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THEORY OF FRACTIONAL DIFFERENTIAL EQUATIONS WITH THREE-POINT BOUNDARY CONDITIONS

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This paper studies existence and uniqueness results in a Banach space for a three-point boundary value problem involving a fractional differential equation given by

$$\begin{aligned} {}^cD^q x(t) &= f(t, x(t)), \quad t \in [0, T], \quad 0 < q < 1, \\ \alpha x(0) + \beta x(T) &= \gamma x(\eta), \qquad 0 < \eta < T, \quad \alpha + \beta \neq \gamma. \end{aligned}$$

The contraction mapping principle and Krasnoselskii's fixed point theorem are employed to establish the results.

Keywords and Phrases. Fractional differential equations, contraction principle, Krasnoselskii's fixed point theorem

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1. INTRODUCTION

In recent years, the application of fractional calculus in physics, continuum mechanics, signal processing, and electromagnetics, with few examples of applications in bioengineering are high lighted in the literature. The methods of fractional calculus, when defined as a Laplace, Sumudu or Fourier convolution product, are suitable for solving many problems in emerging biomedical research. The electrical properties of nerve cell membranes and the propagation of electrical signals are well characterized by differential equations of fractional order. The fractional derivative accurately describes natural phenomena that occur in such common engineering problems as heat transfer, electrode/electrolyte behavior, and sub-threshold nerve propagation. Application of fractional derivatives to viscoelastic materials establishes, in a natural way, hereditary integrals and the power law stress-strain relationship for modeling biomaterials. Fractional operations by following the original approach of Heaviside, demonstrate the basic operations of fractional calculus on well-behaved functions such

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as step, ramp, pulse, and sinusoidal of engineering interest, and can easily be applied in electrochemistry, physics, bioengineering, and biophysics.

The differential equations involving Riemann-Liouville differential operators of fractional order occur in the mathematical modelling of several phenomena in the fields of physics, chemistry, engineering, etc. For details, see [3–6, 11] and the references therein. In consequence, the subject of fractional differential equations is gaining much importance and attention, see for example [1–2, 7–10, 12–15] and the references therein. The definition of Riemann-Liouville fractional derivative, which did certainly play an important role in the development of theory of fractional derivatives and integrals, could hardly produce the physical interpretation of the initial conditions required for the initial value problems involving fractional differential equations. The same difficulty arises for the boundary conditions of the boundary value problems. It was Caputo's definition of fractional derivative:

$$^{c}D^{q}x(t) = \frac{1}{\Gamma(n-q)}\int_{t_0}^{t}(t-s)^{n-q-1}x^{(n)}(s)ds, \quad n-1<q<n,$$

which solved this problem. In fact, Caputo'derivative becomes the conventional n-th derivative as $q \to n$ and the initial conditions for fractional differential equations retain the same form as that of ordinary differential equations with integer derivatives. Another difference is that the Caputo derivative for a constant is zero while the Riemann-Liouville fractional derivative of a constant is nonzero.

In this paper, we prove some existence and uniqueness results for the following three-point fractional boundary value problem

$$\begin{cases} ^{c}D^{q}x(t) = f(t, x(t)), \quad t \in [0,T], T>0,\ 0<q<1, \\ \alpha x(0) + \beta x(T) = \gamma x(\eta),\ 0<\eta<T,\ \alpha+\beta \neq \gamma, \end{cases} \tag{1.1}$$

where $^{c}D^{q}$ denotes Caputo fractional derivative of order q, $f : [0,T] \times X \to X$ and α, β, γ are real constants. Here, $(X, \|\cdot\|)$ is a Banach space and $\mathrm{C} = \mathrm{C}([0,T], X)$ denotes the Banach space of all continuous functions from $[0,T] \to X$ endowed with a topology of uniform convergence with the norm denoted by $\|\cdot\|_{\mathrm{C}}$.

In passing, we remark that the boundary condition in (1.1) appear in certain problems of physics where the controllers at the boundary points dissipate or add energy according to a censor located at an intermediate position.

As argued in [8], the three-point boundary value problem (1.1) is equivalent to the following nonlinear integral equation

$$\begin{aligned} x(t) &= \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1} f(s, x(s))ds \\ &\quad - \frac{1}{(\alpha+\beta-\gamma)\Gamma(q)}\left[\beta\int_0^T (T-s)^{q-1} f(s,x(s))ds - \gamma\int_0^{\eta}(\eta-s)^{q-1} f(s,x(s))ds\right], \end{aligned}$$

where Γ is Gamma function. Now, we state a known result due to Krasnoselskii which is needed to prove the existence of at least one solution of (1.1).

Theorem 1.1. *Let M be a closed convex and nonempty subset of a Banach space X. Let A, B be the operators such that (i) $Ax + By \in M$ whenever $x, y \in M$ (ii) A is compact and continuous (iii) B is a contraction mapping. Then there exists $z \in M$ such that $z = Az + Bz$.*

2. MAIN RESULTS

Theorem 2.1. *Let $f : [0, T] \times X \to X$ be a jointly continuous function satisfying*

($\mathbf{A_1}$) $\|f(t, x) - f(t, y)\| \leq L\|x - y\|, \forall t \in [0, T],\ x, y \in X$;
($\mathbf{A_2}$) $\|f(t, x)\| \leq \mu(t),\ \forall (t, x) \in [0, T] \times X$.

where $\mu \in L^1([0, T], R^+)$. Then the three-point boundary value problem (1.1) *has a unique solution provided*

$$L \leq \frac{\Gamma(q+1)}{2}\left[T^q\left(1 + \frac{|\beta|}{|\alpha+\beta-\gamma|}\right) + \eta^q\frac{|\gamma|}{|\alpha+\beta-\gamma|}\right]^{-1}.$$

Proof. Define $\Theta : \mathrm{C} \to \mathrm{C}$ by

$$(\Theta x)(t) = \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1} f(s, x(s))ds - \frac{1}{(\alpha+\beta-\gamma)\Gamma(q)}$$
$$\times[\beta\int_0^T (T-s)^{q-1} f(s, x(s))ds - \gamma\int_0^\eta (\eta-s)^{q-1} f(s, x(s))ds], t \in [0, T].$$

Setting $\sup_{t\in[0,T]}\|f(t, 0)\| = M$ and choosing

$$r \geq \frac{2M}{\Gamma(q+1)}\left[T^q\left(1 + \frac{|\beta|}{|\alpha+\beta-\gamma|}\right) + \eta^q\frac{|\gamma|}{|\alpha+\beta-\gamma|}\right],$$

we show that $\Theta B_r \subset B_r$, where $B_r = \{x \in \mathrm{C} : \|x\| \leq r\}$. For $x \in B_r$, we have

$$\begin{aligned}
\|(\Theta x)(t)\| \leq\ & \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}[\|f(s, x(s))\|ds \\
&+ \frac{1}{|\alpha+\beta-\gamma|\Gamma(q)}\Bigg[|\beta|\int_0^T (T-s)^{q-1}\|f(s, x(s))\|ds \\
&+ |\gamma|\int_0^\eta (\eta-s)^{q-1}\|f(s, x(s))\|ds\Bigg] \\
&\leq \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}(\|f(s, x(s)) - f(s, 0)\| + \|f(s, 0)\|)ds \\
&+ \frac{1}{|\alpha+\beta-\gamma|\Gamma(q)}\Bigg[|\beta|\int_0^T (T-s)^{q-1}(\|f(s, x(s)) - f(s, 0)\| + \|f(s, 0)\|)ds \\
&+ |\gamma|\int_0^\eta (\eta-s)^{q-1}(\|f(s, x(s)) - f(s, 0)\| + \|f(s, 0)\|)\|ds\Bigg]
\end{aligned}$$

$$
\begin{aligned}
&\leq \ (Lr+M)\frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}ds + (Lr+M)\frac{1}{|\alpha+\beta-\gamma|\Gamma(q)} \\
&\quad \times \left[|\beta|\int_0^T (T-s)^{q-1}ds + |\gamma|\int_0^\eta (\eta-s)^{q-1}ds\right] \\
&\quad \leq (Lr+M)\frac{1}{\Gamma(q+1)}\left[T^q\left(1+\frac{|\beta|}{|\alpha+\beta-\gamma|}\right)+\eta^q\frac{|\gamma|}{|\alpha+\beta-\gamma|}\right] \leq r.
\end{aligned}
$$

Now, for $x, y \in \mathrm{C}$ and for each $t \in [0, T]$, we obtain

$$
\begin{aligned}
\|(\Theta x)(t)-(\Theta y)(t)\| &\leq \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}\|f(s,x(s))-f(s,y(s))\|ds \\
&\quad + \frac{1}{|\alpha+\beta-\gamma|\Gamma(q)}\left[|\beta|\int_0^T (T-s)^{q-1}\|f(s,x(s))-f(s,y(s))\|ds\right. \\
&\quad \left. + |\gamma|\int_0^\eta (\eta-s)^{q-1}\|f(s,x(s))-f(s,y(s))\|ds\right] \\
&\leq L\|x-y\|_{\mathrm{C}}\frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}ds \\
&\quad + L\|x-y\|_{\mathrm{C}}\frac{1}{|\alpha+\beta-\gamma|\Gamma(q)}\left[|\beta|\int_0^T (T-s)^{q-1}ds\right. \\
&\quad \left. + |\gamma|\int_0^\eta (\eta-s)^{q-1}ds\right] \\
&\leq \frac{L}{\Gamma(q+1)}[T^q(1+\frac{|\beta|}{|\alpha+\beta-\gamma|})+\eta^q\frac{|\gamma|}{|\alpha+\beta-\gamma|}]\|x-y\|_{\mathrm{C}} \\
&\leq \Lambda_{\alpha,\beta,\gamma,L,T,q}\|x-y\|_{\mathrm{C}},
\end{aligned}
$$

where

$$
\Lambda_{\alpha,\beta,\gamma,L,T,q} = \frac{L}{\Gamma(q+1)}\left[T^q\left(1+\frac{|\beta|}{|\alpha+\beta-\gamma|}\right)+\eta^q\frac{|\gamma|}{|\alpha+\beta-\gamma|}\right],
$$

which depends only on the parameters involved in the problem. As $\Lambda_{\alpha,\beta,\gamma,L,T,q} < 1$, therefore Θ is a contraction. Thus, the conclusion of the theorem follows by the contraction mapping principle (Banach fixed point theorem). □

Theorem 2.2. *Assume that* $(A_1)-(A_2)$ *hold with*

$$
\frac{L(T^q|\beta|+\eta^q|\gamma|)}{\Gamma(q+1)|\alpha+\beta-\gamma|} < 1.
$$

Then the three-point boundary value problem (1.1) *has at least one solution on* $[0, T]$.

Proof. Let us fix

$$
r \geq \frac{\|\mu\|_{L_1}}{\Gamma(q+1)}\left[T^q\left(1+\frac{|\beta|}{|\alpha+\beta-\gamma|}\right)+\eta^q\frac{|\gamma|}{|\alpha+\beta-\gamma|}\right],
$$

and consider $B_r = \{x \in \mathrm{C} : \|x\| \le r\}$. We define the operators Φ and Ψ on B_r as

$$(\Phi x)(t) = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s, x(s)) ds,$$

$$(\Psi x)(t) = -\frac{1}{(\alpha+\beta-\gamma)\Gamma(q)} \left[\beta \int_0^t (t-s)^{q-1} f(s, x(s)) ds - \gamma \int_0^\eta (\eta-s)^{q-1} f(s, x(s)) ds\right].$$

For $x, y \in B_r$, we find that

$$\|\Phi x + \Psi y\| \le \frac{\|\mu\|_{L_1}}{\Gamma(q+1)} \left[T^q \left(1 + \frac{|\beta|}{|\alpha+\beta-\gamma|}\right) + \eta^q \frac{|\gamma|}{|\alpha+\beta-\gamma|}\right] \le r.$$

Thus, $\Phi x + \Psi y \in B_r$. It follows from the assumption (A_1) that Ψ is a contraction mapping for

$$\frac{L(T^q|\beta| + \eta^q|\gamma|)}{\Gamma(q+1)|\alpha+\beta-\gamma|} < 1.$$

Continuity of f implies that the operator Φ is continuous. Also, Φ is uniformly bounded on B_r as

$$\|\Phi x\| \le \frac{\|\mu\|_{L^1} T^q}{\Gamma(q+1)}.$$

Now we prove the compactness of the operator Φ. Since f is bounded on the compact set $\Omega = [0,T] \times B_r$, we define $\sup_{(t,x)\in\Omega} \|f(t,x)\| = f_{\max}$, and consequently we have

$$\|(\Phi x)(t_1) - (\Phi x)(t_2)\| = \|\frac{1}{\Gamma(q)} \int_0^{t_1} [(t_2-s)^{q-1} - (t_1-s)^{q-1}] f(s, x(s)) ds$$

$$+ \int_{t_1}^{t_2} (t_2-s)^{q-1} f(s, x(s)) ds\| \le \frac{f_{\max}}{\Gamma(q+1)} |2(t_2-t_1)^q + t_1^q - t_2^q|,$$

which is independent of x. So Φ is relatively compact on B_r. Hence, By Arzela Ascoli Theorem, Φ is compact on B_r. Thus all the assumptions of Theorem 1.1 are satisfied and the conclusion of Theorem 1.1 implies that the three-point boundary value problem (1.1) has at least one solution on $[0,T]$. □

Example 1. Consider the following three-point boundary value problem

$$\begin{cases} {}^cD^{\frac{1}{2}} x(t) = \frac{1}{16}(t \sin x(t) - x(t) \cos t), \ t \in [0,1], \\ x(0) + x(1) = x(\frac{1}{2}). \end{cases} \tag{2.1}$$

Here, $f(t, x(t)) = \frac{1}{16}(t \sin x(t) - x(t) \cos t)$, $\alpha = 1$, $\beta = 1$, $\gamma = 1$, $T = 1$, $\eta = \frac{1}{2}$. It can easily be verified that $\|f(t,x) - f(t,y)\| \le \frac{1}{8}\|x - y\|$ which implies that (A_1) is satisfied with $L = \frac{1}{8}$. Further,

$$\frac{2L}{\Gamma(q+1)} \left[T^q \left(1 + \frac{|\beta|}{|\alpha+\beta-\gamma|}\right) + \eta^q \frac{|\gamma|}{|\alpha+\beta-\gamma|}\right] = 0.763661 < 1.$$

Thus, by Theorem 2.1, the three-point boundary value problem (2.1) has a unique solution on $[0,1]$.

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