \to \mathbb{R}_+$ is continuous with $g(y) > 0$ for $y > 0$.

(H_2) $h(j) : N \to \mathbb{R}_+$ is continuous and $\displaystyle\sum_{j=1}^{T} G(k,j)h(j) > 0$.

Theorem 3.4. *Suppose that conditions* ($\mathtt{H}_1$)-($\mathtt{H}_2$) *hold. Then problem* (3.3) *has at least one positive solution y with $y(k) \not\equiv 0$ for $k \in N$ if one of the following conditions holds.*

(h_1) $0 \leq g_0 < A^{-1}$ *and* $B^{-1} < g_\infty \leq \infty$;

(h_2) $0 \leq g_\infty < A^{-1}$ *and* $B^{-1} < g_0 \leq \infty$;

here $g_0 = \displaystyle\lim_{y\to 0^+} \frac{g(y)}{y}, \quad g_\infty = \lim_{y\to\infty} \frac{g(y)}{y}$, *and*

$$A = \max_{k\in N} \sum_{j=1}^{T} G(k,j)h(j), \quad B = \min_{k\in N} \sum_{j=1}^{T} G(k,j)h(j).$$

Proof. To see this, we will apply Theorem 3.3 with $f(k,y) = h(k)g(y)$. We assume that (h_1) holds. The case when (h_2) holds is similar.

From the first part of (h_1), there exists $\beta > 0$ such that $g(y) \leq A^{-1}\beta$ for $0 < y \leq \beta$. Choose $\chi(k) = A^{-1}h(k)$. Then

$$f(k,y) = h(k)g(y) \leq A^{-1}\beta h(k) = \beta\chi(k) \qquad \text{if} \ \ k \in N, \ \ 0 < y \leq \beta,$$

and

$$\sum_{j=1}^{T} G(k,j)\chi(j) = A^{-1}\sum_{j=1}^{T} G(k,j)h(j) \leq A^{-1}\max_{k\in N}\sum_{j=1}^{T} G(k,j)h(j) = 1.$$

Thus hypothesis ($\mathtt{D}_2$) holds.

From the second part of (h_1), there exists $\alpha > 0$ such that $\frac{1}{T+1}\alpha > \beta$ and $g(y) \geq B^{-1}\frac{1}{T+1}\alpha$ for $y \geq \frac{1}{T+1}\alpha$. Choose $\psi(k) = B^{-1}h(k)$, then

$$f(k,y) = h(k)g(y) \geq B^{-1}\frac{1}{T+1}\alpha h(k) = \frac{1}{T+1}\alpha\psi(k), \text{ if } k \in N,\ y \geq \frac{1}{T+1}\alpha,$$

(so in particular for $\frac{1}{T+1}\alpha \leq y \leq \alpha$) and

$$\sum_{j=1}^{T} G(k,j)\psi(j) = B^{-1}\sum_{j=1}^{T} G(k,j)h(j) \geq B^{-1}\min_{k\in N}\sum_{j=1}^{T} G(k,j)h(j) = 1.$$

This implies that hypothesis $(\mathtt{D}_1)$ holds. The result now follows from Theorem 3.3. □

By employing Theorem 3.4, we can easily characterize the eigenvalue intervals for problem (1.5). Since the proof is easy, we only state the results here.

Theorem 3.5. *Suppose that conditions* $(\mathtt{H}_1)$-$(\mathtt{H}_2)$ *hold. Then problem* (1.5) *has at least one positive solution for each*

$$\lambda \in (\frac{1}{Bg_\infty}, \frac{1}{Ag_0}) \tag{3.4}$$

if $\frac{1}{Bg_\infty} < \frac{1}{Ag_0}$. *The same result remains valid for each*

$$\lambda \in (\frac{1}{Bg_0}, \frac{1}{Ag_\infty}) \tag{3.5}$$

if $\frac{1}{Bg_0} < \frac{1}{Ag_\infty}$. *Here we write* $1/g_\alpha = 0$ *if* $g_\alpha = \infty$ *and* $1/g_\alpha = \infty$ *if* $g_\alpha = 0$, *where* $\alpha = 0, \infty$.

4. SECOND ORDER SYSTEMS

In this section we establish the existence of positive solutions to the discrete system (1.3) and characterize the eigenvalue intervals for (1.7).

We take $X_2 = C(N^+,\mathbb{R}) \times C(N^+,\mathbb{R}) \times \ldots \times C(N^+,\mathbb{R})$ (n times) with the norm $\|y\| = \max_{i=1,2,\ldots,n} |y_i|_0$ for $y = (y_1, y_2, \ldots, y_n) \in X_2$, here $|y|_0 = \max_{i\in N^+}|y(i)|$. Then X_2 is a Banach space. Define

$$K_2 = \{y = (y_1, y_2, \ldots, y_n) \in X_2 : \min_{k\in N} y_i(k) \geq q(k)|y_i|_0 \text{ for all } i = 1, 2, \ldots, n\}. \tag{4.1}$$

One can easily verify that K_2 is a cone in X_2. Moreover, let $T : K_2 \to X_2$ be a map with components $(T^1, \ldots, T^n)$, where $T^i, i = 1, 2, \ldots, n$, is defined by

$$(T_i y)(k) = \sum_{j=1}^{T} G(k,j) f^i(j, y(j)), \quad i = 1, 2, \ldots, n;$$

here $G(k,j)$ is given as (3.1).

Lemma 4.1. *Assume that $f^i : N^+ \times \mathbb{R}^n_+ \to \mathbb{R}_+$ is continuous, $i = 1, 2, \ldots, n$. Then T is well defined and maps K_2 into K_2. Moreover, T is continuous and completely continuous.*

Following the ideas in [12, 14], we define the open sets

$$\Omega^r = \{y \in X_2 : \min_{k\in N} y_i(k) < \frac{r}{T+1} \text{ for all } i = 1, 2, \ldots, n\}$$

and

$$B^r = \{y \in X_2 : \|y\| < r\}.$$

Lemma 4.2. Ω^r, B^r *defined above have the following properties:*

(a) $\Omega^r_{K_2}$ *and* $B^r_{K_2}$ *are open relative to* K_2.

(b) $B^{r/T+1}_{K_2} \subset \Omega^r_{K_2} \subset B^r_{K_2}$.

(c) $y \in \partial_{K_2}\Omega^r$ *if and only if* $y \in K_2$ *and* $\min_{k\in N} y_j(k) = \frac{r}{T+1}$ *for some* $j \in \{1, 2, \ldots, n\}$ *and* $\min_{k\in N} y_i(k) \le \frac{r}{T+1}$ *for each* $i \in \{1, 2, \ldots, n\}$.

(d) *If* $y \in \partial_{K_2}\Omega^r$, *then* $\frac{r}{T+1} \le y_j(k) \le r,\ k \in N$ *for some* $j \in \{1, 2, \ldots, n\}$ *and* $0 \le y_i(k) \le r,\ k \in N$ *for each* $i \in \{1, 2, \ldots, n\}$. *Moreover,* $|y_i|_0 \le r$.

Proof. (a) is true since $\min_{k\in N} y_i(k)$ is continuous (discrete topology) for each $i \in \{1, 2, \ldots, n\}$. (c) is clear. Let $y \in \partial_{K_2}\Omega^r$, so we have from (c) that there exists $j \in \{1, 2, \ldots, n\}$ such that

$$\frac{1}{T+1}|y_j|_0 \le q(k)|y_j|_0 \le \min_{k\in N} y_j(k) = \frac{r}{T+1}.$$

Thus $|y_j|_0 \le r$ and $\frac{r}{T+1} \le y_j(k) \le r,\ k \in N$. In addition notice for each $i \in \{1, 2, \ldots, n\}$ that $\frac{1}{T+1}|y_i|_0 \le q(k)|y_i|_0 \le \min_{k\in N} y_i(k) \le \frac{r}{T+1}$, so $|y_i|_0 \le r$ and $0 \le y_i(k) \le r$ for $k \in N$, i. e., (d) holds.

Finally we prove (b). Let $y \in B^{r/T+1}_{K_2}$, then for each $i \in \{1, 2, \ldots, n\}$, we have $|y_i|_0 < \frac{r}{T+1}$, so $\min_{k\in N} y_i(k) < \frac{r}{T+1}$ and $y \in \Omega^r_{K_2}$. If $y \in \Omega^r_{K_2}$, then for each $i \in \{1, 2, \ldots, n\}$, we have $\min_{k\in N} y_i(k) < \frac{r}{T+1}$ and $y_i(k) \ge q(k)|y_i|_0 \ge \frac{1}{T+1}|y_i|_0$ for $k \in N$. This implies $|y_i|_0 < r$, i. e., $\Omega^r_{K_2} \subset B^r_{K_2}$. Hence (b) holds. □

Moreover, it is easy to verify that, for each $\delta > r$, the following relations hold:

$$\Omega^r_{K_2} = (\Omega^r \cap B^\delta)_{K_2} \quad \text{and} \quad \overline{\Omega^r}_{K_2} = (\overline{\Omega^r \cap B^\delta})_{K_2}.$$

Theorem 4.3. *Assume that $f^i : N^+ \times \mathbb{R}^n_+ \to \mathbb{R}_+$ is continuous, $i = 1, 2, \ldots, n$. Furthermore, it is assumed that the following two hypotheses hold:*

(D_3) *For each $i = 1, 2, \ldots, n$, there exist a constant $\alpha > 0$ and a continuous function $\psi_i : N \to (0, \infty)$ such that*

$$f^i(j, y) \ge \frac{1}{T+1}\alpha\psi_i(j),\ \textit{for all } j \in N,\ 0 \le y_l \le \alpha\ (l \in \{1, 2, \ldots, n\}\backslash\{i\})$$

and $\frac{\alpha}{T+1} \le y_i \le \alpha$ *and*

$$\min_{k\in N} \sum_{j=1}^{T} G(k,j)\psi_i(j) \ge 1;$$

(D_4) *For each* $i = 1, 2, \ldots, n$*, there exist a constant* $\beta > 0$ *and a continuous function* $\chi_i : N \to (0, \infty)$ *such that*

$$f^i(j,y) \le \beta\chi_i(j) \qquad \textit{for all } j \in N,\ 0 < y_k \le \beta,\ k \in N$$

and

$$\max_{k\in N} \sum_{j=1}^{T} G(k,j)\chi_i(j) \le 1.$$

Then, the following results hold:

(a) *if* $\beta < \frac{\alpha}{T+1}$*, then problem* (1.3) *has at least one positive solution* y *satisfying*

$$\beta \le \|y\| = \max_{i\in\{1,\ldots,n\}} \max_{k\in N^+} |y_i(k)| \le \alpha;$$

(b) *if* $\alpha < \beta$*, then problem* (1.3) *has at least one positive solution* y *satisfying*

$$\frac{1}{T+1}\alpha \le \|y\| \le \beta.$$

Proof. We show that:

(i) $\|Ty\| \le \|y\|$ for $y \in \partial_{K_2} B^\beta$, and
(ii) there exists $e \in K_2\backslash\{0\}$ such that $y \ne Ty + \lambda e$, for all $y \in \partial_{K_2}\Omega^\alpha$ and all $\lambda > 0$.

We start with (i). Now for any $y \in \partial_{K_2} B^\beta$, we have $|y_i|_0 \le \beta$ for each $i \in \{1, \ldots, n\}$. Fix $i \in \{1, \ldots, n\}$. Then from (D_4) we obtain, for each $k \in N$,

$$\begin{aligned}(T_i y)(k) &= \sum_{j=1}^{T} G(k,j) f^i(j, y(j)) \le \beta \sum_{j=1}^{T} G(k,j)\chi_i(j)\\ &\le \beta \max_{k\in N} \sum_{j=1}^{T} G(k,j)\chi_i(j) \le \beta.\end{aligned}$$

Hence, $|T_i y|_0 \le \|y\|$ for each $i \in \{1, \ldots, n\}$. This implies that (i) holds.

Next we consider part (ii). Let $e(t) \equiv 1$, so $e \in K_2 \setminus \{0\}$. Next, suppose that there exists $y \in \partial_{K_2}\Omega^\alpha$ and $\lambda > 0$ such that $y = Ty + \lambda e$. Since $y \in \partial_{K_2}\Omega^\alpha$, then from Lemma 4.2 (d) there exists $i \in \{1, 2, \ldots, n\}$ with $\frac{\alpha}{T+1} \le y_i(k) \le \alpha,\ k \in N$, and $0 \le y_j(k) \le \alpha$ for $k \in N$ and $j \in \{1, 2, \ldots, n\}\backslash\{i\}$.

From (D_3) we have, for $k \in N$, that

$$\begin{aligned} y_i(k) &= (T_i y)(k) + \lambda = \sum_{j=1}^{T} G(k,j) f^i(j, y(j)) + \lambda \\ &\geq \frac{1}{T+1} \alpha \sum_{j=1}^{T} G(k,j)\psi_i(j) + \lambda \\ &\geq \frac{1}{T+1} \alpha \min_{k\in N} \sum_{j=1}^{T} G(k,j)\psi_i(j) + \lambda \geq \frac{1}{T+1}\alpha + \lambda. \end{aligned}$$

Hence $\min_{k\in N} y_i(k) \geq \dfrac{1}{T+1}\alpha + \lambda > \dfrac{1}{T+1}\alpha$, contradicting the statement of Lemma 4.2 (c). This contradiction proves part (ii) above.

Now suppose that $\beta < \frac{\alpha}{T+1}$. Then one has from Lemma 4.2 that $\overline{B^\beta}_{K_2} \subset B_{K_2}^{\frac{\alpha}{T+1}} \subset \Omega_{K_2}^\alpha$ and therefore it follows from Theorem 2.6 that T has at least one fixed point $y \in \overline{\Omega_{K_2}^\alpha} \setminus B_{K_2}^\beta$. Hence $\|y\| \geq \beta$ and $\frac{1}{T+1}\beta \leq \min_{k\in N} y_i(k) \leq \dfrac{\alpha}{T+1}$. On the other hand, $\frac{1}{T+1}|y_i|_0 \leq \min_{k\in N} y_i(k) \leq \dfrac{\alpha}{T+1}$ and therefore $|y_i|_0 \leq \alpha$ for each $i \in \{1,2,\ldots,n\}$. This implies that $\|y\| \leq \alpha$.

Finally, if $\alpha < \beta$ one has $\overline{\Omega^\alpha}_{K_2} \subset B_{K_2}^\beta$, and then Theorem 2.6 guarantees the existence of at least one fixed point $y \in \overline{B^\beta}_{K_2} \setminus \Omega_{K_2}^\alpha$ of T. Hence we obtain the inequality $\frac{\alpha}{T+1} \leq \|y\| \leq \beta$. $\square$

Next we employ Theorem 4.3 to establish the existence of positive solutions to the following second order discrete system

$$\begin{cases} \Delta^2 y_i(k-1) + h_i(k) g^i(y(k)) = 0, \quad k \in N, \\ y(0) = 0, \quad y(T+1) = 0, \quad i = 1,2,\ldots,n. \end{cases} \tag{4.2}$$

We assume that

(H_3) $g^i : \mathbb{R}_+^n \to \mathbb{R}_+$ is continuous with $g^i(y) > 0$ for $\|y\| > 0$, $i = 1,2,\ldots,n$

(H_4) $h_i(j) : N \to \mathbb{R}_+$ is continuous and $\sum_{j=1}^{T} G(k,j)h_i(j) > 0$, $i = 1,2,\ldots,n$

Theorem 4.4. *Suppose that conditions* (H_3)–(H_4) *hold. Then problem* (4.2) *has at least one positive solution* y *with* $y(k) \not\equiv 0$ *for* $k \in N$ *if one of the following conditions holds.*

(h_3) $0 \leq g_0^i < A_i^{-1}$ *and* $B_i^{-1} < g_\infty^i \leq \infty$, $i = 1,2,\ldots,n$;

(h_4) $0 \leq g_\infty^i < A_i^{-1}$ *and* $B_i^{-1} < g_0^i \leq \infty$, $i = 1,2,\ldots,n$;

here $g_0^i = \lim_{y\to 0^+} \dfrac{g(y)}{\|y\|}$, $\quad g_\infty^i = \lim_{y\to\infty} \dfrac{g(y)}{\|y\|}$, $i = 1,2,\ldots,n$, *and*

$$A_i = \max_{k\in N} \sum_{j=1}^{T} G(k,j)h_i(j), \quad B_i = \min_{k\in N} \sum_{j=1}^{T} G(k,j)h_i(j).$$

Proof. To see this, we will apply Theorem 4.3 with $f^i(k,y) = h_i(k)g^i(y)$, $i = 1,2,\dots,n$. We assume that (h_3) holds. The case when (h_4) holds is similar.

From the first part of (h_3), there exists $\beta > 0$ such that $g^i(y) \le A_i^{-1}\beta$ for $0 < \|y\| \le \beta$. Choose $\chi_i(k) = A_i^{-1}h_i(k)$ for $i = 1,2,\dots,n$. Fix $i \in \{1,\dots,n\}$. Then

$$f^i(k,y) = h_i(k)g^i(y) \le A_i^{-1}\beta h_i(k) = \beta\chi_i(k) \quad \text{ if } \ k \in N \text{ and } \ 0 < y_j \le \beta,$$

for $j \in \{1,\dots,n\}$ and

$$\sum_{j=1}^{T} G(k,j)\chi_i(j) = A_i^{-1}\sum_{j=1}^{T} G(k,j)h_i(j) \le A_i^{-1}\max_{k\in N}\sum_{j=1}^{T} G(k,j)h_i(j) = 1.$$

Thus hypothesis $(\mathtt{D}_4)$ holds.

From the second part of (h_3), there exists $\alpha > 0$ such that $\frac{1}{T+1}\alpha > \beta$ and $g^i(y) \ge B_i^{-1}\frac{1}{T+1}\alpha$ for $\|y\| \ge \frac{1}{T+1}\alpha$, $i = 1,2,\dots,n$.

Thus $g^i(y) \ge B_i^{-1}\frac{1}{T+1}\alpha$ for $y_i \ge \frac{1}{T+1}\alpha$, $i = 1,2,\dots,n$. Choose $\psi_i(k) = B_i^{-1}h_i(k)$, then

$$f^i(k,y) = h_i(k)g^i(y) \ge B_i^{-1}\frac{1}{T+1}\alpha h_i(k) = \frac{1}{T+1}\alpha\psi_i(k), \text{ if } k \in N,\ y_i \ge \frac{1}{T+1}\alpha,$$

(so in particular for $\frac{1}{T+1}\alpha \le y_i \le \alpha$) and

$$\sum_{j=1}^{T} G(k,j)\psi_i(j) = B_i^{-1}\sum_{j=1}^{T} G(k,j)h_i(j) \ge B_i^{-1}\min_{k\in N}\sum_{j=1}^{T} G(k,j)h_i(j) = 1.$$

This implies that hypothesis $(\mathtt{D}_3)$ holds. The result now follows from Theorem 4.3. $\square$

Now we characterize the eigenvalue intervals for the systems (1.7). Let

$$A = \max\{A_i, i = 1,2,\dots,n\}, \quad B = \min\{B_i, i = 1,2,\dots,n\}.$$

Theorem 4.5. *Suppose that conditions* $(\mathtt{H}_3)$-$(\mathtt{H}_4)$ *hold. Then problem* (1.7) *has at least one positive solution for each*

$$\lambda \in \Big(\frac{1}{B\min\limits_{i=1,2,\dots,n}\{g_\infty^i\}}, \frac{1}{A\max\limits_{i=1,2,\dots,n}\{g_0^i\}}\Big) \tag{4.3}$$

if $\dfrac{1}{B\min\limits_{i=1,2,\dots,n}\{g_\infty^i\}} < \dfrac{1}{A\max\limits_{i=1,2,\dots,n}\{g_0^i\}}$. *The same result remains valid for each*

$$\lambda \in \Big(\frac{1}{B\min\limits_{i=1,2,\dots,n}\{g_0^i\}}, \frac{1}{A\max\limits_{i=1,2,\dots,n}\{g_\infty^i\}}\Big) \tag{4.4}$$

if $\dfrac{1}{B\min\limits_{i=1,2,\dots,n}\{g_0^i\}} < \dfrac{1}{A\max\limits_{i=1,2,\dots,n}\{g_\infty^i\}}$. *Here we write* $1/g_\alpha^i = 0$ *if* $g_\alpha^i = \infty$ *and* $1/g_\alpha^i = \infty$ *if* $g_\alpha^i = 0$, *where* $\alpha = 0, \infty$.

Proof. We consider the case (4.3). The case (4.4) is similar. If λ satisfies (4.3), then

$$\lambda g_0^i \le \lambda \max_{i=1,2,\dots,n} \{g_0^i\} < \frac{1}{A} \le \frac{1}{A_i}, \qquad i = 1, 2, \dots, n,$$

and

$$\lambda g_\infty^i \ge \lambda \min_{i=1,2,\dots,n} \{g_\infty^i\} > \frac{1}{B} \ge \frac{1}{B_i}, \qquad i = 1, 2, \dots, n.$$

So Theorem 4.4 applies directly. □

5. ONE-DIMENSIONAL p-LAPLACIAN

In this section we establish the existence of positive solutions to the one-dimensional discrete $p-$Laplacian problem (1.2) and characterize the eigenvalue intervals for systems (1.6).

Define the operator

$$Ty(k) = \begin{cases} 0, \qquad k = 0 \text{ or } T+1, \\ \displaystyle\sum_{s=k}^{T} \phi^{-1}(\tau + \sum_{r=1}^{s} f(r, y(r))), \; k \in N, \end{cases}$$

where τ is a solution of the equation

$$\phi^{-1}(\tau) + \sum_{k=1}^{T} \phi^{-1}(\tau + \sum_{r=1}^{k} f(r, u(r))) = 0.$$

Lemma 5.1 ([10]). *Assume that $f : N^+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is continuous. Then T is well defined and maps K_1 into K_1. Moreover, T is continuous and completely continuous; here K_1 is defined by* (3.2)*.*

Theorem 5.2. *Assume that $f : N^+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is continuous. Furthermore, it is assumed that the following two hypotheses hold:*

(F_1) *there exist a constant $\alpha > 0$ and a continuous function $\psi : N \to (0, \infty)$ such that*

$$f(j, y) \ge \phi(\frac{1}{T+1}\alpha)\psi(j), \qquad \text{for all } j \in N, \; \frac{\alpha}{T+1} \le y \le \alpha$$

and

$$\min_{k \in N} P(k) \ge 1;$$

here we assume $P(k)$ is the unique solution of

$$\begin{cases} \Delta(\phi(\Delta P(k-1))) + \psi(k) = 0, \; k \in N, \\ P(0) = 0, \quad P(T+1) = 0; \end{cases}$$

(F_2) *there exist a constant* $\beta > 0$ *and a continuous function* $\chi : N \to (0, \infty)$ *such that*

$$f(j, y) \leq \phi(\beta)\chi(j) \qquad \text{for all } j \in N,\ 0 < y \leq \beta$$

and

$$\max_{k \in N} Q(k) \leq 1;$$

here we assume $Q(k)$ *is the unique solution of*

$$\begin{cases} \Delta(\phi(\Delta Q(k-1))) + \chi(k) = 0,\ k \in N, \\ Q(0) = 0, \quad Q(T+1) = 0. \end{cases}$$

Then, the following results hold:

(a) *if* $\beta < \frac{\alpha}{T+1}$, *then problem* (1.2) *has at least one positive solution* y *satisfying*

$$\beta \leq |y|_0 \leq \alpha;$$

(b) *if* $\alpha < \beta$, *then problem* (1.2) *has at least one positive solution* y *satisfying*

$$\frac{1}{T+1}\alpha \leq |y|_0 \leq \beta.$$

Remark 5.3 ([7, 16])**.** For each continuous function $\alpha : N \to (0, \infty)$, the linear problem

$$\begin{cases} \Delta(\phi(\Delta y(k-1))) + \alpha(k) = 0,\ k \in N, \\ y(0) = 0, \quad y(T+1) = 0, \end{cases}$$

has exactly one positive solution. This guarantees the existence and uniqueness of $P(k)$ and $Q(k)$.

Proof of Theorem 5.2. We show that :

(i) $|Ty|_0 \leq |y|_0$ for $y \in \partial_{K_1} B^\beta$, and
(ii) there exists $e \in K_1 \backslash \{0\}$ such that $y \neq Ty + \lambda e$, for all $y \in \partial_{K_1} \Omega^\alpha$ and all $\lambda > 0$.

We start with (i). Now for any $y \in \partial_{K_1} B^\beta$, we have $|y|_0 = \beta$. Then from (F_2), $f(k, y) \leq \phi(\beta)\chi(k)$for each $k \in N$. By Lemma 2.4, we obtain

$$Ty(k) \leq \beta Q(k) \leq \beta \max_{k \in N} Q(k) \leq \beta.$$

Hence, $|Ty|_0 \leq |y|_0$ for each $y \in \partial_{K_1} B^\beta$ and so (i) holds.

Next we consider part (ii). Let $e(t) \equiv 1$, so $e \in K_1 \setminus \{0\}$. Next, suppose that there exists $y \in \partial_{K_1} \Omega^\alpha$ and $\lambda > 0$ such that $y = Ty + \lambda e$. Since $y \in \partial_{K_1} \Omega^\alpha$, then from Lemma 3.2 (d), we have $\frac{1}{T+1}\alpha \leq y(k) \leq \alpha,\ k \in N$.

From (F_1), $f(k,y(k)) \geq \phi(\frac{1}{T+1}\alpha)\psi(k)$ for $k \in N$. By Lemma 2.4, for each $k \in N$, we have

$$\begin{aligned} y(k) = Ty(k) + \lambda &\geq \frac{1}{T+1}\alpha P(k) + \lambda \\ &\geq \frac{1}{T+1}\alpha \min_{k\in N} P(k) + \lambda \geq \frac{1}{T+1}\alpha + \lambda. \end{aligned}$$

Hence $\min_{k\in N} y(k) \geq \dfrac{1}{T+1}\alpha + \lambda > \dfrac{1}{T+1}\alpha$, contradicting the statement of Lemma 3.2 (c). This contradiction proves part (ii) of our claim.

The rest of the proof is similar to that in the proof of Theorem 3.3, so we omit the details. □

Next we employ Theorem 5.2 to establish the existence of positive solutions to the following discrete problem

$$\begin{cases} \Delta(\phi(\Delta y(k-1))) + h(k)g(y(k)) = 0, \ k \in N, \\ y(0) = 0, \quad y(T+1) = 0. \end{cases} \tag{5.1}$$

We assume (H_1) holds. Moreover, it is assumed that

(H_5) $H(k) > 0$ for $k \in N$; here we assume $H(k)$ is the unique solution of

$$\begin{cases} \Delta(\phi(\Delta y(k-1))) + h(k) = 0, \ k \in N, \\ y(0) = 0, \quad y(T+1) = 0. \end{cases}$$

Theorem 5.4. *Suppose that conditions* ($\mathtt{H}_1$) *and* ($\mathtt{H}_5$) *hold. Then problem* (5.1) *has at least one positive solution* y *with* $y(k) \not\equiv 0$ *for* $k \in N$ *if one of the following conditions holds.*

(f_1) $0 \leq g_0 < (\frac{1}{M})^{p-1}$ *and* $(\frac{1}{m})^{p-1} < g_\infty \leq \infty$;

(f_2) $0 \leq g_\infty < (\frac{1}{M})^{p-1}$ *and* $(\frac{1}{m})^{p-1} < g_0 \leq \infty$;

here $g_0 = \displaystyle\lim_{y\to 0^+} \frac{g(y)}{y^{p-1}}, \quad g_\infty = \lim_{y\to\infty} \frac{g(y)}{y^{p-1}}$, *and*

$$M = \max_{k\in N} H(k), \quad m = \min_{k\in N} H(k).$$

Proof. To see this, we will apply Theorem 5.2 with $f(k,y) = h(k)g(y)$. We assume that (f_1) holds. The case when (f_2) holds is similar.

From the first part of (f_1), there exists $\beta > 0$ such that $g(y) \leq (\frac{1}{M})^{p-1}\beta^{p-1}$ for $0 < y \leq \beta$. Choose $\chi(k) = (\frac{1}{M})^{p-1}h(k)$. Then

$$f(k,y) = h(k)g(y) \leq (\frac{1}{M})^{p-1}\beta^{p-1}h(k) = \beta^{p-1}\chi(k) \qquad \text{if} \ \ k \in N \ \ \text{and} \ \ 0 < y \leq \beta,$$

and

$$Q(k) = M^{-1}H(k) \leq M^{-1}\max_{k\in N} H(k) = 1.$$

Thus hypothesis ($\mathtt{F}_2$) holds.

From the second part of (f_1), there exists $\alpha > 0$ such that $\frac{1}{T+1}\alpha > \beta$ and $g(y) \geq (\frac{1}{m})^{p-1}(\frac{1}{T+1}\alpha)^{p-1}$ for $y \geq \frac{1}{T+1}\alpha$. Choose $\psi(k) = (\frac{1}{m})^{p-1}h(k)$, then

$$f(k,y) = h(k)g(y) \geq (\frac{1}{m})^{p-1}(\frac{1}{T+1}\alpha)^{p-1}h(k) = (\frac{1}{T+1}\alpha)^{p-1}\psi(k),$$

$k \in N,\ y \geq \frac{1}{T+1}\alpha,$ (so in particular for $\frac{1}{T+1}\alpha \leq y \leq \alpha$) and

$$P(k) = m^{-1}H(k) \geq m^{-1}\min_{k\in N} H(k) = 1.$$

This implies that hypothesis $(\mathtt{F}_1)$ holds. The result now follows from Theorem 5.2. □

By employing Theorem 5.4, we can easily characterize the eigenvalue intervals for (1.6).

Theorem 5.5. *Suppose that conditions* $(\mathtt{H}_1)$ *and* $(\mathtt{H}_5)$ *hold. Then problem* (1.6) *has at least one positive solution for each*

$$\lambda \in (\frac{1}{m^{p-1}g_\infty}, \frac{1}{M^{p-1}g_0})$$

if $\frac{1}{m^{p-1}g_\infty} < \frac{1}{M^{p-1}g_0}$. *The same result remains valid for each*

$$\lambda \in (\frac{1}{m^{p-1}g_0}, \frac{1}{M^{p-1}g_\infty})$$

if $\frac{1}{m^{p-1}g_0} < \frac{1}{M^{p-1}g_\infty}$.

6. p-LAPLACIAN SYSTEMS

In this section we consider the discrete p-Laplacian system (1.4) and (1.8). Define the operator $T : X_2 \to X_2$ by

$$Ty = (T_1y, T_2y, \ldots, T_ny);$$

here

$$(T_iy)(k) = \begin{cases} 0, & k = 0 \text{ or } T+1, \\ \displaystyle\sum_{s=k}^{T} \phi^{-1}(\tau_i + \sum_{r=1}^{s} f^i(r, y(r))), & k \in N, \end{cases} \qquad i = 1, 2, \ldots, n,$$

and τ_i is a solution of the equation

$$\phi^{-1}(\tau) + \sum_{k=1}^{T} \phi^{-1}(\tau + \sum_{r=1}^{k} f^i(r, u(r))) = 0, \quad i = 1, 2, \ldots, n.$$

Lemma 6.1. *Assume that* $f^i : N^+ \times \mathbb{R}^n_+ \to \mathbb{R}_+$ *is continuous,* $i = 1, 2, \ldots, n$. *Then* T *is well defined and maps* K_2 *into* K_2. *Moreover,* T *is continuous and completely continuous; here* K_2 *is defined by* (4.1).

Theorem 6.2. *Assume that* $f^i : N^+ \times \mathbb{R}^n_+ \to \mathbb{R}_+$ *is continuous,* $i = 1, 2, \ldots, n$. *Furthermore, it is assumed that the following two hypotheses hold:*

(F$_3$) *For each* $i = 1, 2, \ldots, n$, *there exist a constant* $\alpha > 0$ *and a continuous function* $\psi_i : N \to (0, \infty)$ *such that*

$$f^i(j, y) \geq \phi(\frac{1}{T+1}\alpha)\psi_i(j),\ \text{for all}\ j \in N,\ 0 \leq y_l \leq \alpha\ (l \in \{1, 2, \ldots, n\}\backslash\{i\})$$

and $\frac{\alpha}{T+1} \leq y_i \leq \alpha$ *and*

$$\min_{k\in N} P_i(k) \geq 1;$$

here we assume $P_i(k)$ *is the unique solution of*

$$\begin{cases} \Delta(\phi(\Delta P(k-1))) + \psi_i(k) = 0,\ k \in N, \\ P(0) = 0, \quad P(T+1) = 0; \end{cases}$$

(F$_4$) *For each* $i = 1, 2, \ldots, n$, *there exist a constant* $\beta > 0$ *and a continuous function* $\chi_i : N \to (0, \infty)$ *such that*

$$f^i(j, y) \leq \phi(\beta)\chi_i(j) \qquad \text{for all}\ j \in N,\ 0 < y_k \leq \beta,\ k \in N$$

and

$$\max_{k\in N} Q_i(k) \leq 1;$$

here we assume $Q_i(k)$ *is the unique solution of*

$$\begin{cases} \Delta(\phi(\Delta Q(k-1))) + \chi_i(k) = 0,\ k \in N, \\ Q(0) = 0, \quad Q(T+1) = 0. \end{cases}$$

Then, the following results hold:

(a) *if* $\beta < \frac{\alpha}{T+1}$, *then problem* (1.4) *has at least one positive solution* y *satisfying*

$$\beta \leq \|y\| = \max_{i\in\{1,\ldots,n\}} \max_{k\in N^+} |y_i(k)| \leq \alpha;$$

(b) *if* $\alpha < \beta$, *then problem* (1.4) *has at least one positive solution* y *satisfying*

$$\frac{1}{T+1}\alpha \leq \|y\| \leq \beta.$$

Proof. We show that :

(i) $\|Ty\| \leq \|y\|$ for $y \in \partial_{K_2} B^\beta$, and
(ii) there exists $e \in K_2\backslash\{0\}$ such that $y \neq Ty + \lambda e$, for all $y \in \partial_{K_2}\Omega^\alpha$ and all $\lambda > 0$.

We start with (i). Now for any $y \in \partial_{K_2} B^\beta$, we have $|y_i|_0 \leq \beta$ for each $i \in \{1, 2, \ldots, n\}$. Fix $i \in \{1, 2, \ldots, n\}$. Then from (F$_4$), $f^i(k, y) \leq \phi(\beta)\chi_i(k)$for each $k \in N$. By Lemma 2.4, we obtain

$$(T_i y)(k) \leq \beta Q_i(k) \leq \beta \max_{k\in N} Q_i(k) \leq \beta.$$

Hence, $|T_i y|_0 \leq \|y\|$ for each $i \in \{1, 2, \ldots, n\}$. This implies that (i) holds.

Next we consider inequality (ii). Let $e(t) \equiv 1$, so $e \in K_2 \setminus \{0\}$. Next, suppose that there exists $y \in \partial_{K_2}\Omega^\alpha$ and $\lambda > 0$ such that $y = Ty + \lambda e$. Since $y \in \partial_{K_2}\Omega^\alpha$, then from Lemma 4.2 (d), there exists $i \in \{1, 2, \dots, n\}$ with $\frac{1}{T+1}\alpha \le y_i(k) \le \alpha,\ k \in N$ and $0 \le y_j(k) \le \alpha$ for $j \in \{1, 2, \dots, n\} \backslash \{i\}$.

From (F_3), $f^i(k, y(k)) \ge \phi(\frac{1}{T+1}\alpha)\psi_i(k)$ for $k \in N$. By Lemma 2.4, for each $k \in N$, we have

$$\begin{aligned} y_i(k) &= (T_i y)(k) + \lambda \ge \frac{1}{T+1}\alpha P_i(k) + \lambda \\ &\ge \frac{1}{T+1}\alpha \min_{k\in N} P_i(k) + \lambda \ge \frac{1}{T+1}\alpha + \lambda. \end{aligned}$$

Hence $\min_{k\in N} y_i(k) \ge \dfrac{1}{T+1}\alpha + \lambda > \dfrac{1}{T+1}\alpha$, contradicting the statement of Lemma 4.2 (c). This contradiction proves part (ii) above.

The rest of the proof is similar to that in the proof of Theorem 4.3, so we omit the details. □

Next we employ Theorem 5.2 to establish the existence of positive solutions to the following discrete system

$$\begin{cases} \Delta(\phi(\Delta y(k-1))) + h_i(k) g^i(y(k)) = 0,\ k \in N, \\ y(0) = 0, \quad y(T+1) = 0. \end{cases} \tag{6.1}$$

We assume that (H_3) holds. Moreover, it is assumed that

(H_6) $\widetilde{H_i}(k) > 0$ for $k \in N,\ i = 1, 2, \dots, n$; here we assume $\widetilde{H_i}(k)$ is the unique solution of
$$\begin{cases} \Delta(\phi(\Delta y(k-1))) + h_i(k) = 0,\ k \in N, \\ y(0) = 0, \quad y(T+1) = 0. \end{cases}$$

Theorem 6.3. *Suppose that conditions* ($\mathtt{H}_3$) *and* ($\mathtt{H}_6$) *hold. Then problem* (6.1) *has at least one positive solution* y *with* $y(k) \not\equiv 0$ *for* $k \in N$ *if one of the following conditions holds.*

(f_3) $0 \le g_0^i < (\frac{1}{M_i})^{p-1}$ *and* $(\frac{1}{m_i})^{p-1} < g_\infty^i \le \infty,\ i = 1, 2, \dots, n;$

(f_4) $0 \le g_\infty^i < (\frac{1}{M_i})^{p-1}$ *and* $(\frac{1}{m_i})^{p-1} < g_0^i \le \infty,\ i = 1, 2, \dots, n;$

here $g_0^i = \lim_{y\to 0^+} \dfrac{g^i(y)}{y^{p-1}}, \quad g_\infty^i = \lim_{y\to\infty} \dfrac{g^i(y)}{y^{p-1}}$, *and*

$$M_i = \max_{k\in N} \widetilde{H_i}(k), \quad m_i = \min_{k\in N} \widetilde{H_i}(k).$$

Proof. To see this, we will apply Theorem 6.2 with $f^i(k, y) = h_i(k) g^i(y),\ i = 1, 2, \dots, n$. We assume that ($f_3$) holds. The case when ($f_4$) holds is similar.

From the first part of (f_3), there exists $\beta > 0$ such that $g^i(y) \le (\frac{1}{M_i})^{p-1}\beta^{p-1}$ for $0 < \|y\| \le \beta$. Choose $\chi_i(k) = (\frac{1}{M_i})^{p-1} h_i(k)$ for $i = 1, 2, \dots, n$. Fix $i \in \{1, \dots, n\}$. Choose $\chi_i(k) = (\frac{1}{M_i})^{p-1}$. Then

$$f^i(k, y) = h_i(k) g^i(y) \le (\frac{1}{M_i})^{p-1}\beta^{p-1} h_i(k) = \beta^{p-1}\chi_i(k) \qquad \text{if } \ k \in N$$

and $0 < y_j \le \beta,\ j \in \{1, \dots, n\}$ and

$$Q_i(k) = M_i^{-1}\widetilde{H_i}(k) \le M_i^{-1} \max_{k\in N} \widetilde{H_i}(k) = 1.$$

Thus hypothesis $(\mathtt{F}_4)$ holds.

From the second part of (f_3), there exists $\alpha > 0$ such that $\frac{1}{T+1}\alpha > \beta$ and $g^i(y) \ge (\frac{1}{m_i})^{p-1}(\frac{1}{T+1}\alpha)^{p-1}$ for $\|y\| \ge \frac{1}{T+1}\alpha,\ i = 1, 2, \dots, n$. Thus $g^i(y) \ge (\frac{1}{m_i})^{p-1}(\frac{1}{T+1}\alpha)^{p-1}$ for $y_i \ge \frac{1}{T+1}\alpha,\ i = 1, 2, \dots, n$. Fix $i \in \{1, \dots, n\}$. Choose $\psi_i(k) = (\frac{1}{m_i})^{p-1}h_i(k)$, then

$$f^i(k, y) = h_i(k)g^i(y) \ge (\frac{1}{m_i})^{p-1}(\frac{1}{T+1}\alpha)^{p-1}h_i(k) = (\frac{1}{T+1}\alpha)^{p-1}\psi_i(k), \quad k \in N,$$

$y_i \ge \frac{1}{T+1}\alpha$, (so in particular for $\frac{1}{T+1}\alpha \le y_i \le \alpha$) and

$$P_i(k) = m_i^{-1}\widetilde{H_i}(k) \ge m_i^{-1} \min_{k\in N} \widetilde{H_i}(k) = 1.$$

This implies that hypothesis $(\mathtt{F}_3)$ holds. The result now follows from Theorem 6.2. □

Finally we employ Theorem 6.3 we characterize the eigenvalue intervals for system (1.8). Here we only state the results and omit the proof since it is similar to that of Theorem 4.5.

Let

$$M = \max\{M_i, i = 1, 2, \dots, n\}, \quad m = \min\{m_i, i = 1, 2, \dots, n\}.$$

Theorem 6.4. *Suppose that conditions* $(\mathtt{H}_3)$ *and* $(\mathtt{H}_6)$ *hold. Then problem* (1.8) *has at least one positive solution for each*

$$\lambda \in \left(\frac{1}{m^{p-1} \min\limits_{i=1,2,\dots,n}\{g_\infty^i\}}, \frac{1}{M^{p-1} \max\limits_{i=1,2,\dots,n}\{g_0^i\}}\right)$$

if $\dfrac{1}{m^{p-1} \min\limits_{i=1,2,\dots,n}\{g_\infty^i\}} < \dfrac{1}{M^{p-1} \max\limits_{i=1,2,\dots,n}\{g_0^i\}}$. *The same result remains valid for each*

$$\lambda \in \left(\frac{1}{m^{p-1} \min\limits_{i=1,2,\dots,n}\{g_0^i\}}, \frac{1}{M^{p-1} \max\limits_{i=1,2,\dots,n}\{g_\infty^i\}}\right)$$

if $\dfrac{1}{m^{p-1} \min\limits_{i=1,2,\dots,n}\{g_0^i\}} < \dfrac{1}{M^{p-1} \max\limits_{i=1,2,\dots,n}\{g_\infty^i\}}$.

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