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EXISTENCE RESULTS FOR SECOND ORDER HYPERBOLIC PARTIAL DIFFERENTIAL INCLUSIONS INVOLVING DISCONTINUOUS MULTIFUNCTIONS

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ABSTRACT. The present paper studies the existence results for the second order hyperbolic partial differential inclusions in the nonconvex case and without assuming any kind of the continuity conditions on the multi-valued functions. The existence of the extremal solutions is also established under certain monotonicity conditions.

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1. STATEMENT OF THE PROBLEMS

Let $\mathbb{R}$ denote the real line and let $J_a = [0, a]$ and $J_b = [0, b]$ be two closed and bounded intervals in $\mathbb{R}$ for some real numbers $a > 0$ and $b > 0$. By $\mathcal{P}_p(\mathbb{R})$ we denote the class of all non-empty subsets of $\mathbb{R}$ with property p. Now consider the second order hyperbolic partial differential inclusion (in short HPDI)

$$\left.\begin{aligned} \frac{\partial^2 u(x,y)}{\partial x \partial y} &\in F(x, y, u(x, y)) \text{ a.e. } (x, y) \in J_a \times J_b, \\ u(x, 0) = \phi(x),&\ u(0, y) = \psi(y), \end{aligned}\right\} \tag{1.1}$$

where, $F : J_a \times J_b \times \mathbb{R} \to \mathcal{P}_p(\mathbb{R})$ and the functions $\phi : J_a \to \mathbb{R}$ and $\psi : J_b \to \mathbb{R}$ are continuous with $\phi(0) = \psi(0)$.

By a *solution* of the HPDI (1.1) we mean a function $u \in AC(J_a \times J_b, \mathbb{R})$ such that there exists a function $v \in L^1(J_a \times J_b, \mathbb{R})$ such that $v(x, y) \in G(x, y, u(x, y))$ a.e. $(x, y) \in J_a \times J_b$ satisfying

$$\frac{\partial^2 u(x,y)}{\partial x \partial y} = v(x, y),$$

$$u(x, 0) = \phi(x),\ u(0, y) = \psi(y),$$

where $AC(J_a \times J_b, \mathbb{R})$ is the space of absolutely continuous real-valued functions on $J_a \times J_b$ and the functions $\phi : J_a \to \mathbb{R}$ and $\psi : J_b \to \mathbb{R}$ are continuous with $\phi(0) = \psi(0)$.

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The HPDI (1.1) has been discussed in the literature by several authors for the existence theorems under different continuity conditions on the multi-valued functions F. The details may be found in Dawidowski and Kubiaczyk [4] and the references therein. In this paper we discuss the HPDI (1.1) for the existence of solutions as well as the existence of extremal solutions under certain monotonicity conditions without assuming any kind of continuity conditions on the multi-valued functions F.

Later, we consider the perturbed second order hyperbolic partial differential inclusions (in short HPDI)

$$\left.\begin{array}{r} \dfrac{\partial^2 u(x,y)}{\partial x \partial y} \in F(x,y,u(x,y)) + G(x,y,u(x,y)) \ \text{ a.e. } \ (x,y) \in J_a \times J_b, \\ u(x,0) = \phi(x), \ u(0,y) = \psi(y), \end{array}\right\} \tag{1.2}$$

where, $F, G : J_a \times J_b \times \mathbb{R} \to \mathcal{P}_p(\mathbb{R})$ and the functions $\phi : J_a \to \mathbb{R}$ and $\psi : J_b \to \mathbb{R}$ are continuous with $\phi(0) = \psi(0)$.

By a *solution* of the HPDI (1.2) we mean a function $u \in AC(J_a \times J_b, \mathbb{R})$ such that there exist functions $v_1, v_2 \in L^1(J_a \times J_b, \mathbb{R})$ such that $v_1(x,y) \in F(x,y,u(x,y))$ and $v_2(x,y) \in G(x,y,u(x,y))$ almost everywhere for $(x,y) \in J_a \times J_b$ satisfying

$$\frac{\partial^2 u(x,y)}{\partial x \partial y} = v_1(x,y) + v_2(x,y),$$

$$u(x,0) = \phi(x), \ u(0,y) = \psi(y),$$

where $AC(J_a \times J_b, \mathbb{R})$ is the space of absolutely continuous real-valued functions on $J_a \times J_b$ and the functions $\phi : J_a \to \mathbb{R}$ and $\psi : J_b \to \mathbb{R}$ are continuous with $\phi(0) = \psi(0)$.

The HPDI (1.2) has been discussed in Belarbi and Benchohra [1] for the existence results under the mixed Lipschitz and Carathéodory conditions on the multi-valued functions. Here, we prove the existence of solutions as well as the existence of the extremal solutions for the HPDI (1.2) under different monotonicity conditions on the multi-valued functions. Here, we do not require any kind of continuity condition on one of the multi-valued functions F and G.

2. PRELIMINARIES

In this section, we introduce notations, definitions, and some basic results from multi-valued analysis which are used in the remaining part of this paper.

Let $C(J_a \times J_b, \mathbb{R})$ be the Banach space of all continuous functions from $J_a \times J_b$ into $\mathbb{R}$ with the norm

$$\|u\|_\infty = \sup\{|u(x,y)| : (x,y) \in J_a \times J_b\}, \tag{2.1}$$

for each $u \in C(J_a \times J_b, \mathbb{R})$.

Let $L^1(J_a \times J_b, \mathbb{R})$ denote the Banach space of measurable functions $u : J_a \times J_b \to \mathbb{R}$ which are Lebesgue integrable normed by

$$\|u\|_{L^1} = \int_0^a \int_0^b |u(x,y)|\, dx\, dy \tag{2.2}$$

for each $u \in L^1(J_a \times J_b, \mathbb{R})$.

We equip the space $X = C(J_a \times J_b, \mathbb{R})$ with the order relation $\leq$ defined by the cone K in X, that is,

$$K = \{u \in X : u(x,y) \geq 0, \forall\, (x,y) \in J_a \times J_b\}. \tag{2.3}$$

It is known that the cone K is normal in X. Details on cones and their properties may be found in Heikkilä and Lakshmikantham [12]. Let $a, b \in X$ be such that $a \leq b$. Then, by an order interval $[a, b]$, we mean the set of points in X given by

$$[a, b] = \{u \in X : a \leq u \leq b\}.$$

Let $A, B \in \mathcal{P}_p(X)$. Denote

$$A \pm B = \{a \pm b : a \in A \quad \text{and} \quad b \in B\},$$

$$\lambda A = \{\lambda a : \lambda \in \mathbb{R} \quad \text{and} \quad a \in A\}.$$

Also denote

$$\|A\| = \{\|a\| : a \in A\}$$

and

$$\|A\|_{\mathcal{P}} = \sup\{\|a\| : a \in A\}.$$

Let the Banach space X be equipped with the order relation $\leq$ and define the order relation in $\mathcal{P}_p(X)$ as follows.

Let $A, B \in \mathcal{P}_p(X)$. Then by $A \overset{i}{\leq} B$ we mean " for every $a \in A$ there exists a $b \in B$ such that $a \leq b$ ". Again $A \overset{d}{\leq} B$ means for each $b \in B$ there exists a $a \in A$ such that $a \leq b$. Further, we have $A \overset{id}{\leq} B \iff A \overset{i}{\leq} B$ and $A \overset{d}{\leq} B$. Finally $A \leq B$ implies that $a \leq b$ for all $a \in A$ and $b \in B$. Note that if $A \leq A$, then it follows that A is a singleton set. See Dhage [6] and references therein.

Definition 2.1. A mapping $Q : X \to \mathcal{P}_p(X)$ is called right monotone increasing (resp. left monotone increasing) if $Qx \overset{i}{\leq} Qy$ (resp. $Qx \overset{d}{\leq} Qy$) for all $x, y \in X$ for which $x \leq y$. Similarly, Q is called monotone increasing if it is left as well as right monotone increasing on X. Finally, Q is strict monotone increasing if $Qx \leq Qy$ for all $x, y \in X$ for which $x \leq y$, $x \neq y$.

Remark 2.2. Note that every strict monotone increasing multi-valued mapping is right monotone increasing, but the converse may not be true.

It is known that the monotone technique is a very useful tool for proving the existence of the extremal solutions for differential equations and inclusions. The exhaustive treatment of this method for discontinuous differential equations may be found in Heikkila and Lakshmikantham [12]. The monotone method blended with the lower and upper solutions method has been employed in Lashmikantham and Pandit [13] and Pandit [15] for proving existence results for hyperbolic partial differential equations under continuity and certain compactness type conditions. The method of lower and upper solutions has been discussed in Halidias and Papageorgiou [11] for proving the existence results for second order ordinary differential inclusions under upper semi-continuity of the multi-valued functions. But the use of the monotone technique in the theory of differential inclusions involving discontinuous multi-valued functions is relatively new to the literature. Some recent results in this direction appear in Dhage [5, 6, 7, 9]. In the methods of monotone technique for differential inclusions, the operator in question is required to satisfy certain monotonicity condition with respect to certain order relation on the domains of definition. The following two fixed point theorems are fundamental in the monotone theory for discontinuous differential inclusions involving right or strict monotone increasing multi-valued functions.

Theorem 2.3 (Dhage [5])**.** *Let $[a,b]$ be an order interval in a subset Y of an ordered Banach space X and let $Q : [a,b] \to \mathcal{P}_{cp}([a,b])$ be a right monotone increasing multi-valued mapping. If every sequence $\{y_n\} \subset \bigcup Q([a,b])$ defined by $y_n \in Qx_n$, $n \in \mathbb{N}$ has a cluster point, whenever $\{x_n\}$ is a monotone increasing sequence in $[a,b]$, then Q has a fixed point.*

Theorem 2.4 (Dhage [7])**.** *Let $[a,b]$ be an order interval in a subset Y of an ordered Banach space X and let $Q : [a,b] \to \mathcal{P}_{cp}([a,b])$ be a strict monotone increasing multi-valued mapping. If every sequence $\{y_n\} \subset \bigcup Q([a,b])$ defined by $y_n \in Qx_n$, $n \in \mathbb{N}$ has a cluster point, whenever $\{x_n\}$ is a monotone sequence in $[a,b]$, then Q has a least fixed point x_* and a greatest fixed point x^* in $[a,b]$. Moreover,*

$$x_* = \min\{y \in [a,b] \mid Qy \leq y\} \quad \textit{and} \quad x^* = \max\{y \in [a,b] \mid y \leq Qy\}.$$

Let X be a metric space. A multi-valued operator $T : X \to \mathcal{P}(X)$ is convex (closed) valued if $T(x)$ is convex (closed) for all $x \in X$. T is bounded on bounded sets if $T(S) = \cup_{x\in S} T(x)$ is a bounded subset of X for all $S \in \mathcal{P}_{bd}(X)$ (i.e. $\sup_{x\in S}\{\sup\{|y| : y \in T(x)\}\} < \infty$). T is called upper semi-continuous (u.s.c.) on X if for each $x_0 \in X$ the set $T(x_0)$ is a nonempty closed subset of X and if for each open set N of X containing $T(x_0)$, there exists an open neighborhood N_0 of x_0 such that $T(N_0) \subseteq N$. T is called **totally bounded** if $T(S)$ is a totally bounded subset of X for each $S \in \mathcal{P}_{bd}(X)$. T is called **compact** if $T(S)$ is a relatively compact subset of X for each $S \in \mathcal{P}_{bd}(X)$. T is said to be completely continuous if it is continuous and compact on X. Note that every compact operator is totally bounded, but the converse may not be true. However, these two notions are equivalent in a complete metric space X. If the multi-valued map T is completely continuous with nonempty compact values, then T is u.s.c. if and only if T has a closed graph (i.e. $x_n \to x_*$, $y_n \to y_*$, $y_n \in T(x_n)$ imply $y_* \in T(x_*)$). Finally, T has a fixed point if there is $x \in X$ such that $x \in T(x)$.

The following hybrid fixed point theorems are useful in the sequel.

Theorem 2.5 (Dhage [10]). *Let $[a, b]$ be an order interval in an ordered Banach space X. Let $A, B : [a, b] \to \mathcal{P}_{cp}(X)$ be two right monotone increasing multi-valued operators satisfying*

(*a*) *A is completely continuous,*
(*b*) *B is totally bounded, and*
(*c*) *$Ax + By \subset [a, b]$ for all $x, y \in [a, b]$.*

Further, if the cone K in X is normal, then the operator inclusion $x \in Ax + Bx$ has a solution in $[a, b]$.

Theorem 2.6 (Dhage [7]). *Let $[a, b]$ be an order interval in an ordered Banach space X. Let $A, B : [a, b] \to \mathcal{P}_{cp}(X)$ be two strict monotone increasing multi-valued operators satisfying*

(*a*) *A is completely continuous,*
(*b*) *B is totally bounded, and*
(*c*) *$Ax + By \subset [a, b]$ for all $x, y \in [a, b]$.*

Further, if the cone K in X is normal, then the operator inclusion $x \in Ax + Bx$ has a least and a greatest solution in $[a, b]$.

Let (X, d) be a metric space induced from the normed space $(X, |\cdot|)$. Consider the function $d_H : \mathcal{P}_p(X) \times \mathcal{P}_p(X) \to \mathbb{R}_+ \cup \{\infty\}$ defined by

$$d_H(A, B) = \max\Big\{\sup_{a\in A} d(a, B), \sup_{b\in B} d(A, b)\Big\},$$

where $d(A, b) = \inf_{a\in A} d(a, b)$, $d(a, B) = \inf_{b\in B} d(a, b)$. Then $(\mathcal{P}_{cl}(X), d_H)$ is a metric space and $(\mathcal{P}_{cl,bd}(X), d_H)$ is called a generalized metric space (see [4]).

Definition 2.7. A multi-valued operator $T : X \to \mathcal{P}_{cl}(X)$ is called a multi-valued contraction on X if exists a real number $\alpha < 1$ such that

$$d_H(T(x), T(y)) \le \alpha d(x, y)$$

for all x, $y \in X$. The number α is called the contraction constant of T on X.

We also need the following two hybrid fixed point theorems in the sequel.

Theorem 2.8 (Dhage [10])**.** *Let $[a, b]$ be an order interval in an ordered Banach space X. Let $A, B : [a, b] \to \mathcal{P}_{cp}(X)$ be two right monotone increasing multi-valued operators satisfying*

(a) *A is multi-valued contraction,*
(b) *B is totally bounded, and*
(c) *$Ax + By \subset [a, b]$ for all $x, y \in [a, b]$.*

Further, if the cone K in X is normal, then the operator inclusion $x \in Ax + Bx$ has a solution in $[a, b]$.

Theorem 2.9 (Dhage [7])**.** *Let $[a, b]$ be an order interval in an ordered Banach space X. Let $A, B : [a, b] \to \mathcal{P}_{cp}(X)$ be two strict monotone increasing multi-valued operators satisfying*

(a) *A is multi-valued contraction,*
(b) *B is totally bounded, and*
(c) *$Ax + By \subset [a, b]$ for all $x, y \in [a, b]$.*

Further, if the cone K in X is normal, then the operator inclusion $x \in Ax + Bx$ has a least and a greatest solution in $[a, b]$.

Remark 2.10. Notice that hypothesis (c) of Theorems 2.3, 2.4, 2.5 and 2.6 hold if the multi-valued operators A and B are right monotone increasing and the elements a and b satisfy $a \le Aa + Ba$ and $Ab + Bb \le b$.

3. EXISTENCE THEORY FOR THE PROBLEM (1.1)

Definition 3.1. A multi-valued map $\beta : J_a \times J_b \to \mathcal{P}_{cl}(\mathbb{R})$ is said to be measurable if for every $z \in \mathbb{R}$, the function $(x, y) \mapsto d(z, \beta(x, y)) = \inf\{|z - u| : u \in \beta(x, y)\}$ is measurable.

Let $\beta : J_a \times J_b \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ be a multi-valued map with nonempty compact values. Assign to β, the multi-valued operator

$$S^1_\beta : C(J_a \times J_b, \mathbb{R}) \to \mathcal{P}(L^1(J_a \times J_b, \mathbb{R}))$$

defined by

$$S^1_\beta(u) = \{w \in L^1(J_a \times J_b, \mathbb{R}) : w(x, y) \in \beta(x, y, u(x, y)) \text{ a.e. } (x, y) \in J_a \times J_b\}. \quad (3.1)$$

The operator S^1_β is called the Niemytsky operator associated with the multi-valued function β and $S^1_\beta(u)$ is called the *selection set* of functions of the multi-valued function β at $u \in C(J_a \times J_b, \mathbb{R})$.

The integral of the multi-valued function F is defined as

$$\int_0^x \int_0^y \beta(t, s, u(t, s))\, dt\, ds = \left\{ \int_0^x \int_0^y v(t, s)\, ds\, dt : v \in S^1_\beta(u) \right\}.$$

If $S_F^1(u) \neq \emptyset$ for each $u \in C(J_a \times J_b, \mathbb{R})$, then the HPDI (1.1) is equivalent to the integral inclusion

$$u(x,y) \in z_0(x,y) + \int_0^x \int_0^y F(t,s,u(t,s))\, ds\, dt \tag{3.2}$$

for all $(x,y) \in J_a \times J_b$, where $z_0(x,y) = \phi(x) + \psi(y) - \phi(0)$. It is clear that $z_0 \in C(J_a \times J_b, \mathbb{R})$.

Definition 3.2. A multi-valued function $F(x,y,u)$ is called right monotone increasing in u almost everywhere for $(x,y) \in J_a \times J_b$ if $F(x,y,u_1) \overset{i}{\leq} F(x,y,u_2)$ almost everywhere for $(x,y) \in J_a \times J_b$ for all $u_1,\ u_2 \in \mathbb{R}$ with $u_1 \leq u_2$.

3.1 Existence result. We need the following definition in the sequel.

Definition 3.3. A function $u \in AC(J_a \times J_b, \mathbb{R})$ is called a solution of the HPDI (1.1) if there exists a function $v \in L^1(J_a \times J_b, \mathbb{R})$ such that $v(x,y) \in G(x,y,u(x,y))$ a.e. $(x,y) \in J_a \times J_b$ satisfying

$$\begin{gathered} \frac{\partial^2 u(x,y)}{\partial x \partial y} = v(x,y),\ (x,y) \in J_a \times J_b, \\ u(x,0) = \phi(x),\ u(0,y) = \psi(y), \end{gathered}$$

where $AC(J_a \times J_b, \mathbb{R})$ is the space of absolutely continuous real-valued functions on $J_a \times J_b$.

Definition 3.4. A function $\underline{u}(\cdot,\cdot) \in AC(J_a \times J_b, \mathbb{R})$ is said to be a strict lower solution of the HPDI (1.1) if for all $v \in S_F^1(\underline{u})$, we have

$$\begin{gathered} \frac{\partial^2 \underline{u}(x,y)}{\partial x \partial y} \leq v(x,y), \\ \underline{u}(x,0) \leq \phi(x),\ \underline{u}(0,y) \leq \psi(y), \end{gathered}$$

for all $(x,y) \in J_a \times J_b$. Similarly, a function $\overline{u}(\cdot,\cdot) \in AC(J_a \times J_b, \mathbb{R})$ is said to be a strict upper solution of the HPDI (1.1) if for all $v \in S_F^1(\overline{u})$, one has

$$\begin{gathered} \frac{\partial^2 \overline{u}(x,y)}{\partial x \partial y} \geq v(x,y), \\ \overline{u}(x,0) \geq \phi(x),\ \overline{u}(0,y) \geq \psi(y), \end{gathered}$$

for all $(x,y) \in J_a \times J_b$.

Definition 3.5. A solution u_M of the problem (1.1) is said to be maximal if for any other solution u to the problem (1.1), we have $u(x,y) \leq u_M(x,y)$ for all $(x,y) \in J_a \times J_b$. Again a solution u_m of the problem (1.1) is said to be minimal if $u_m(x,y) \leq u(x,y)$ for all $(x,y) \in J_a \times J_b$, where u is any solution of the problem (1.1) on $J_a \times J_b$.

We consider the following set of hypotheses in the sequel.

(A_1) $F(x,y,u)$ is closed and bounded for each $(x,y,u) \in J_a \times J_b \times \mathbb{R}$.

(A_2) The multi-valued function $(x,y) \mapsto F(x,y,u)$ is measurable for each $u \in \mathbb{R}$ and right monotone increasing in u almost everywhere for $(x,y) \in J_a \times J_b$.

(A_3) $S_F^1(u) \neq \emptyset$ for all $u \in C(J_a \times J_b, \mathbb{R})$.

(A_4) The multi-valued map $u \mapsto S_F^1(u)$ is right monotone increasing in $C(J_a \times J_b, \mathbb{R})$.

(A_5) FDI (1.1) has a strict lower solution $\underline{u}$ and a strict upper solution $\overline{u}$ with $\underline{u} \leq \overline{u}$.

(A_6) The function $h : J_a \times J_b \to \mathbb{R}$ defined by

$$h(x,y) = \|F(x,y,\underline{u})\|_{\mathcal{P}} + \|F(x,y,\overline{u})\|_{\mathcal{P}}$$

is Lebesgue integrable.

Remark 3.6. Note that if (A_2) and (A_5)–(A_6) hold, then we have

$$\|F(x,y,u(x,y))\|_{\mathcal{P}} \leq h(x,y) \quad \text{a.e.} \quad (x,y) \in J_a \times J_b$$

for all $u \in [a,b]$.

Hypotheses (A_1)–(A_3) are common in the literature. Some nice sufficient conditions guaranteeing that (A_3) holds are given by Deimling [4] and Lasota and Opial [14]. A mild form of (A_5) is used in Halidias and Papageorgiou [11]. Hypotheses (A_4) and (A_5) are relatively new to the literature, but special cases of them have been appeared in the works of several authors. Note that (A_5) holds, in particular, if F is bounded on $J_a \times J_b \times \mathbb{R}$ (see Dhage [5, 6] and references therein). Hypothesis (A_2) is assumed in order for (A_4) to make sense.

Theorem 3.7. *Assume that (A_1)–(A_6) hold. Then the HPDI* (1.1) *has a solution in* $[\underline{u},\overline{u}]$ *defined on* $J_a \times J_b$.

Proof. Let $X = C(J_a \times J_b, \mathbb{R})$ and let $Y = AC(J_a \times J_b, \mathbb{R}) \subset X$. Define an order interval $[\underline{u},\overline{u}]$ in Y, which is well defined in view of hypothesis (A_5). Define the operators Q on $[\underline{u},\overline{u}]$ by

$$Qu(x,y) = z_0(x,y) + \int_0^x \int_0^y F(t,s,u(t,s))\,ds\,dt, \quad (x,y) \in J_a \times J_b. \tag{3.3}$$

Clearly, the operator Q is well defined in view of hypothesis (A_3). We will show that Q satisfies all the conditions of Theorem 2.3.

Step I : First, we show that Q has compact values on $[\underline{u},\overline{u}]$. Observe that if $(x,y) \in J_a \times J_b$, then operator Q is equivalent to the composition $\mathcal{K} \circ S_G^1$ of two operators on $L^1(J_a \times J_b, \mathbb{R})$, where $\mathcal{K} : L^1(J_a \times J_b, \mathbb{R}) \to X$ is the continuous operator defined by

$$\mathcal{K}v(x,y) = z_0(x,y) + \int_0^x \int_0^y v(t,s)\,ds\,dt \tag{3.4}$$

for $(x,y) \in J_a \times J_b$. To show Q has compact values, it then suffices to prove that the composition operator $\mathcal{K} \circ S_F^1$ has compact values on $[\underline{u},\overline{u}]$. Let $u \in [\underline{u},\overline{u}]$ be arbitrary and let $\{v_n\}$ be a sequence in $S_F^1(u)$. By the definition of S_F^1, we have

$v_n(x,y) \in F(x,y,u(x,y))$ a.e. $(x,y) \in J_a \times J_b$. Since $F(x,y,u(x,y))$ is compact, there is a convergent subsequence of $v_n(x,y)$ (for simplicity call it $v_n(x,y)$ itself) that converges in measure to some $v(x,y)$, where $v(x,y) \in F(x,y,u(x,y))$ a.e. for $(x,y) \in J_a \times J_b$. From the continuity of $\mathcal{K}$, it follows that $\mathcal{K}v_n(x,y) \to \mathcal{K}v(x,y)$ pointwise on $J_a \times J_b$ as $n \to \infty$. In order to show that the convergence is uniform, we first show that $\{\mathcal{K}v_n\}$ is an equi-continuous sequence. Let $(x_1,y_1),(x_2,y_2) \in J_a \times J_b$; then

$$\begin{aligned}
|\mathcal{K}v_n(x_1,y_1) - \mathcal{K}v_n(x_2,y_2)| &\le |z_0(x_1,y_1) - z_0(x_2,y_2)| \\
&\quad + \left| \int_0^{x_1}\int_0^{y_1} v_n(t,s)\,dt\,ds - \int_0^{x_2}\int_0^{y_2} v_n(t,s)\,dt\,ds \right| \\
&\le |z_0(x_1,y_1) - z_0(x_2,y_2)| + \left| \int_{x_2}^{x_1}\int_{y_2}^{y_1} v_n(t,s)\,dt\,ds \right| \\
&\le |z_0(x_1,y_1) - z_0(x_2,y_2)| + \left| \int_{x_2}^{x_1}\int_{y_2}^{y_1} |v_n(t,s)|\,dt\,ds \right|.
\end{aligned}$$

Since $v_n \in L^1(J_a \times J_b, \mathbb{R})$, the right hand side of the above inequality tends to 0 as $(x_1,y_1) \to (x_2,y_2)$. Hence, $\{\mathcal{K}v_n\}$ is equi-continuous, and an easy application of the Ascoli theorem implies that $\{\mathcal{K}v_n\}$ has a uniformly convergent subsequence. We then have $\mathcal{K}v_{n_j} \to \mathcal{K}v \in (\mathcal{K} \circ S_F^1)(u)$ as $j \to \infty$, and so $(\mathcal{K} \circ S_F^1)(u)$ is a compact set in X. Therefore, Q is a compact-valued multi-valued operator on $[\underline{u}, \overline{u}]$.

Step II : Next, we show that Q is right monotone increasing and maps $[\underline{u}, \overline{u}]$ into itself. Let $u_1, u_2 \in [\underline{u}, \overline{u}]$ be such that $u_1 \le u_2$. Since $S_F^1(u_1) \overset{i}{\le} S_F^1(u_2)$, we have that $Q(u_1) \overset{i}{\le} Q(u_2)$. From (A_5), it follows that $\underline{u} \le Q\underline{u}$ and $Q\overline{u} \le \overline{u}$. Now Q is right monotone increasing, so we have

$$\underline{u} \le Q\underline{u} \overset{i}{\le} Qu \overset{i}{\le} Q\overline{u} \le \overline{u}$$

for all $u \in [\underline{u}, \overline{u}]$. Hence, Q defines a right monotone increasing multi-valued operator $Q : [\underline{u}, \overline{u}] \to \mathcal{P}_{cp}([\underline{u}, \overline{u}])$.

Step III : Finally, let $\{u_n\}$ be a monotone sequence in $[\underline{u}, \overline{u}]$ and let $\{z_n\}$ be a sequence in $B([\underline{u}, \overline{u}])$ defined by $z_n \in Bu_n$, $n \in \mathbb{N}$. We shall show that $\{z_n\}$ has a cluster point. This is achieved by showing that $\{z_n\}$ is a uniformly bounded and equi-continuous sequence.

First we show that $\{z_n\}$ is uniformly bounded sequence. By the definition of $\{z_n\}$, there is a $v_n \in S_G^1(u_n)$ such that

$$z_n(x,y) = z_0(x,y) + \int_0^x \int_0^y v_n(t,s))\,ds\,dt.$$

Then, we have

$$
\begin{aligned}
|z_n(x,y)| &\le |z_0(x,y)| + \int_0^a \int_0^b |v_n(t,s)| ds dt \\
&\le |z_0(x,y)| + \int_0^a \int_0^b h(t,s)\, ds\, dt \\
&\le \|z_0\|_\infty + \|h_r\|_{L^1}
\end{aligned}
$$

for all $(x_1,y_1),(x_2,y_2) \in J_a \times J_b$. This shows that $\{z_n\}$ is uniformly bounded sequence. Next, we show that $\{z_n\}$ is an equicontinuous sequence in X. Let $(x_1,y_1),(x_2,y_2) \in J_a \times J_b$. Then we have

$$
\begin{aligned}
|z_n(x_1,y_1) - z_n(x_2,y_2)| &\le |z_0(x_1,y_1) - z_0(x_2,y_2)| + \left| \int_{x_1}^{x_2} \int_{y_1}^{y_2} |v_n(t,s)|\, ds\, dt \right| \\
&\le |z_0(x_1,y_1) - z_0(x_2,y_2)| + \left| \int_{x_1}^{x_2} \int_{y_1}^{y_2} h(t,s)\, ds\, dt \right|.
\end{aligned}
$$

The right hand side tends to zero as $(x_2,y_2) \to (x_1,y_1)$. So, $\{z_n\}$ is an equicontinuous sequence in X.

Now $\{z_n\}$ is uniformly bounded and equi-continuous, so it has a cluster point in view of Arzelà-Ascoli theorem. Now an application of Theorem 2.3 yields that the hyperbolic partial differential inclusion (1.1) has a minimal and a maximal solution in $[\underline{u}, \overline{u}]$ defined on $J_a \times J_b$. This completes the proof. □

3.2 Extremal solutions. Next, we prove a result concerning the extremal solutions for the HPDI (1.1) on $J_a \times J_b$. We need the following definitions in the sequel.

Definition 3.8. A multi-valued function $F(x,y,u)$ is called strict monotone increasing in u almost everywhere for $(x,y) \in J_a \times J_b$ if $F(x,y,u_1) \le F(x,y,u_2)$ almost everywhere $(x,y) \in J_a \times J_b$ for all $u_1,\ u_2 \in \mathbb{R}$ for which $u_1 < u_2$.

Definition 3.9. A multi-valued function $\beta : J_a \times J_b \times \mathbb{R} \to \mathcal{P}_p(\mathbb{R})$ is called L^1-Chandrabhan if

(i) $(x,y) \mapsto \beta(x,y,u(x,y))$ is measurable for all $u \in C(J_a \times J_b, \mathbb{R})$,

(ii) The function $u \mapsto \beta(x,y,u)$ is strict monotone increasing almost everywhere for $(x,y) \in J_a \times J_b$, and

(iii) for each $r > 0$, there exists a function $h_r \in L^1(J_a \times J_b, \mathbb{R})$ such that

$$
\|\beta(x,y,u)\|_{\mathcal{P}} = \sup\{|z| : z \in \beta(x,y,u)\} \le h_r(x,y) \text{ a.e. } (x,y) \in J_a \times J_b
$$

for all $u \in \mathbb{R}$ with $|u| \le r$.

We need the following additional hypothesis in the sequel.

(A_7) The multi-valued function F is L^1-Chandrabhan on $J_a \times J_b \times \mathbb{R}$.

Remark 3.10. Note that if the multi-valued function $F(x, y, u)$ is L^1-Chandrabhan and (A_5) holds, then it is measurable in x and y and integrably bounded on $J_a \times J_b \times [-r, r]$, where $r = \max\{\|u\| : u \in [\underline{u}, \overline{u}]\}$. Such a real number r exists since the order interval $[\underline{u}, \overline{u}]$ is norm-bounded in view of the normality of the cone K in X. It follows from a selection theorem (see Deimling [4]) that S^1_F is non-empty and has closed values on $[\underline{u}, \overline{u}]$, i.e.,

$$S^1_F(u) = \{u \in L^1(J_a \times J_b, \mathbb{R}) \mid u(x, y) \in F(x, y, u(x, y)) \text{ a.e. } (x, y) \in J_a \times J_b\} \neq \emptyset$$

for all $u \in [a, b] \subset C(J_a \times J_b, \mathbb{R})$.

Theorem 3.11. *Assume that* (A_1), (A_5) *and* (A_7) *hold. Then the HPDI* (1.1) *has a minimal and a maximal solution in* $[\underline{u}, \overline{u}]$ *defined on* $J_a \times J_b$.

Proof. The proof is quite similar to that of Theorem 3.1. Now, $S^1_F(u) \neq \emptyset$ for each $u \in [\underline{u}, \overline{u}]$ in view of Remark 3.10. Also, the multi-valued map $u \mapsto S^1_F(u)$ is strictly monotone increasing on $[\underline{u}, \overline{u}]$. Consequently, the multi-valued operator Q defined by (3.3) is strictly monotone increasing on $[\underline{u}, \overline{u}]$. Hence, the desired result follows by an application of Theorem 2.2. □

4. EXISTENCE THEORY FOR THE PROBLEM (1.2)

4.1 Existence result. We need the following definitions in the sequel.

Definition 4.1. A function $u \in AC(J_a \times J_b, \mathbb{R})$ is called a solution of the HPDI (1.2) if there exist a functions $v_1, v_2 \in L^1(J_a \times J_b, \mathbb{R})$ such that $v_1(x, y) \in F(x, y, u(x, y))$ and $v_2(x, y) \in G(x, y, u(x, y))$ a.e. $(x, y) \in J_a \times J_b$ satisfying

$$\begin{aligned} \frac{\partial^2 u(x, y)}{\partial x \partial y} &= v_1(x, y) + v_2(x, y),\ (x, y) \in J_a \times J_b, \\ u(x, 0) &= \phi(x),\ u(0, y) = \psi(y). \end{aligned}$$

Definition 4.2. A function $\underline{u}(\cdot, \cdot) \in AC(J_a \times J_b, \mathbb{R})$ is said to be a strict lower solution of the HPDI (1.2) if for all $v_1 \in S^1_F(\underline{u})$ and $v_2 \in S^1_G(\underline{u})$, we have

$$\begin{aligned} \frac{\partial^2 \underline{u}(x, y)}{\partial x \partial y} &\leq v_1(x, y) + v_2(x, y), \\ \underline{u}(x, 0) \leq \phi(x),&\ \underline{u}(0, y) \leq \psi(y), \end{aligned}$$

for all $(x, y) \in J_a \times J_b$. Similarly, a function $\overline{u}(\cdot, \cdot) \in AC(J_a \times J_b, \mathbb{R})$ is said to be a strict upper solution of the HPDI (1.2) if for all $v_1 \in S^1_F(\overline{u})$ and $v_2 \in S^1_G(\overline{u})$, we have

$$\begin{aligned} \frac{\partial^2 \overline{u}(x, y)}{\partial x \partial y} &\geq v_1(x, y) + v_2(x, y), \\ \overline{u}(x, 0) \geq \phi(x),&\ \overline{u}(0, y) \geq \psi(y), \end{aligned}$$

for all $(x, y) \in J_a \times J_b$.

Definition 4.3. A solution u_M of the problem (1.2) is said to be maximal if for any other solution u to the problem (1.2) we have $u(x,y) \leq u_M(x,y)$ for all $(x,y) \in J_a \times J_b$. Again, a solution u_m of the problem (1.2) is said to be minimal if $u_m(x,y) \leq u(x,y)$ for all $(x,y) \in J_a \times J_b$, where u is any other solution of the problem (1.2) on $J_a \times J_b$.

Definition 4.4. A multi-valued mapping $\beta : J_a \times J_b \times \mathbb{R} \to \mathcal{P}_{cp}(\mathbb{R})$ is said to be Carathéodory if

(i) $(x,y) \mapsto \beta(x,y,z)$ is measurable for each $z \in \mathbb{R}$;
(ii) $z \mapsto \beta(x,y,z)$ is upper semi-continuous for almost each $(x,y) \in J_a \times J_b$.

A Carathéodory multi-valued mapping β is called L^1-Carathéodory if

(iii) for each real number $r > 0$ there exists a $h_r \in L^1(J_a \times J_b, \mathbb{R})$ such that
$$\|\beta(x,y,z)\|_{\mathcal{P}} \leq h_r(x,y) \text{ a.e. } (x,y) \in J_a \times J_b,$$
for all $z \in \mathbb{R}$ with $|z| \leq r$.

Finally, a Carathéodory multi-valued map β is called $L^1_{\mathbb{R}}$-Carathéodory if

(iv) there exists a $h \in L^1(J_a \times J_b, \mathbb{R})$ such that
$$\|\beta(x,y,z)\|_{\mathcal{P}} \leq h(x,y) \text{ a.e. } (x,y) \in J_a \times J_b,$$
for all $z \in \mathbb{R}$.

Then we have the following lemmas due to Lasota and Opial [14].

Lemma 4.5. *If* $\dim(X) < \infty$ *and* $\beta : J \times X \times X \to \mathcal{P}_{cp}(X)$ *is* L^1*-Carathéodory, then* $S^1_\beta(x) \neq \emptyset$ *for each* $x \in X$.

Lemma 4.6 (Lasota and Opial [14])**.** *Let* X *be a Banach space. Let* $\beta : J_a \times J_b \times X \to \mathcal{P}_{cp}(X)$ *be an* L^1*-Carathéodory multi-valued map with* $S^1_\beta \neq \emptyset$*, and let* $\mathcal{L}$ *be a linear continuous mapping from* $L^1(J_a \times J_b, X)$ *into* $C(J_a \times J_b, X)$*, then the operator*
$$\mathcal{L} \circ S^1_\beta : C(J_a \times J_b, X) \to \mathcal{P}_{cp,cv}(C(J_a \times J_b, X))$$
$$u \mapsto (\mathcal{L} \circ S^1_\beta)(u) := \mathcal{L}(S^1_\beta(u))$$
is a closed graph operator in $C(J_a \times J_b, X) \times C(J_a \times J_b, X)$.

We consider the following set of hypotheses in the sequel.

(B_1) $F, G : J_a \times J_b \times \mathbb{R} \to \mathcal{P}_{cp,cv}(\mathbb{R})$.
(B_2) G is L^1- Carathéodory.
(B_3) The multi-valued mapping $(x,y) \mapsto G(x,y,z(x,y))$ is measurable and there exists a function $k \in L^1(J_a \times J_b, \mathbb{R})$ such that
$$d_H(G(x,y,z(x,y)), G(x,y,\overline{z}(x,y))) \leq k(x,y)|z(x,y) - \overline{z}(x,y)| \text{ a.e. } (x,y) \in J_a \times J_b$$
for all $z, \overline{z} \in \mathbb{R}$.

(B_4) The multi-valued map $u \mapsto S^1_G(u)$ is right monotone increasing in $C(J_a \times J_b, \mathbb{R})$.

(B_5) HPDI (1.2) has a strict lower solution $\underline{u}$ and a strict upper solution $\overline{u}$ with $\underline{u} \leq \overline{u}$.

Our main existence theorem for the problem (1.1) is

Theorem 4.7. *Assume that the hypotheses (A_2)-(A_4), (A_6) and (B_1)-(B_2) and (B_4)-(B_5) hold. Then the HPDI* (1.2) *has a solution in $[\underline{u}, \overline{u}]$ defined on $J_a \times J_b$.*

Proof. Let $X = C(J_a \times J_b, \mathbb{R})$ and define an order interval $[\underline{u}, \overline{u}]$ in X which is well defined in view of hypothesis (B_5). Define two multi-valued maps A, B on $[\underline{u}, \overline{u}]$ by

$$A(u) = \left\{ z \in X : z(x,y) = \int_0^x \int_0^y v_1(t,s)\, ds\, dt,\ v_1 \in S^1_G(u) \right\} \tag{4.1}$$

and

$$B(u) = \left\{ z \in X : z(x,y) = z_0(x,y) + \int_0^x \int_0^y v_2(t,s)\, ds\, dt,\ v_2 \in S^1_F(u) \right\}. \tag{4.2}$$

Then the HPDI (1.2) is transformed into the operator inclusion

$$u(x,y) \in Au(x,y) + Bu(x,y),\ (x,y) \in J_a \times J_b. \tag{4.3}$$

Clearly, $A, B : [\underline{u}, \overline{u}] \to X$. We shall show that the operators A and B satisfy all the conditions of Theorem 2.5. The proof will be given in the following steps.

Step I : First, we show that A has compact values on $[\underline{u}, \overline{u}]$. Observe that if $(x,y) \in J_a \times J_b$, then the operator A is equivalent to the composition $\mathcal{L} \circ S^1_G$ of two operators on $L^1(J_a \times J_b, \mathbb{R})$, where $\mathcal{L} : L^1(J_a \times J_b, \mathbb{R}) \to X$ is the continuous operator defined by

$$\mathcal{L}v(x,y) = \int_0^x \int_0^y v(t,s)\, ds\, dt \tag{4.4}$$

for $(x,y) \in J_a \times J_b$. To show A has compact values, it then suffices to prove that the composition operator $\mathcal{L} \circ S^1_G$ has compact values on $[\underline{u}, \overline{u}]$. Now proceeding with the arguments similar to that of Step I in the proof of Theorem 3.1, it can be shown that $(\mathcal{L} \circ S^1_G)(u)$ is a compact set in X for each $u \in [\underline{u}, \overline{u}]$. Therefore, A is a compact-valued multi-valued operator on $[\underline{u}, \overline{u}]$.

Step II : Next we show that A is completely continuous on $[\underline{u}, \overline{u}]$. First, we show that B is compact on $[\underline{u}, \overline{u}]$. Let $u \in [\underline{u}, \overline{u}]$ be arbitrary. Then, for each $z \in B(u)$, there exists $v \in S^1_G(u)$ such that for each $(x,y) \in J_a \times J_b$ we have

$$z(x,y) = \int_0^x \int_0^y v(t,s)\, ds\, dt.$$

From (H5), we have

$$|z(x,y)| \leq \int_0^a \int_0^b |v(t,s)|\, ds\, dt \leq \int_0^a \int_0^b h_r(t,s)\, ds\, dt \leq \|h_r\|_{L^1}.$$

Next, we show that B maps $[\underline{u}, \overline{u}]$ into an equicontinuous subset of X. Let $u \in [\underline{u}, \overline{u}]$ be arbitrary and let $(x_1, y_1), (x_2, y_2) \in J_a \times J_b$. Then for each $z \in B(u)$,

$$|z(x_2, y_2) - z(x_1, y_1)| \leq \left| \int_{x_1}^{x_2} \int_{y_1}^{y_2} |v(t,s)|\, ds\, dt \right| \leq \left| \int_{x_1}^{x_2} \int_{y_1}^{y_2} h_r(t,s)\, ds\, dt \right|.$$

The right hand side tends to zero as $(x_2, y_2) \to (x_1, y_1)$. An application of Arzelá-Ascoli Theorem yields that the operator $B : [\underline{u}, \overline{u}] \to \mathcal{P}_p(X)$ is compact. Next we prove that B has a closed graph. Let $u_n \to u_*$, $z_n \in B(u_n)$ and $z_n \to z_*$. We need to show that $z_* \in B(u_*)$. $z_n \in B(u_n)$ implies that there exists $v_n \in S_{G,u_n}$ such that for each $(x, y) \in J_a \times J_b$,

$$z_n(x, y) = \int_0^x \int_0^y v_n(t,s)\, ds\, dt.$$

We must show that there exists $v_* \in S^1_G(u_*)$ such that for each $(x, y) \in J_a \times J_b$,

$$z_*(x, y) = \int_0^x \int_0^y v_*(t,s)\, ds\, dt.$$

Clearly, we have

$$\|z_n - z_*\|_\infty \to 0 \quad \text{as } n \to \infty.$$

Consider the continuous linear operator $\mathcal{L} : L^1(J_a \times J_b, \mathbb{R}) \to C(J_a \times J_b, \mathbb{R})$ given by

$$v \mapsto \mathcal{L}(v)(x, y) = \int_0^x \int_0^y v(t,s)\, ds\, dt.$$

From Lemma 4.5, it follows that $\mathcal{L} \circ S^1_G$ is a closed graph operator. Moreover, we have

$$z_n(x, y) \in \mathcal{L}(S^1_G(u_n)).$$

Since $u_n \to u_*$, it follows from Lemma 4.5 that

$$z_*(x, y) = \int_0^x \int_0^y v_*(t,s)\, ds\, dt$$

for some $v_* \in S^1_G(u_*)$. This shows that A is a completely continuous multi-valued operator on $[\underline{u}, \overline{u}]$.

Step III : Next we show that B is a totally bounded multi-valued operator on $[\underline{u}, \overline{u}]$. Let $z \in \bigcup B([\underline{u}, \overline{u}])$ be arbitrary. Then there is a $v \in S^1_F(u)$ for some $u \in [\underline{u}, \overline{u}]$ such that

$$z(x, y) = z_0(x, y) + \int_0^x \int_0^y v(t,s)\, ds\, dt.$$

Then, we have

$$\begin{aligned} |z(x, y)| &\leq |z_0(x, y)| + \int_0^a \int_0^b |v(t,s)|\, ds\, dt \\ &\leq |z_0(x, y)| + \int_0^a \int_0^b h(t,s)\, ds\, dt \\ &\leq \|z_0\|_\infty + \|h\|_{L^1}. \end{aligned}$$

This shows that $\bigcup B([\underline{u},\overline{u}])$ is a uniformly bounded subset of X. Next, we show that $\bigcup B([\underline{u},\overline{u}])$ is an equicontinuous set in X. Let $(x_1,y_1),(x_2,y_2)\in J_a\times J_b$. Then we have

$$\begin{aligned}|z(x_2,y_2)-z(x_1,y_1)| &\le |z_0(x_1,y_1)-z_0(x_2,y_2)|+\left|\int_{x_1}^{x_2}\int_{y_1}^{y_2}|v(t,s)|\,ds\,dt\right|\\ &\le |z_0(x_1,y_1)-z_0(x_2,y_2)|+\left|\int_{x_1}^{x_2}\int_{y_1}^{y_2}h(t,s)\,ds\,dt\right|.\end{aligned}$$

The right hand side tends to zero as $(x_2,y_2)\to(x_1,y_1)$. So, $\bigcup B([\underline{u},\overline{u}])$ is an equicontinuous sequence in X. Now $\bigcup B([\underline{u},\overline{u}])$ is uniformly bounded and equi-continuous, so it totally bounded in view of Arzelà-Ascoli theorem.

Step IV : From the hypotheses (A_4) and (B_4), it follows that the multi-valued maps $u\mapsto S^1_F(u)$ and $u\mapsto S^1_G(u)$ are right monotone increasing in $[\underline{u},\overline{u}]$. Therefore,

$$S^1_F(u_1)\overset{i}{\le} S^1_F(u_2)\quad\text{and}\quad S^1_G(u_1)\overset{i}{\le} S^1_G(u_2)$$

for all $u_1,u_2\in[\underline{u},\overline{u}]$ for which $u_1\le u_2$. Consequently, the multi-valued operators A and B are right monotone increasing on $[\underline{u},\overline{u}]$.

Thus, the multi-valued operators A and B satisfy all the conditions of Theorem 2.5 and hence the problem (1.2) has a solution in $[\underline{u},\overline{u}]$ defined on $J_a\times J_b$. This completes the proof. □

Theorem 4.8. *Assume that the hypotheses (A_2)-(A_4), (A_6), (B_1),(B_3) and (B_4)-(B_5) hold. Then the HPDI* (1.2) *has a solution in $[\underline{u},\overline{u}]$ defined on $J_a\times J_b$.*

Proof. Let $X=C(J_a\times J_b,\mathbb{R})$ and define an order interval $[\underline{u},\overline{u}]$ in X which is well defined in view of hypothesis (B_5). Define two multi-valued maps A,B on $[\underline{u},\overline{u}]$ by (4.1) and (4.2) respectively.

Define an equivalent norm $\|\cdot\|$ in $C(J_a\times J_b,\mathbb{R})$ by

$$\|u\|=\sup_{(x,y)\in J_a\times J_b} e^{-2K(x,y)}|u(x,y)| \tag{4.5}$$

where

$$K(x,y)=\int_0^x\int_0^y k(t,s)\,ds\,dt.$$

Then we have

$$|u(x,y)|\le e^{2K(x,y)}\|u\|. \tag{4.6}$$

See Dhage [8] and the references therein.

We show that A is a multi-valued contraction on $[\underline{u},\overline{u}]$. Let $u_1,u_2\in[\underline{u},\overline{u}]$ and $z_1\in A(u_1)$. Then, there exists $v_1(x,y)\in G(x,y,u_1(x,y))$ such that for each $(x,y)\in J_a\times J_b$,

$$z_1(x,y)=\int_0^x\int_0^y v_1(t,s)\,ds\,dt.$$

From (B_3), it follows that

$$d_H(G(x,y,u_1(x,y)),G(x,y,u_2(x,y))) \le k(x,y)|u_1(x,y)-u_2(x,y)|.$$

Hence, there exists $w \in F(x,y,u_2(x,y))$ such that

$$|v_1(x,y)-w| \le k(x,y)|u_1(x,y)-u_2(x,y)|.$$

Consider $U : J_a \times J_b \to \mathcal{P}_p(\mathbb{R})$ given by

$$U(x,y) = \{w \in \mathbb{R} : |v_1(x,y)-w| \le k(x,y)|u_1(x,y)-u_2(x,y)|\}.$$

Since the multi-valued mapping $V(x,y) = U(x,y) \cap G(x,y,u_2(x,y))$ is measurable (see Proposition III.4 in [2]), there exists a function $v_2(x,y)$ which is a measurable selection for V. So, $v_2(x,y) \in G(x,y,u_2(x,y))$ and for each $(x,y) \in J_a \times J_b$,

$$|v_1(x,y)-v_2(x,y)| \le k(x,y)|u_1(x,y)-u_2(x,y)|.$$

Let us define for each $(x,y) \in J_a \times J_b$,

$$z_2(x,y) = \int_0^x \int_0^y v_2(t,s)\,ds\,dt.$$

Thus, we have by (B_3),

$$\begin{aligned}
e^{-2K(x,y)}|z_1(x,y)-z_2(x,y)| &\le e^{-2K(x,y)} \int_0^x \int_0^y k(t,s)|u_1(t,s)-u_2(t,s)|\,ds\,dt \\
&\le e^{-2K(x,y)} \int_0^x \int_0^y k(t,s)\{e^{-2K(t,s)}|u_1(t,s)-u_2(t,s)|\}e^{2K(t,s)}\,ds\,dt \\
&\le e^{-2K(x,y)} \int_0^x \int_0^y k(t,s)\|u_1-u_2\|e^{2K(t,s)}\,ds\,dt \\
&\le e^{-2K(x,y)} \int_0^x \int_0^y \frac{1}{2}\frac{\partial^2}{\partial t \partial s}\left[e^{2K(t,s)}\right]\|u_1-u_2\|\,ds\,dt \\
&= e^{-2K(x,y)} \left(\frac{e^{2K(x,y)}}{2} \Bigg|_{(0,0)}^{(x,y)} \right) \|u_1-u_2\| \\
&\le \frac{1}{2}\|u_1-u_2\|.
\end{aligned}$$

Taking supremum over t we obtain

$$\|z_1-z_2\| \le \frac{1}{2}\|u_1-u_2\|.$$

By an analogous relation, obtained by interchanging the roles of u_1 and u_2, it follows that

$$d_H(A(u_1),A(u_2)) \le \frac{1}{2}\|u_1-u_2\|.$$

So, A is a multi-valued contraction on X. The rest of the proof is similar to that of Theorem 3.3 and now the conclusion follows by an application of Theorem 2.6. This completes the proof. □

4.2 Extremal solutions. Finally, we prove the existence theorems for the extremal solutions for HPDI (1.2) between the given strict upper and lower solutions on $J_a \times J_b$ under suitable conditions.

We need the following hypothesis in the sequel.

(B_6) The multi-valued function $G(x, y, u)$ is strict monotone increasing in u almost everywhere for $(x, y) \in J_a \times J_b$.

Theorem 4.9. *Assume that the hypotheses (A_7), (B_1),(B_2) and (B_4)-(B_6) hold. Then the HPDI* (1.2) *has a minimal and a maximal solution in* $[\underline{u}, \overline{u}]$ *defined on* $J_a \times J_b$.

Proof. Let $X = C(J_a \times J_b, \mathbb{R})$ and define an order interval $[\underline{u}, \overline{u}]$ in X which is well defined in view of hypothesis (B_5). Define two multi-valued maps A, B on $[\underline{u}, \overline{u}]$ by (4.1) and (4.2) respectively. It can be shown as in the proof of Theorem 4.2 that A and B are respectively completely continuous and totally bounded compact-valued multi-valued operators on $[\underline{u}, \overline{u}]$. Since the multi-valued functions $F(x, y, u)$ and $G(x, y, u)$ are strictly monotone increasing in u almost everywhere $(x, y) \in J_a \times J_b$, the multi-valued maps $u \mapsto S^1_F(u)$ and $u \mapsto S^1_G(u)$ are strictly monotone increasing in $[\underline{u}, \overline{u}]$. As a result the multi-valued operators A and B are strictly monotone increasing in $[\underline{u}, \overline{u}]$. Now the desired conclusion follows by an application of Theorem 2.8. This completes the proof. □

Theorem 4.10. *Assume that* (A_7)*,* (B_1)*,*(B_3) *and* (B_4)*-*(B_5) *hold. Then the HPDI* (1.2) *has a minimal and a maximal solution in* $[\underline{u}, \overline{u}]$ *defined on* $J_a \times J_b$.

Proof. Let $X = C(J_a \times J_b, \mathbb{R})$ and define an order interval $[\underline{u}, \overline{u}]$ in X which is well defined in view of hypothesis (B_5). Define two multi-valued maps A, B on $[\underline{u}, \overline{u}]$ by (4.1) and (4.2) respectively. It can be shown as in the proof of Theorem 4.2 that A and B are respectively contraction and totally bounded compact-valued multi-valued operators on $[\underline{u}, \overline{u}]$. Since the multi-valued functions $F(x, y, u)$ and $G(x, y, u)$ are strictly monotone increasing in u almost everywhere $(x, y) \in J_a \times J_b$, the multi-valued maps $u \mapsto S^1_F(u)$ and $u \mapsto S^1_G(u)$ are are strictly monotone increasing in $[\underline{u}, \overline{u}]$. As a result, the multi-valued operators A and B are strictly monotone increasing in $[\underline{u}, \overline{u}]$. Now the desired conclusion follows by an application of Theorem 2.9. This completes the proof. □

In the following we give an example illustrating the abstract theory developed for the problem (1.1).

5. AN EXAMPLE

Example 5.1. Let $J_a = [0, 1] = J_b$ and define two functions $\phi, \psi : [0, 1] \to \mathbb{R}$ by $\phi(x) = x$ and $\psi(y) = y^2$ so that $\phi(0) = \psi(0)$. Now consider the hyperbolic differential inclusion

$$\begin{cases} \dfrac{\partial^2 u(x,y)}{\partial x \partial y} \in G(x,y,u(x,y)), \quad (x,y) \in J_a \times J_b, \\ \\ \qquad u(x,0) = x, \quad u(0,y) = y^2, \end{cases} \tag{5.1}$$

where the multi-valued function $G : J_a \times J_b \times \mathbb{R} \to \mathcal{P}_p(\mathbb{R})$ is defined by

$$G(x,y,u) = \begin{cases} \left\{\frac{1}{30}\right\}, & \text{if } u < 0, \\ \left[\frac{1}{20}, \frac{1}{18}\right], & \text{if } u = 0, \\ \left\{\frac{1}{17}\right\}, & \text{if } u \in (0,1), \\ \left\{\frac{[u]}{15+[u]}\right\}, & \text{if } u \geq 1, \end{cases} \tag{5.2}$$

for all $x, y \in [0,1]$ and $u \in \mathbb{R}$, where $[u]$ is the greatest integer not greater than u. Now the multi-valued map G satisfies all the conditions of Theorem 3.2 with $\underline{u} \equiv 0$ and $\overline{u} \equiv 3$ on J. Hence, the above HPDI (16) has a minimal and a maximal solution in $[0,3]$ defined on $[0,1] \times [0,1]$.

Remark 5.2. Notice that we do not require the multi-valued functions involved in the hyperbolic partial differential inclusions (1.1) or (1.2) of this paper to have convex values for any of our existence results which is the usual case with the most of the existence results for differential inclusions. This is the advantage of the monotone method over those of topological methods in the existence theory for such differential inclusions. Finally, while concluding this paper, we mention that the monotone method of this paper can also be applied to problems of higher order hyperbolic partial differential inclusions with appropriate modifications for proving the various aspects of the solutions. Some of the results in this direction will be reported elsewhere.

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