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NONLOCAL BOUNDARY VALUE PROBLEMS WITH TWO NONLINEAR BOUNDARY CONDITIONS

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ABSTRACT. We establish new results on the existence of positive solutions for some nonlocal boundary value problems subject to two nonlinear boundary conditions. Our approach relies on the classical fixed point index theory for compact maps.

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1. INTRODUCTION

In this paper we discuss the existence of positive solutions for the second order differential equation

$$u''(t) + g(t)f(t, u(t)) = 0,\ t \in (0, 1), \tag{1.1}$$

subject to some nonlinear boundary conditions (BCs). Problems of this type have been studied recently by several authors, see for example [2, 6, 8, 13, 14, 24, 25].

In particular we deal with the BCs

$$u'(0) + H_1(\alpha[u]) = 0,\ \sigma u'(1) + u(\eta) = H_2(\beta[u]),\ \eta \in [0, 1], \tag{1.2}$$

where H_1, H_2 are continuous functions such that there exist $h_{11}, h_{12}, h_{21}, h_{22} \in [0, \infty)$ with

$$h_{11}v \le H_1(v) \le h_{12}v \quad \text{and} \quad h_{21}v \le H_2(v) \le h_{22}v, \tag{1.3}$$

for every $v \ge 0$. Here $\alpha[u], \beta[u]$ are linear functionals given by

$$\alpha[u] = \int_0^1 u(s)\, dA(s), \quad \beta[u] = \int_0^1 u(s)\, dB(s), \tag{1.4}$$

involving Lebesgue-Stieltjes integrals.

When $H_1(v) = H_2(v) = v$, this type of BCs includes, as special cases, the so-called m-point problems

$$\alpha[u] = \sum_{i=1}^{m} \alpha_i u(\xi_i), \quad \beta[u] = \sum_{i=1}^{m} \beta_i u(\tau_i),$$

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and integral BCs

$$\alpha[u] = \int_0^1 \alpha(s)u(s)\,ds, \quad \beta[u] = \int_0^1 \beta(s)u(s)\,ds.$$

Multi-point and integral BCs are widely studied objects, see for example [3, 10, 11, 12, 17, 21, 23, 26] and the reference therein.

One motivation for studying this type of BVP is that it occurs in some heat flow problems. For example, the special case $\alpha[u] = u(\xi)$, $\beta[u] = u(\tau)$, namely,

$$u'(0) + H_1(u(\xi)) = 0,\ \sigma u'(1) + u(\eta) = H_2(u(\tau)),\ \xi, \eta, \tau \in [0,1], \tag{1.5}$$

arises in the study of the steady states of a heated bar of length 1. Here two controllers at $t = 0$ and $t = 1$ add or remove heat according to the temperatures detected by three sensors at $t = \xi$, $t = \eta$ and $t = \tau$.

Heat flow problems of this type have been investigated recently in [7, 8, 9, 18, 19, 20]. In particular Infante and Webb [9], motivated by an earlier work of Guidotti and Merino [5], studied the existence of positive solutions of the BVP

$$-u''(t) = f(t, u(t)),\ t \in (0,1), \quad u'(0) = 0,\ \sigma u'(1) + u(\eta) = 0, \tag{1.6}$$

that models a bar, insulated at $t = 0$, with a controller in $t = 1$ adding or removing heat depending on the temperature detected by a sensor at $t = \eta$.

The approach in [9] is to find the Green's function k corresponding to the BVP (1.6) and seek solutions as fixed points of an Hammerstein integral operator of the type

$$Fu(t) := \int_0^1 k(t,s)g(s)f(s,u(s))\,ds. \tag{1.7}$$

Our idea is to rewrite the BVP (1.1)–(1.2) as a perturbation of the integral equation $u(t) = Fu(t)$, that is

$$u(t) = \gamma(t)H_1(\alpha[u]) + \delta(t)H_2(\beta[u]) + \int_0^1 k(t,s)g(s)f(s,u(s))\,ds, \tag{1.8}$$

and seek fixed points of the perturbed Hammerstein operator

$$Tu(t) := \gamma(t)H_1(\alpha[u]) + \delta(t)H_2(\beta[u]) + Fu(t).$$

Our main ingredient is the classical theory of fixed poind index, that we utilize on a suitable cone of continuous functions, combined with some results from the paper [22].

2. FIXED POINT INDEX CALCULATIONS

We make the following assumptions on the terms that occur in (1.8):

- $f : [0,1] \times [0,\infty) \to [0,\infty)$ is continuous.
- $k : [0,1] \times [0,1] \to [0,\infty)$ is continuous.
- There exist a subinterval $[a,b] \subseteq [0,1]$, a function $\Phi \in L^\infty[0,1]$, and a constant $c_1 \in (0,1]$ such that
$$k(t,s) \le \Phi(s) \text{ for } t \in [0,1] \text{ and almost every } s \in [0,1],$$
$$k(t,s) \ge c_1\Phi(s) \text{ for } t \in [a,b] \text{ and almost every } s \in [0,1].$$
- $g\,\Phi \in L^1[0,1]$, $g \ge 0$ a.e., and $\int_a^b \Phi(s)g(s)\,ds > 0$.
- A, B are functions of bounded variation. Here dA and dB are *positive* measures and we use the notation
$$\mathcal{K}_A(s) := \int_0^1 k(t,s)\,dA(t) \text{ and } \mathcal{K}_B(s) := \int_0^1 k(t,s)\,dB(t).$$
- $\gamma \in C[0,1]$, $\gamma(t) \ge 0$, $h_{12}\alpha[\gamma] < 1$. There exists $c_2 \in (0,1]$ such that
$$\gamma(t) \ge c_2\|\gamma\| \text{ for } t \in [a,b].$$
- $\delta \in C[0,1]$, $\delta(t) \ge 0$, $h_{22}\beta[\delta] < 1$. There exists $c_3 \in (0,1]$ such that
$$\delta(t) \ge c_3\|\delta\| \text{ for } t \in [a,b].$$
- $D_2 := (1 - h_{12}\alpha[\gamma])(1 - h_{22}\beta[\delta]) - h_{12}h_{22}\alpha[\delta]\beta[\gamma] > 0.$

Under the hypotheses above, we can work in the cone
$$K = \{u \in C[0,1],\ u \ge 0 : \min_{t\in[a,b]} u(t) \ge c\|u\|\}, \tag{2.1}$$
where $c = \min\{c_1, c_2, c_3\}$. This type of cone has been used firstly by Krasnosel'skiĭ, see e.g. [15], and D. Guo, see e.g. [4], and later by several authors.

A routine check shows that, under the above hypotheses, T maps K into K and is compact.

Remark 2.1. In [22] the authors work with *linear* BCs and are able to handle *sign-changing* measures. Here, since we want our functionals α, β to be inequality preserving, we deal with positive measures only.

Before our index calculations, we recall some useful facts concerning real 2×2 matrices:

Definition 2.2 ([22])**.** A 2×2 matrix $\mathcal{M}$ is said to be order preserving (or positive) if $p_1 \ge p_0$, $q_1 \ge q_0$ imply
$$\mathcal{M}\begin{pmatrix} p_1 \\ q_1 \end{pmatrix} \ge \mathcal{M}\begin{pmatrix} p_0 \\ q_0 \end{pmatrix},$$
in the sense of components.

Lemma 2.3. ([22]) *Let*

$$\mathcal{M} = \begin{pmatrix} a & -b \\ -c & d \end{pmatrix}$$

with $a, b, c, d \geq 0$ *and* $\det \mathcal{M} > 0$. *Then* $\mathcal{M}^{-1}$ *is order preserving.*

Lemma 2.4. ([22]) *Let* $\mathcal{M}$ *satisfy the hypotheses of Lemma 2.3. Suppose* $p \geq 0, q \geq 0$ *and*

$$\mathcal{M}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} p \\ q \end{pmatrix} \quad \text{and} \quad \mathcal{M}_\mu \begin{pmatrix} x_\mu \\ y_\mu \end{pmatrix} = \begin{pmatrix} p \\ q \end{pmatrix},$$

where $\mathcal{M}_\mu = \mu I + \mathcal{M}$ *with* $\mu \geq 0$. *Then* $x_\mu \leq x$ *and* $y_\mu \leq y$.

We make use of the following open bounded sets (relative to K):

$$K_\rho = \{u \in K : \|u\| < \rho\},\ V_\rho = \{u \in K : \min_{t\in[a,b]} u(t) < \rho\}.$$

The set V_ρ is equal to the set called $\Omega_{\rho/c}$ in [16] (c here is from (2.1)). Note that $K_\rho \subset V_\rho \subset K_{\rho/c}$.
We also utilize the quantity

$$D_1 := (1 - h_{11}\alpha[\gamma])(1 - h_{21}\beta[\delta]) - h_{11}h_{21}\alpha[\delta]\beta[\gamma].$$

The condition $D_2 > 0$ implies $D_1 > 0$.

The following Lemma shows that the index is 0 on the set V_ρ.

Lemma 2.5. *Assume that there exists* $\rho > 0$ *such that*

$$\begin{aligned} f_{\rho,\rho/c}\Big(\big(\frac{c_2\|\gamma\|}{D_1}(1 - h_{21}\beta[\delta]) + \frac{c_3\|\delta\|}{D_1}h_{11}\beta[\gamma]\big) \int_a^b \mathcal{K}_A(s)g(s)\,ds \\ + \big(\frac{c_2\|\gamma\|}{D_1}h_{21}\alpha[\delta] + \frac{c_3\|\delta\|}{D_1}(1 - h_{11}\alpha[\gamma])\big) \int_a^b \mathcal{K}_B(s)g(s)\,ds + \frac{1}{M}\Big) > 1, \end{aligned} \tag{2.2}$$

where

$$f_{\rho,\rho/c} = \inf\Big\{\frac{f(t,u)}{\rho} :\ (t,u) \in [a,b]\times[\rho,\rho/c]\Big\} \text{ and } \frac{1}{M} = \inf_{t\in[a,b]} \int_a^b k(t,s)g(s)\,ds. \tag{2.3}$$

Then $i_K(T, V_\rho) = 0$.

Proof. Let $e(t) \equiv 1$ for $t \in [0,1]$. Then $e \in K$. We prove that

$$u \neq Tu + \lambda e \quad \text{for } u \in \partial V_\rho \quad \text{and } \lambda \geq 0,$$

which ensures that the index is 0 on the set V_ρ. In fact, if this is not so, there exist $u \in \partial V_\rho$ and $\lambda \geq 0$ such that $u = Tu + \lambda e$. Then we have

$$u(t) = \gamma(t)H_1(\alpha[u]) + \delta(t)H_2(\beta[u]) + Fu(t) + \lambda$$

and therefore

$$u(t) \geq h_{11}\gamma(t)\alpha[u] + h_{21}\delta(t)\beta[u] + Fu(t) + \lambda. \tag{2.4}$$

Applying α and β to both sides of (2.4) gives

$$\alpha[u] \geq h_{11}\alpha[\gamma]\alpha[u] + h_{21}\alpha[\delta]\beta[u] + \alpha[Fu] + \lambda\alpha[1],$$
$$\beta[u] \geq h_{11}\beta[\gamma]\alpha[u] + h_{21}\beta[\delta]\beta[u] + \beta[Fu] + \lambda\beta[1].$$

This can be written in the form

$$\begin{pmatrix} 1 - h_{11}\alpha[\gamma] & -h_{21}\alpha[\delta] \\ -h_{11}\beta[\gamma] & 1 - h_{21}\beta[\delta] \end{pmatrix} \begin{pmatrix} \alpha[u] \\ \beta[u] \end{pmatrix} \geq \begin{pmatrix} \alpha[Fu] + \lambda\alpha[1] \\ \beta[Fu] + \lambda\beta[1] \end{pmatrix} \geq \begin{pmatrix} \alpha[Fu] \\ \beta[Fu] \end{pmatrix}. \tag{2.5}$$

Setting

$$\underline{\mathcal{M}} = \begin{pmatrix} 1 - h_{11}\alpha[\gamma] & -h_{21}\alpha[\delta] \\ -h_{11}\beta[\gamma] & 1 - h_{21}\beta[\delta] \end{pmatrix}$$

gives

$$\underline{\mathcal{M}}^{-1} = \frac{1}{D_1} \begin{pmatrix} 1 - h_{21}\beta[\delta] & h_{21}\alpha[\delta] \\ h_{11}\beta[\gamma] & 1 - h_{11}\alpha[\gamma] \end{pmatrix}.$$

Note that $\underline{\mathcal{M}}^{-1}$ is order preserving by Lemma 2.3. Thus, if we apply $\underline{\mathcal{M}}^{-1}$ to both sides of the inequality (2.5), we obtain

$$\begin{pmatrix} \alpha[u] \\ \beta[u] \end{pmatrix} \geq \frac{1}{D_1} \begin{pmatrix} 1 - h_{21}\beta[\delta] & h_{21}\alpha[\delta] \\ h_{11}\beta[\gamma] & 1 - h_{11}\alpha[\gamma] \end{pmatrix} \begin{pmatrix} \alpha[Fu] \\ \beta[Fu] \end{pmatrix}$$

and therefore

$$\begin{aligned} u(t) \geq & \Big(\frac{\gamma(t)}{D_1}(1 - h_{21}\beta[\delta]) + \frac{\delta(t)}{D_1} h_{11}\beta[\gamma]\Big)\alpha[Fu] \\ & + \Big(\frac{\gamma(t)}{D_1} h_{21}\alpha[\delta] + \frac{\delta(t)}{D_1}(1 - h_{11}\alpha[\gamma])\Big)\beta[Fu] + Fu(t) + \lambda. \end{aligned}$$

Then we have, for $t \in [a,b]$,

$$\begin{aligned} u(t) \geq & \Big(\frac{\gamma(t)}{D_1}(1 - h_{21}\beta[\delta]) + \frac{\delta(t)}{D_1} h_{11}\beta[\gamma]\Big) \int_0^1 \mathcal{K}_A(s) g(s) f(s,u(s))\, ds \\ & + \Big(\frac{\gamma(t)}{D_1} h_{21}\alpha[\delta] + \frac{\delta(t)}{D_1}(1 - h_{11}\alpha[\gamma])\Big) \int_0^1 \mathcal{K}_B(s) g(s) f(s,u(s))\, ds \\ & + \int_0^1 k(t,s) g(s) f(s,u(s))\, ds + \lambda \\ \geq & \Big(\frac{c_2\|\gamma\|}{D_1}(1 - h_{21}\beta[\delta]) + \frac{c_3\|\delta\|}{D_1} h_{11}\beta[\gamma]\Big) \int_a^b \mathcal{K}_A(s) g(s) f(s,u(s))\, ds \\ & + \Big(\frac{c_2\|\gamma\|}{D_1} h_{21}\alpha[\delta] + \frac{c_3\|\delta\|}{D_1}(1 - h_{11}\alpha[\gamma])\Big) \int_a^b \mathcal{K}_B(s) g(s) f(s,u(s))\, ds \\ & + \int_a^b k(t,s) g(s) f(s,u(s))\, ds + \lambda. \end{aligned}$$

Using the hypothesis (2.2) we obtain $\min_{t\in[a,b]} u(t) > \rho + \lambda \geq \rho$, contradicting the fact that $u \in \partial V_\rho$. $\square$

We now prove that the index is 1 on the set K_ρ.

Lemma 2.6. *Assume that there exists $\rho > 0$ such that*

$$f^{0,\rho}\Big(\big(\frac{\|\gamma\|}{D_2}(1-h_{22}\beta[\delta])+\frac{\|\delta\|}{D_2}h_{12}\beta[\gamma]\big)\int_0^1 \mathcal{K}_A(s)g(s)\,ds + \big(\frac{\|\gamma\|}{D_2}h_{22}\alpha[\delta]+\frac{\|\delta\|}{D_2}(1-h_{12}\alpha[\gamma])\big)\int_0^1 \mathcal{K}_B(s)g(s)\,ds+\frac{1}{m}\Big)<1, \tag{2.6}$$

where

$$f^{0,\rho}=\sup\Big\{\frac{f(t,u)}{\rho}:\ (t,u)\in[0,1]\times[0,\rho]\Big\}\ \text{and}\ \frac{1}{m}=\sup_{t\in[0,1]}\int_0^1 k(t,s)g(s)\,ds. \tag{2.7}$$

Then $i_K(T,K_\rho)=1$.

Proof. (i) We show that $Tu\neq\lambda u$ for all $\lambda\geq 1$ when $u\in\partial K_\rho$, which implies that $i_K(T,K_\rho)=1$. In fact, if this does not happen, there exist u with $\|u\|=\rho$ and $\lambda\geq 1$ such that $\lambda u(t)=Tu(t)$. Then we have

$$\lambda u(t)=\gamma(t)H_1(\alpha[u])+\delta(t)H_2(\beta[u])+Fu(t)$$

and therefore

$$\lambda u(t)\leq h_{12}\gamma(t)\alpha[u]+h_{22}\delta(t)\beta[u]+Fu(t). \tag{2.8}$$

Applying α and β to both sides of (2.8) gives

$$\lambda\alpha[u]\leq h_{12}\alpha[\gamma]\alpha[u]+h_{22}\alpha[\delta]\beta[u]+\alpha[Fu],$$
$$\lambda\beta[u]\leq h_{12}\beta[\gamma]\alpha[u]+h_{22}\beta[\delta]\beta[u]+\beta[Fu].$$

Thus we have

$$\begin{pmatrix}\lambda-h_{12}\alpha[\gamma] & -h_{22}\alpha[\delta]\\ -h_{12}\beta[\gamma] & \lambda-h_{22}\beta[\delta]\end{pmatrix}\begin{pmatrix}\alpha[u]\\ \beta[u]\end{pmatrix}\leq\begin{pmatrix}\alpha[Fu]\\ \beta[Fu]\end{pmatrix}. \tag{2.9}$$

Setting

$$\overline{\mathcal{M}}_\lambda=\begin{pmatrix}\lambda-h_{12}\alpha[\gamma] & -h_{22}\alpha[\delta]\\ -h_{12}\beta[\gamma] & \lambda-h_{22}\beta[\delta]\end{pmatrix},$$

we have

$$(\overline{\mathcal{M}}_\lambda)^{-1}=\frac{1}{D_\lambda}\begin{pmatrix}\lambda-h_{22}\beta[\delta] & h_{22}\alpha[\delta]\\ h_{12}\beta[\gamma] & \lambda-h_{12}\alpha[\gamma]\end{pmatrix},$$

where

$$D_\lambda:=(\lambda-h_{12}\alpha[\gamma])(\lambda-h_{22}\beta[\delta])-h_{22}h_{12}\alpha[\delta]\beta[\gamma]\geq D_2>0.$$

Note that $(\overline{\mathcal{M}}_\lambda)^{-1}$ is order preserving by Lemma 2.3. Thus, if we apply $(\overline{\mathcal{M}}_\lambda)^{-1}$ to both sides of the inequality (2.9) we obtain

$$\begin{pmatrix}\alpha[u]\\ \beta[u]\end{pmatrix}\leq\frac{1}{D_\lambda}\begin{pmatrix}\lambda-h_{22}\beta[\delta] & h_{22}\alpha[\delta]\\ h_{12}\beta[\gamma] & \lambda-h_{12}\alpha[\gamma]\end{pmatrix}\begin{pmatrix}\alpha[Fu]\\ \beta[Fu]\end{pmatrix},$$

and as a consequence of Lemma 2.4, we have

$$\begin{pmatrix}\alpha[u]\\ \beta[u]\end{pmatrix} \le \frac{1}{D_2}\begin{pmatrix}1-h_{22}\beta[\delta] & h_{22}\alpha[\delta]\\ h_{12}\beta[\gamma] & 1-h_{12}\alpha[\gamma]\end{pmatrix}\begin{pmatrix}\alpha[Fu]\\ \beta[Fu]\end{pmatrix}.$$

Hence we obtain

$$\begin{aligned}\lambda u(t) \le \frac{\gamma(t)}{D_2}\Big[(1-h_{22}\beta[\delta])\alpha[Fu] + h_{22}\alpha[\delta]\beta[Fu]\Big]\\ + \frac{\delta(t)}{D_2}\Big[(1-h_{12}\alpha[\gamma])\beta[Fu]) + h_{12}\beta[\gamma]\alpha[Fu]\Big] + Fu(t).\end{aligned}$$

Using the inequality $f(s,u(s)) \le \rho f^{0,\rho}$ and taking the supremum over $[0,1]$ gives

$$\begin{aligned}\lambda\rho \le \rho f^{0,\rho}\,\Big(\Big(\frac{\|\gamma\|}{D_2}(1-h_{22}\beta[\delta]) + \frac{\|\delta\|}{D_2}h_{12}\beta[\gamma]\Big)\int_0^1 \mathcal{K}_A(s)g(s)\,ds\\ + \Big(\frac{\|\gamma\|}{D_2}h_{22}\alpha[\delta] + \frac{\|\delta\|}{D_2}(1-h_{12}\alpha[\gamma])\Big)\int_0^1 \mathcal{K}_B(s)g(s)\,ds + \frac{1}{m}\Big),\end{aligned}$$

contradicting (2.6). □

3. THE BOUNDARY VALUE PROBLEM

We now turn our attention to the differential equation

$$u''(t) + g(t)f(t,u(t)) = 0,\ t\in(0,1), \tag{3.1}$$

with the BCs

$$u'(0) + H_1(\alpha[u]) = 0,\ \sigma u'(1) + u(\eta) = H_2(\beta[u]). \tag{3.2}$$

The solution of $-u'' = y$ under these BCs can be written in the form

$$\begin{aligned}u(t) =&(\sigma+\eta-t)H_1(\alpha[u]) + H_2(\beta[u])\\ &+\sigma\int_0^1 y(s)ds + \int_0^\eta(\eta-s)y(s)ds - \int_0^t (t-s)y(s)ds.\end{aligned}$$

The solution of our BVP is

$$u(t) = \gamma(t)H_1(\alpha[u]) + \delta(t)H_2(\beta[u]) + \int_0^1 k(t,s)g(s)f(s,u(s))ds, \tag{3.3}$$

where

$$\gamma(t) = \sigma+\eta-t, \quad \delta(t) = 1$$

and

$$k(t,s) = \sigma + \begin{cases}\eta - s, & s\le\eta\\ 0, & s>\eta\end{cases} - \begin{cases}t-s, & s\le t\\ 0, & s>t.\end{cases}$$

Here we focus on the case $\beta > 0$ and $\sigma+\eta\ge 1$ that leads to the existence of *positive solutions*. The case $\beta>0$ and $\sigma+\eta<1$, that leads to solutions that are *positive on a sub-interval* has been investigated in the case of one nonlinear perturbation in [8], and for the case of no perturbations in [9].

Upper bounds for $k(t,s)$ and $\gamma(t)$ were given in [7, 9, 19] as follows:

$$\|\gamma\| = \sigma + \eta, \quad \Phi(s) = \begin{cases} \sigma, & s > \eta, \\ \sigma + \eta - s, & s \le \eta. \end{cases} \tag{3.4}$$

In [9, 19] it has been shown that

$$k(t,s) \ge c_1 \Phi(s), \text{ for } t \in [0,b],$$

where $c_1 = 1 - \frac{b}{\sigma+\eta}$. Here $b \in (0,1)$ may be arbitrary and $b = 1$ is allowed if $\sigma + \eta > 1$.

A direct calculation shows that the inequality

$$\gamma(t) \ge c_2 \|\gamma\| \text{ for } t \in [0,b],$$

is satisfied when $c_2 = c_1$. Note that $\delta(t) \equiv 1$, so $\|\delta\| = 1$ and one may take $c_3 = 1$. This leads to the choice

$$c = 1 - \frac{b}{\sigma + \eta}. \tag{3.5}$$

Hence, we work on the cone

$$K = \{u \in C[0,1] : u(t) \ge 0 \text{ for every } t \in [0,1] \text{ and } \min_{t \in [0,b]} u(t) \ge c\|u\|\},$$

with c as in (3.5).

By means of the two Lemmas above, provided that the function f has a suitable oscillatory behavior, one may establish the existence of multiple positive solutions (we refer the reader to [16] to see the type of results that may be stated). Here, for brevity, we state a result for the case of one positive solution.

Theorem 3.1. *Let* $[a,b] = [0,b]$ *and* c *be as in* (3.5). *Then the equation* (3.3) *has a nonzero solution in* K *if either of the following conditions hold.*

- (S_1) *There exist* $\rho_1, \rho_2 \in (0,\infty)$ *with* $\rho_1 < \rho_2$ *such that* (2.6) *is satisfied for* ρ_1 *and* (2.2) *is satisfied for* ρ_2.
- (S_2) *There exist* $\rho_1, \rho_2 \in (0,\infty)$ *with* $\rho_1 < c\rho_2$ *such that* (2.2) *is satisfied for* ρ_1 *and* (2.6) *is satisfied for* ρ_2.

The next example sheds some light on the conditions that occur in our theory.

Example 3.2. Consider the BVP

$$-u''(t) = f(t, u(t)), \text{ a.e. on } [0,1], \tag{3.6}$$

$$u'(0) + H_1(\alpha u(\xi)) = 0, \ \sigma u'(1) + u(\eta) = H_2(\beta u(\tau)), \tag{3.7}$$

where $\xi, \eta, \tau \in [0,1]$ and $\xi < \eta < \tau$. In this case $g \equiv 1$ and we may take dA to be the Dirac measure of weight α at ξ and dB the Dirac measure of weight β at τ.

We need

$$h_{12}\alpha[\gamma] < 1, \ \ h_{22}\beta[\delta] < 1, \ \text{ and } \ D_2 > 0.$$

For our BVP we have

$$\alpha[\gamma] = \alpha(\sigma+\eta-\xi),\ \alpha[\delta] = \alpha,\ \beta[\gamma] = \beta(\sigma+\eta-\tau),\ \beta[\delta] = \beta,$$

and therefore the total requirement is

$$\begin{aligned} &h_{12}\alpha(\sigma+\eta-\xi) < 1,\ \ h_{22}\beta < 1,\\ &(1-h_{12}\alpha(\sigma+\eta-\xi))(1-h_{22}\beta) - h_{12}h_{22}\alpha\beta > 0. \end{aligned}$$

Furthermore, by direct computation, we have

$$\frac{1}{m} = \sigma + \eta^2/2,$$

and

$$\int_0^1 \mathcal{K}_A(s)\,ds = \frac{1}{2}\alpha(2\sigma+\eta^2-\xi^2), \quad \int_0^1 \mathcal{K}_B(s)\,ds = \frac{1}{2}\beta(2\sigma+\eta^2-\tau^2).$$

Note that different choices of b affect the growth condition on f in (2.2). Webb [19] computed the minimal M with the corresponding optimal interval $[0,b]$, for the special case

$$u'(0) = 0, \quad \sigma u'(1) + u(\eta) = 0,$$

as follows:

$$\frac{1}{M} = \begin{cases} (\sigma+\eta)^2/4 \quad (\text{for } b = (\sigma+\eta)/2), & \text{if } \sigma < \eta,\\ (\sigma^2+\eta^2)/2 \quad (\text{for } b = \sigma), & \text{if } \eta \le \sigma \le 1,\\ (2\sigma-1+\eta^2)/2 \quad (\text{for } b = 1), & \text{if } \sigma > 1. \end{cases}$$

In order to illustrate the remaining terms in (2.2), we now suppose $\xi < \eta < \sigma < \tau$ and set $[a,b] = [0,\sigma]$. This gives

$$\int_0^\sigma \mathcal{K}_A(s)\,ds = \frac{1}{2}\alpha(2\sigma^2+\eta^2-\xi^2), \quad \int_0^\sigma \mathcal{K}_B(s)\,ds = \frac{1}{2}\beta(3\sigma^2+\eta^2-2\tau\sigma).$$

Thus all the ingredients that appear in (2.2) and (2.6) can be computed.

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