Communications in Applied Analysis **12** (2008), no. 3, 341–352

MULTIPLE PERIODIC SOLUTIONS FOR A FIRST ORDER NONLINEAR FUNCTIONAL DIFFERENTIAL EQUATION WITH APPLICATIONS TO POPULATION DYNAMICS

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ABSTRACT. In this paper, we use Leggett-Williams multiple fixed point theorem to obtain several different sufficient conditions for the existence of at least three positive periodic solutions for the first order functional differential equations of the form

$$y'(t) = -a(t)y(t) + \lambda f(t, y(h(t))).$$

Some applications to mathematical ecological models and population models are also given.

AMS (MOS) Subject classification: 34K13, 34C10, 34C27

1. INTRODUCTION

Consider the nonlinear first order functional differential equation of the form

$$y'(t) = -a(t)y(t) + \lambda f(t, y(h(t))) \tag{1.1}$$

where λ is a positive parameter, $h, a \in C(R, R_+)$ are T-periodic with $a(t) \not\equiv 0$, and $f \in C(R \times R_+, R_+)$ is T-periodic with respect to the first variable, where $R = (-\infty, \infty), R_+ = [0, \infty)$ and T is a positive constant. Our aim of this paper is to study the existence of at least three positive T-periodic solutions of (1.1) using Leggett-Williams multiple fixed point theorem [12].

If $h(t) = t - \tau(t)$ and $\tau \in C(R, R_+)$ with $\tau(t) \leq t$, then (1.1) takes the form

$$y'(t) = -a(t)y(t) + \lambda f(t, y(t - \tau(t))). \tag{1.2}$$

As the existence of positive periodic solutions of (1.1) is regarded, one can find from the arguments in the succeeding sections that some similar results can be derived for (1.2). Functional differential equation of the form (1.2) includes many mathematical

Received August 14, 2008

ecological and population models (directly or after some transformation), such as:
(a) Lasota-Wazewska model [4, 6, 13, 16, 17, 19, 23, 24, 26]

$$y'(t) = -a(t)y(t) + p(t)e^{-\gamma(t)y(t-\tau(t))}. \tag{1.3}$$

(b) Models for blood cell production [3, 5, 14, 15, 22]

$$y'(t) = -a(t)y(t) + b(t)\frac{y^m(t-\tau(t))}{1+y^n(t-\tau(t))}. \tag{1.4}$$

(c) Nicholson's blowflies models [7, 9, 11, 15, 22]

$$y'(t) = -a(t)y(t) + b(t)y^m(t-\tau(t))e^{-\gamma(t)y^n(t-\tau(t))}. \tag{1.5}$$

The variation of the environment, in particular the periodic variation of the environment, plays an important role in many biological and ecological systems. Hence it will be interesting to study the existence of at least one periodic solution of (1.3)–(1.5). Many authors have used fixed point theorem of cone expansion and cone compression method, upper-lower solution method and iterative technique to find out at least one and at least two positive periodic solutions of (1.1) although it is very difficult to find upper and lower solutions for a general differential equation. For instance, see [2, 7, 10, 16, 21, 22, 25, 26]. In [26], Zhang et al. have used fixed point theorem of cone expansion and cone compression to investigate the existence of multiple positive periodic solutions for the first order differential equation. On the other hand, it has been observed that very few papers exist in the literature on the existence of at least three periodic solutions of (1.1). For example, see [1]. The use of Leggett-Williams multiple fixed point theorem for the existence of at least three periodic solutions of (1.1) is relatively scarce in the literature. In this paper, by using Leggett-Williams multiple fixed point theorem, we provide some results on the existence of at least three positive periodic solutions of (1.1) and then apply our results to obtain some new criteria for the existence of at least three positive periodic solutions of the models (1.3)–(1.5). Some explicit intervals on the parameter λ are given while proving our results. The results of this paper can be extended to

$$y'(t) = -a(t)y(t) + f(t, y(t-\tau_1(t)), \ldots, y(t-\tau_n(t))) \tag{1.6}$$

where $0 \le \tau_i(t) \le t$, $i = 0, 1, \ldots, n$, $f \in C(R \times R_+^n, R_+)$ is periodic with respect to the first variable, $\tau_i(t+T) = \tau_i(t)$, $1 \le i \le n$.

For the last two decades, there has been a rich literature on the use of the fixed point theorems on the existence of positive solutions of boundary value problems. The existence of periodic solutions of this type equation is closely related to the existence of general boundary value problems. Many ideas in the paper and the references come from those for general boundary value problems.

2. PRELIMINARY

First, observe that the solution of (1.1) may be written in the form

$$y(t) = \lambda \int_t^{t+T} G(t,s) f(s, y(h(s)))\, ds,$$

where $G(t,s) = \frac{e^{\int_t^s a(\theta)\, d\theta}}{e^{\int_0^T a(\theta)\, d\theta} - 1}$ is the Green's kernel. The Green's kernel $G(t,s)$ used in this paper is well known in the literature. As in many articles, its lower bound, being positive, is used for defining a cone. It is easy to verify that $G(t,s)$ satisfies the property

$$0 < \alpha = \frac{1}{\delta - 1} \le G(t,s) \le \frac{\delta}{\delta - 1} = \beta, \;\; for\ s \in [t, t+T] \tag{2.1}$$

where

$$\delta = e^{\int_0^T a(\theta)\, d\theta}.$$

The following concept from Leggett-Williams multiple fixed point theorem [12] is needed for our use in the sequel: Let X be a Banach space and K be a cone in X. A mapping ψ is said to be a concave nonnegative continuous functional on K if $\psi : K \to [0, \infty)$ is continuous and

$$\psi(\mu x + (1-\mu) y) \ge \mu \psi(x) + (1-\mu)\psi(y), \;\; x, y \in K, \;\; \mu \in [0,1].$$

Let $a, b, c > 0$ be constants with K and X as defined above. Define

$$K_a = \{y \in K; \|y\| < a\}, \qquad K(\psi, b, c) = \{y \in K; \;\psi(y) \ge b, \;\|y\| \le c\}.$$

Theorem 2.1 (Leggett-Williams multiple fixed point theorem, [12]). *Let $X = (X, \|.\|)$ be a Banach space and $K \subset X$ a cone, and $c_4 > 0$ a constant. Suppose there exists a concave nonnegative continuous function ψ on K with $\psi(u) \le u$ for $u \in \overline{K}_{c_4}$ and let $A : \overline{K}_{c_4} \to \overline{K}_{c_4}$ be a continuous compact map. Assume that there are numbers c_1, c_2 and c_3 with $0 < c_1 < c_2 < c_3 \le c_4$ such that*
(i) $\{u \in K(\psi, c_2, c_3); \psi(u) > c_2\} \ne \phi$ and $\psi(Au) > c_2$ for all $u \in K(\psi, c_2, c_3)$;
(ii) $\|Au\| < c_1$ for all $u \in \overline{K}_{c_1}$;
(iii) $\psi(Au) > c_2$ for all $u \in K(\psi, c_2, c_4)$ with $\|Au\| > c_3$.

Then A has at least three fixed points u_1, u_2 and u_3 in $\overline{K}_{c_4}$. Furthermore, we have $u_1 \in \overline{K}_{c_1}$, $u_2 \in \{u \in K(\psi, c_2, c_4); \psi(u) > c_2\}$, $u_3 \in \overline{K}_{c_4} \backslash \{K(\psi, c_2, c_4) \cup \overline{K}_{c_1}\}$.

Let $X = \{y(t); y(t) \in C(R,R), y(t) = y(t+T)\}$ with the norm $\|y\| = sup_{t\in[0,T]}|y(t)|$; then X is a Banach space with the norm $\|\cdot\|$. Define a cone K in X by

$$K = \{y(t); y \in X, y(t) \ge \frac{1}{\delta}\|y\| \quad \forall t \in [0,T]\}$$

and an operator A_λ on X by

$$(A_\lambda y)(t) = \lambda \int_t^{t+T} G(t,s) f(s, y(h(s)))\, ds.$$

Lemma 2.2. *$A_\lambda(K) \subset K$ and $A_\lambda : K \to K$ is compact and continuous.*

Lemma 2.3. *Existence of positive periodic solutions of* (1.1) *is equivalent to the existence of fixed point problem of A_λ in K.*

The proofs of the Lemma 2.2 and Lemma 2.3 are straightforward and hence we omit their proof.

3. MAIN RESULTS

Let

$$f^h = \limsup_{y\to h} \max_{0\le t\le T} \frac{f(t,y)}{a(t)y}.$$

Theorem 3.1. *Let $f^\infty < T$. Further, assume that there are constants $0 < c_1 < c_2$ such that*
(H_1) $f(t,y) \ge 2\delta c_2$ for $y \in K$ with $c_2 \le y \le \delta c_2$ and $0 \le t \le T$
and
(H_2) $f(t,y) < \frac{(\delta-1)c_1}{\delta}$ for $y \in K$ with $y \le c_1$ and $0 \le t \le T$
hold. Then (1.1) *has at least three positive T-periodic solutions for*

$$\frac{\delta-1}{2\delta T} < \lambda < \frac{1}{T}.$$

Proof. From $f^\infty < T$, it follows that there exist an $\epsilon \in (0,T)$ and $\theta > 0$ such that $f(t,y) \le \epsilon\, a(t)y$ for $y \ge \theta$ and $0 \le t \le T$. Let $\gamma = \max_{0\le y\le\theta, 0\le t\le T} f(t,y)$. Then $f(t,y) \le \epsilon\, a(t)y + \gamma$ for $y \ge 0$ and $0 \le t \le T$. Choose

$$c_4 > \max\left\{\frac{\delta\,\gamma\, T}{(\delta-1)(T-\epsilon)}, \delta c_2\right\}.$$

Then for $y \in \overline{K}_{c_4}$, we have

$$\begin{aligned}
\|A_\lambda y\| &= \sup_{0\le t\le T} \lambda \int_t^{t+T} G(t,s) f(s, y(h(s)))\, ds \\
&\le \sup_{0\le t\le T} \lambda \int_t^{t+T} G(t,s)(a(s)y(h(s))\,\epsilon + \gamma)\, ds \\
&\le \sup_{0\le t\le T} \lambda \int_t^{t+T} G(t,s)(a(s)\|y\|\,\epsilon + \gamma)\, ds \\
&\le \lambda\left[\epsilon c_4 \sup_{0\le t\le T} \int_t^{t+T} a(s) G(t,s)\, ds + \gamma \int_t^{t+T} G(t,s)\, ds\right] \\
&\le \lambda\left[\epsilon\, c_4 + \frac{\gamma\,\delta}{\delta-1} T\right] \\
&< \frac{1}{T}\left[\epsilon\, c_4 + \frac{\gamma\,\delta}{\delta-1} T\right] \\
&< c_4.
\end{aligned}$$

From Lemma 2.2 and the above, it follows that $A_\lambda : \overline{K}_{c_4} \to \overline{K}_{c_4}$. Next, define a nonnegative continuous function ψ on K by $\psi(y) = \min_{t\in[0,T]} y(t)$. Then $\psi(y) \le \|y\|$. Let $c_3 = \delta c_2$ and $\phi_0(t) = \phi_0$, ϕ_0 is any given number satisfying $c_2 < \phi_0 < c_3$. Then $\phi_0 \in \{y : y \in K(\psi, c_2, c_3), \psi(y) > c_2\}$. Further, for $y \in K(\psi, c_2, c_3)$, we have by (H_1)

$$\begin{aligned}
\psi(A_\lambda y) &= \min_{0\le t\le T} \lambda \int_t^{t+T} G(t,s) f(s, y(h(s)))\, ds \\
&\ge \frac{1}{\delta - 1} \lambda \int_0^T f(s, y(h(s))) ds \\
&\ge \frac{\lambda}{\delta - 1} 2\, \delta\, c_2\, T \\
&> c_2.
\end{aligned}$$

Now, let $y \in \overline{K}_{c_1}$. Then, using (H_2)

$$\begin{aligned}
\|A_\lambda y\| &= \sup_{0\le t\le T} \lambda \int_t^{t+T} G(t,s) f(s, y(h(s)))\, ds \\
&\le \frac{\lambda\, \delta}{\delta - 1} \int_0^T f(s, y(h(s))) ds \\
&\le \frac{\lambda\, \delta}{\delta - 1}\, c_1\, \frac{\delta - 1}{\delta} T \\
&< \frac{1}{T}\, c_1\, T \\
&< c_1,
\end{aligned}$$

that is, $A_\lambda y \in \overline{K}_{c_1}$. Finally, for $y \in K(\psi, c_2, c_4)$ and $\|A_\lambda y\| > c_3$, we have

$$c_3 < \|A_\lambda y\| \le \frac{\delta}{\delta - 1}\, \lambda \int_0^T f(s, y(h(s)))\, ds$$

which in turn implies that

$$\begin{aligned}
\psi(A_\lambda y) &\ge \frac{1}{\delta - 1} \lambda \int_0^T f(s, y(h(s)))\, ds \\
&> \frac{c_3}{\delta} = c_2.
\end{aligned}$$

Hence all the conditions of Theorem 2.1 are satisfied. Consequently, (1.1) has at least three positive T-periodic solutions. This completes the proof of the theorem. □

Theorem 3.2. *Let $f^\infty < T$. Assume that there exist constants $0 < c_1 < c_2$ such that*
(H_3) $f(t, y) \ge 2(\delta - 1)c_2$ for $y \in K$ with $c_2 \le y \le \delta c_2$ and $0 \le t \le T$
and (H_2) hold. Then (1.1) has at least three positive T-periodic solutions for

$$\frac{1}{2T} < \lambda < \frac{1}{T}.$$

The proof of the theorem is as same as the proof of Theorem 3.1. Here we use (H_3) in place of (H_1) in the following way to prove the condition (i) of Theorem 2.1. Define ψ on K by $\psi(y) = \min_{t\in[0,T]} y(t)$ and set $c_3 = \delta c_2$. Then

$$\begin{aligned}
\psi(A_\lambda y) &= \min_{0\le t\le T} \lambda \int_t^{t+T} G(t,s) f(s, y(h(s)))\, ds \\
&\ge \frac{1}{\delta-1}\lambda \int_0^T f(s, y(h(s)))\, ds \\
&\ge \frac{\lambda}{\delta-1} 2\,(\delta-1)\, c_2\, T \\
&> \frac{1}{2T(\delta-1)} 2\,(\delta-1)\, c_2\, T \\
&> c_2.
\end{aligned}$$

Thus, by Theorem 2.1, (1.1) has at least three positive T-periodic solutions.

Theorem 3.3. *Let $f^\infty < T$ and $f^0 < T$. Further, assume that there exists a constant $c_2 > 0$ such that (H_1) holds. Then there exist at least three positive T-periodic solutions of (1.1) for*

$$\frac{\delta-1}{2\delta T} < \lambda < \frac{1}{T}.$$

Proof. Since $f^\infty < T$, then there exist $0 < \delta_1 < T$ and $\xi_1 > 0$ such that

$$f(t,y) \le \delta_1 a(t) y \ \ \textit{for } y \ge \xi_1 \textit{ and } 0 \le t \le T.$$

Let

$$\gamma = \max_{0\le y\le \xi_1, 0\le t\le T} f(t,y).$$

Then $f(t,y) \le \delta_1 a(t) y + \gamma$ for $y \ge 0$ and $0 \le t \le T$. Choosing c_4 as in Theorem 3.1, one may prove that $A_\lambda : \overline{K}_{c_4} \to \overline{K}_{c_4}$. Further, defining a continuous function ψ on K by $\psi(y) = y$ and using (H_1), we can prove that the condition (i) of Theorem 2.1 holds. From $f^0 < T$, there exist $\delta_2, 0 < \delta_2 < T$ and $\frac{c_2}{2} > \xi_2 > 0$ such that

$$f(t,y) \le \delta_2 a(t) y \ \textit{for } 0 \le y \le \xi_2 \textit{ and } 0 \le t \le T.$$

Set $0 < c_1 = \xi_2$. Then for $y \in \overline{K}_{c_1}$, we have

$$\begin{aligned}
\|A_\lambda y\| &= \sup_{0\le t\le T} \lambda \int_t^{t+T} G(t,s) f(s, y(h(s)))\, ds \\
&= \sup_{0\le t\le T} \lambda \int_0^{T} G(t,s) f(s, y(h(s)))\, ds \\
&< \lambda\, \delta_2 \sup_{0\le t\le T} \int_0^T G(t,s) a(s) \|y\|\, ds \\
&< \lambda\, c_1\, \delta_2 \sup_{0\le t\le T} \int_0^T a(s) G(t,s)\, ds \\
&< \lambda\, c_1\, \delta_2
\end{aligned}$$

$$
\begin{aligned}
&< \frac{1}{T} c_1\, \delta_2 \\
&< c_1,
\end{aligned}
$$

that is, the condition (ii) of Theorem 2.1 is satisfied. In a similar way to Theorem 3.1, it can be shown that the condition (iii) of Theorem 2.1 is satisfied. Hence there exist at least three positive T-periodic solutions of (1.1). Thus the theorem is proved. □

Theorem 3.4. *Let $f^\infty < T$, $f^0 < T$ and (H_3) hold. Then there exist at least three positive T-periodic solutions of* (1.1) *for*

$$\frac{1}{2T} < \lambda < \frac{1}{T}.$$

In [26], Zhang et.al. proved the following interesting result:

Theorem 3.5. *Let $f^0 < 1$ and $f^\infty < 1$. Further, assume that there exists $\rho > 0$ such that $f(t,y) > a(t)|y|$ for $\mu\rho < |u| < \rho$, where $\mu = exp\{-\int_0^T a(s)\,ds\}$. Then* (1.6) *has at least two positive T-periodic solutions y_1 and y_2 such that*

$$0 < \|y_1\| < \rho < \|y_2\|.$$

We note that our results can be applied to (1.6). Applying Theorem 3.3 or Theorem 3.4 to (1.6), we have the following:

Corollary 3.6. *Let $f^0 < T$ and $f^\infty < T$. Further, assume that there exists a constant $c_2 > 0$ such that either*

(H_4) $f(t,y) \geq 2\delta\, c_2$ *for* $y \in K$ *with* $c_2 \leq y \leq \delta c_2$ *and* $0 \leq t \leq T$

or

(H_5) $f(t,y) \geq 2(\delta - 1)c_2$ *for* $y \in K$ *with* $c_2 \leq y \leq \delta c_2$ *and* $0 \leq t \leq T$.

hold. Then (1.6) *has at least three positive T-periodic solutions for*

$$\frac{\delta - 1}{2\,\delta\, T} < \lambda < \frac{1}{T}.$$

Our Corollary 3.6 is different from Theorem 3.5. Indeed, the upper bound on f^0 and f^∞ is the general period T and there exist at least three T-periodic solutions in Corollary 3.6.

Theorem 3.7. *Let $f^\infty < T$. Assume that there are constants $0 < c_1 < c_2$ such that (H_1) holds and*

(H_6) $f(t,y) < y$ *for* $0 \leq y \leq c_1$ *and* $0 \leq t \leq T$.

Then there exist at least three positive T-periodic solution of (1.1) *for*

$$\frac{\delta - 1}{2\delta T} < \lambda \leq \frac{\delta - 1}{\delta T}.$$

The proof of the theorem is same as Theorem 3.3. One is required to choose

$$c_4 > \max\{\frac{\gamma}{(\delta-1)\delta_1 + T}, \delta c_2\},$$

to show that $A_\lambda : \overline{K}_{c_4} \to \overline{K}_{c_4}$. However, we need the following argument to prove the condition (ii) of Theorem 2.1: from (H_6), we have, for $x \in \overline{K_{c_1}}$

$$\begin{aligned} \|A_\lambda y\| &= \sup_{0\le t\le T} \lambda \int_t^{t+T} G(t,s) f(s, y(h(s)))\, ds \\ &\le \lambda \frac{\delta}{\delta - 1} c_1 T \le c_1. \end{aligned}$$

Example 3.8. Consider the equation

$$y'(t) = -\frac{1}{6\pi}(2+\cos t)y(t) + \frac{3}{28\pi} e^4 y^2(t-\tau) e^{-y(t-\tau)} \tag{3.1}$$

where $\tau > 0$ is a constant. Here $a(t) = \frac{1}{6\pi}(2+\cos t)$, $T = 2\pi$. Set $f(t,y) = \frac{3}{4\pi} e^4 y^2 e^{-y}$ and $\lambda = \frac{1}{7}$. It is easy to verify that $\frac{1}{2T} < \lambda < \frac{1}{T}$, that is, $\frac{1}{4\pi} < \frac{1}{7} < \frac{1}{2\pi}$. Now, $\delta = e^{\int_0^{2\pi} a(t)\, dt} = e^{\frac{2}{3}} = 1.947734041$, $\alpha = \frac{1}{\delta - 1} = 1.05514834$ and $\beta = \frac{\delta}{\delta-1} = 2.05514834$. Further, $f^\infty < 2\pi$ holds. Choose $c_1 = \frac{3}{100}$, $c_2 = \frac{1}{2}$. Then $c_3 = c_2\delta = 0.97386702$. For $c_2 \le y \le c_2\delta$, we obtain $\frac{3}{4\pi} e^4 y^2 e^{-y} > \frac{3}{4\pi} e^4 c_2^2 e^{-c_2\delta}$. Since $\frac{3}{4\pi} e^4 c_2 e^{-c_2\delta} > 2(\delta - 1)$, then $f(t,y) > 2(\delta-1)c_2$ for $c_2 \le y \le c_2\delta$, that is, (H_3) holds. Further, for $0 \le y \le c_1$, $f(t,y) = \frac{3}{4\pi} e^4 y^2 e^{-y} < \frac{3}{4\pi} e^4 c_1^2 < \frac{1}{\beta} c_1$ holds. This in turn implies that (H_2) holds. By Theorem 3.2, (3.1) has at least three positive 2π-periodic solutions. Theorem 3.4 can be applied to this example.

4. APPLICATIONS

Example 4.1. Consider the Lasota-Wazewska model

$$y'(t) = -ay(t) + pe^{-\gamma y(t-\tau)}. \tag{4.1}$$

If $a(t) \equiv a, p(t) \equiv p, \tau(t) \equiv \tau$ *and* $\gamma(t) \equiv \gamma$ are positive constants, then (1.3) reduces to (4.1). In [6, 26] the authors have proved that (4.1) has at least one positive periodic solution. However, to the best of our knowledge, no result exists for the existence of at least three positive periodic solutions of (4.1). It would be interesting to obtain results, directly or after some analysis, for the existence of at least three positive periodic solutions of (4.1). The following result follows from Theorem 3.2.

Theorem 4.2. *Let* $\gamma < 2e$, $\delta \le \frac{2e}{2e-\gamma}$ *and* $\gamma\delta^2 < \delta - 1$ *hold. Then* (4.1) *has at least three positive* T*-periodic solutions for* $\frac{1}{2T} < p < \frac{1}{T}$.

Proof. Let $f(t,y) = e^{-\gamma y}$. Then $f(t,y) > e^{-\gamma\delta c_2}$ for $c_2 \le y \le \delta c_2$. Thus, (H_3) holds if and only if $e^{-\gamma\delta c_2} \ge 2(\delta-1)c_2$ for $c_2 \le y \le \delta c_2$. Now choose $c_2 = \frac{1}{\delta\gamma}$. Then $\delta \le \frac{2e}{2e-\gamma}$ and $c_2 = \frac{1}{\delta\gamma}$ imply that $e^{-\gamma\delta c_2} \ge 2(\delta-1)c_2$ for $c_2 \le y \le \delta c_2$. Hence (H_3) is satisfied. It is clear that $f^\infty < T$. In order to apply Theorem 3.2, we are required to show the existence of a constant c_1 such that $0 < c_1 < c_2$ and (H_2) holds. Since $f(t,y) < 1$, then (H_2) holds if $c_1 > \frac{\delta}{\delta-1}$. Indeed, (H_2) holds if $1 < \frac{\delta-1}{\delta}c_1$ for $0 \le y \le c_1$, that is, $c_1 > \frac{\delta}{\delta-1}$. Now we show the existence of c_1. Clearly, $\gamma\delta^2 < \delta - 1$ implies that $\frac{\delta}{\delta-1} < \frac{1}{\gamma\delta} = c_2$. Thus, there exists a real $c_1 \in (\frac{\delta}{\delta-1}, \frac{1}{\gamma\delta})$ such that $\frac{\delta}{\delta-1} < c_1 < c_2 = \frac{1}{\gamma\delta}$. Thus $f(t,y)$ satisfy the property (H_2). By Theorem 3.2, (4.1) has at least three positive T-periodic solutions for $\frac{1}{2T} < p < \frac{1}{T}$. The theorem is proved. □

Example 4.3. If $a(t) \equiv a$, $b(t) \equiv b$ and $\tau(t) \equiv \tau$ are constants, then (1.4) reduces to

$$y'(t) = -ay(t) + b\frac{y^m(t-\tau)}{1+y^n(t-\tau)}. \tag{4.2}$$

Set

$$\mu = 2(\delta-1)\delta^{2m-1}\frac{n}{1+n-m}(\frac{1+n-m}{m-1})^{\frac{m-1}{n}}. \tag{4.3}$$

Applying Theorem 3.4 to Eq. (4.2), we obtain

Theorem 4.4. *Let $0 < m-1 < n$. Eq. (4.2) has at least three positive T-periodic solutions for $\frac{\mu}{2T} < b < \frac{\mu}{T}$, where μ is given in (4.3).*

Proof. Since $\delta > 1$ and $0 < m-1 < n$, then $\mu > 0$. Equation (4.1) can be written as

$$y'(t) = -ay(t) + \frac{b}{\mu}\mu\frac{y^m(t-\tau)}{1+y^n(t-\tau)}. \tag{4.4}$$

Let $f(t,y) = \mu\frac{y^m}{1+y^n}$. Since $m > 1$, then $f^0 = 0 < T$ and $f^\infty = 0 < T$. To complete the proof of the theorem, inview of Theorem 3.4, we need to find a $c_2 > 0$ such that (H_3) holds. Set $c_2 = \frac{1}{\delta}(\frac{m-1}{1+n-m})^{\frac{1}{n}}$. Now, for $c_2 \le \|y\| \le \delta c_2$, we have

$$\mu\frac{y^m}{1+y^n} \ge \mu\frac{(\|y\|/\delta)^m}{1+\delta^n c_2^n} \ge \frac{\mu}{\delta^m}.\frac{c_2^m}{1+\delta^n c_2^n} \tag{4.5}$$

Since $c_2 = \frac{1}{\delta}(\frac{m-1}{1+n-m})^{\frac{1}{n}}$, then $1+\delta^n c_2^n = \frac{n}{1+n-m}$. Then, from (4.5) we have, using (4.3)

$$\begin{aligned}
\mu\frac{y^m}{1+y^n} &\ge \frac{c_2^m}{\delta^m}.\frac{n-m+1}{n}2(\delta-1)\delta^{2m-1}.\frac{n}{n-m+1}.(\frac{1+n-m}{m-1})^{\frac{m-1}{n}} \\
&\ge 2(\delta-1)c_2^m\delta^{m-1}.(\frac{1+n-m}{m-1})^{\frac{m-1}{n}} \\
&\ge 2(\delta-1)c_2^m\delta^{m-1}.\frac{1}{\delta^{m-1}c_2^{m-1}} \\
&\ge 2(\delta-1)c_2.
\end{aligned}$$

This completes the proof of the theorem. □

Example 4.5. If $a(t) \equiv a, b(t) \equiv b, \tau(t) \equiv \tau$ *and* $\gamma(t) \equiv \gamma$ are positive constants, then (1.5) reduces to

$$y'(t) = -ay(t) + by^m(t-\tau)e^{-\gamma y^n(t-\tau)} \tag{4.6}$$

Theorem 4.6. *Let $m > 1$ and $2e(\delta-1)\delta^{(m-1)}\gamma^{\frac{m-1}{n}} \le 1$. Then Eq. (4.6) has at least three positive T-periodic solutions for $\frac{1}{2T} < b < \frac{1}{T}$.*

Proof. Let $f(t,y) = y^m e^{-\gamma y}$. Set $c_2 = \frac{1}{\delta\,\gamma^{1/n}}$. Then it is easy to observe that $c_2 = \frac{1}{\delta\,\gamma^{1/n}}$ and $2e(\delta-1)\delta^{(m-1)}\gamma^{\frac{m-1}{n}} \leq 1$ imply that $c_2^m e^{-\delta\,\gamma^n\, c_2^n} > 2(\delta-1)c_2$ for $c_2 \leq y \leq \delta c_2$ and hence (H_3) is satisfied. Further, $f^\infty = 0 < T$, and $f^0 = 0 < T$ hold. Then by Theorem 3.4, (4.6) has at least three positive T-periodic solutions for $\frac{1}{2T} < b < \frac{1}{T}$. The proof is complete. □

Although, the condition in Theorem 4.6 looks complicated, it is easy to verify. The following corollary follows from Theorem 4.6.

Corollary 4.7. *Let $m > 1$ and $1 < \delta \leq \frac{1}{\gamma^{1/n}} \leq \frac{1+2e}{e}$. Then (4.6) has at least three positive T-periodic solutions for $\frac{1}{2T} < b < \frac{1}{T}$.*

Proof. In fact, $1 < \delta \leq \frac{1}{\gamma^{1/n}} \leq \frac{1+2e}{e}$ implies that $2e(\delta-1)\delta^{(m-1)}\gamma^{\frac{m-1}{n}} \leq 1$ and hence by Theorem 4.6, (4.6) has at least three positive T-periodic solutions for $\frac{1}{2T} < b < \frac{1}{T}$. This completes the proof of the theorem. □

Remark 4.8. One may find that Theorem 4.4 do not work for $m = 1$ and $m = n+1$ and Theorem 4.6 do not work for $m = 1$. Thus, it would be interesting to obtain sufficient conditions on the coefficient functions for the existence of atleast three positive T-periodic solutions of the equations (4.2) and (4.6) for general m and n.

ACKNOWLEDGMENTS

The work of the first and the third authors is supported by *National Board for Higher Mathematics (DAE) Govt. of India,* under sponsored research scheme, vide grant no. 48/5/2006-R&D-II/1350, Dated February 26, 2007.

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