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HARTMAN TYPE OSCILLATION CRITERIA FOR LINEAR MATRIX HAMILTONIAN SYSTEMS

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ABSTRACT. In this paper, some new oscillation criteria for linear matrix Hamiltonian systems are established, which involves the maximum eigenvalue of the coefficients. These results improve and generalize some known oscillation criteria due to G. J. Butler, L. H. Erbe and A. B. Mingarelli [1], N. Parhi and P. Praharaj [2] for self-adjoint second order matrix differential systems, and Yang et. al. [4] for linear matrix Hamiltonian systems.

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1. INTRODUCTION

In this paper, we consider oscillatory properties for the linear matrix Hamiltonian system

$$\begin{cases} X' &= A(t)X + B(t)U, \\ U' &= C(t)X - A^*(t)U, \quad t \geq a, \end{cases} \tag{1.1}$$

where $A(t), B(t), C(t)$ are real $n \times n$ matrix-valued functions, B, C are symmetric, B is positive definite. By M^* we mean the transpose of the matrix M, for any $n \times n$ symmetric matrix M, its eigenvalues are real numbers, and we always denote them by $\lambda_1[M] \geq \lambda_2[M] \geq \cdots \geq \lambda_n[M]$. The trace of M is denoted by $\mathrm{tr}(M)$ and $\mathrm{tr}(M) = \sum_{k=1}^{n} \lambda_k(M)$.

For any two solutions $(X_1(t), U_1(t))$ and $(X_2(t), U_2(t))$ of system (1.1), the Wronskian matrix $X_1^*(t)U_2(t) - U_1^*(t)X_2(t)$ is a constant matrix. In particular, for any solution $(X(t), U(t))$ of system (1.1), $X^*(t)U(t) - U^*(t)X(t)$ is a constant matrix.

A solution $(X(t), U(t))$ of system (1.1) is said to be nontrivial, if $\det X(t) \neq 0$ is fulfilled for at least one $t \geq a$. A nontrivial solution $(X(t), U(t))$ of system (1.1) is said to be prepared if $X^*(t)U(t) - U^*(t)X(t) \equiv 0$, $t \geq a$. A nontrivial prepared solution $(X(t), U(t))$ of system (1.1) is said to be oscillatory in case the determinant of $X(t)$

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vanishes on $[T, \infty)$ for each $T \geq 0$. By Sturm's separation theorem (see, for example, [16]), we know that all prepared solutions of system (1.1) are either oscillatory or non-oscillatory, so we can divide system (1.1) into two cases: if all prepared solutions of system (1.1) are oscillatory, system (1.1) is oscillatory; otherwise, system (1.1) is non-oscillatory.

When $A(t) \equiv 0$, system (1.1) reduces to the second order self-adjoint matrix differential system

$$(P(t)Y')' + Q(t)Y = 0, \qquad t \geq a \tag{1.2}$$

with $P(t) = B^{-1}(t)$, $Q(t) = -C(t)$. Oscillation and non-oscillation of system (1.2) and its special cases

$$Y'' + Q(t)Y = 0, \qquad t \geq a \tag{1.3}$$

have been extensively studied by many authors (see [1, 2, 7, 8, 15] and references contained therein). Many of these criteria involve the integral of the coefficients modelled on either the criteria due to Wintner[9] or Kamenev[10] for the scalar equation

$$y'' + q(t)y = 0, \qquad t \geq a, \tag{1.4}$$

$$(p(t)y')' + q(t)y = 0, \qquad t \geq a. \tag{1.5}$$

Wintner[9] proved that

$$\lim_{t\to\infty} \int_a^t \int_a^s q(\tau)d\tau ds = \infty \tag{1.6}$$

is sufficient condition for (1.4) to be oscillatory. Hartman[11] weakened this hypothesis to the following oscillatory condition.

$$-\infty < \liminf_{t\to\infty} \frac{1}{t} \int_a^t \int_a^s q(\tau)d\tau < \limsup_{t\to\infty} \frac{1}{t} \int_a^t \int_a^s q(\tau)d\tau \leq \infty. \tag{1.7}$$

By using the maximum eigenvalue of the coefficients or positive linear functional, oscillation criteria of Hartman type have been given for system (1.3) by Butler et. al. [1] and system (1.2) by N. Parhi and P. Praharaj [2]. Here, we list the results due to N. Parhi and P. Praharaj.

Theorem 1.1. *Suppose that $P^{-1}(t) \geq I$ for $t \geq a$, and*

$$\liminf_{t\to\infty} \frac{1}{t} \int_a^t \left(\operatorname{tr} \int_a^s Q(\tau)d\tau \right) ds > -\infty. \tag{1.8}$$

If one of the conditions

$$(\mathrm{C})_1 \ \limsup_{t\to\infty} \frac{1}{t} \int_a^t \left(\operatorname{tr} \int_a^s Q(\tau)d\tau \right) ds = \infty; \tag{1.9}$$

$$(\mathrm{C})_2 \ \limsup_{t\to\infty} \frac{1}{t} \int_a^t \left[\operatorname{tr} \int_a^s Q(\tau)d\tau \right]^2 ds = \infty; \tag{1.10}$$

$$\text{(C)}_3\ \lim \operatorname{approx} \sup_{t\to\infty} \left[\operatorname{tr} \int_a^t Q(\tau) d\tau \right] = \infty; \tag{1.11}$$

$$\text{(C)}_4\ \lim \operatorname{approx} \inf_{t\to\infty} \left[\operatorname{tr} \int_a^t Q(\tau) d\tau \right] = -\infty \tag{1.12}$$

holds, then system (1.2) *is oscillatory.*

Theorem 1.1 is a generalization of paper [1], where equation (1.3) is considered, and the positive linear functional $\operatorname{tr}\int_a^t Q(\tau)d\tau$ is replaced by the convex functional $\lambda_1(\int_a^t Q(\tau)d\tau)$.

Recently, using positive linear functional, Yang et. al. [4] obtain the following oscillation criteria for system (1.1).

Theorem 1.2. *Assume there exist a positive function $v \in \mathrm{C}^1([a,\infty),(0,\infty))$ and a positive linear functional satisfying g on the collections of real $n\times n$ matrix, such that*

$$\lim_{t\to\infty} \int_a^t \frac{1}{v(s)g(B^{-1}(s))} ds = \infty,$$

and

$$\lim_{t\to\infty} g(J_0(t)) = \infty,$$

where

$$\begin{aligned} J_0(t) = \int_a^t & \left[-v(A^*B^{-1}A + C) + \frac{v'}{2}(A^*B^{-1} + B^{-1}A) - \frac{v'^2}{4v} B^{-1} \right](s)ds \\ & - v(t)B^{-1}(t)A(t) + \frac{v'(t)}{2} B^{-1}(t). \end{aligned}$$

Then system (1.2) *is oscillatory.*

Following the paper [15], we see that any positive linear functional g can be characterized by a semi-positive definite matrix $[\alpha_{ij}]$ as

$$g(A) = \langle([\alpha_{ij}] \otimes A)u, u\rangle,$$

with $u = (1,1,\ldots,1)^T$, where A is an arbitrary $n\times n$ matrix. Moreover $\|g\| = g(I)$, where the norm is defined in Euclidean space. A positive linear functional g with $\|g\| = 1$ satisfies

$$\lambda_n(D) \le g(D) \le \lambda_1(D), \tag{1.13}$$

where D is an arbitrary $n\times n$ symmetric matrix. So the positive linear functional $g(A)$ in Theorem 1.2 is equivalent to the positive linear functional $\operatorname{tr}A$. Moreover, we note that $J_0(t)$ in Theorem 1.2 is not symmetric.

In this paper, we use the convex functional $\lambda_1(\cdot)$ to obtain oscillation criteria for system (1.1), which extend and improve the oscillation criteria mentioned above.

2. MAIN RESULTS

If $(X(t),U(t))$ is a prepared solution of system (1.1) such that $X(t)$ is nonsingular for t sufficiently large, without loss of generality, we may suppose that $X(t)$ is nonsingular for $t \geq a$. Set

$$W(t) = -v(t)U(t)X^{-1}(t), \quad t \geq a.$$

Then $W(t)$ satisfies the Riccati matrix equation

$$\begin{aligned} W'(t) &= \left(\frac{v'}{v}W - vC + WA + A^*W + \frac{1}{v}WBW\right)(t) \\ &= D(t) + (\frac{1}{v}R^*BR)(t), \ t \geq a, \end{aligned} \tag{2.1}$$

where $D(t) = \left(-v(A^*B^{-1}A + C) + \frac{v'}{2}(A^*B^{-1} + B^{-1}A) - \frac{v'^2}{4v}B^{-1}\right)(t)$, and $R(t) = \left(W - vB^{-1}A + \frac{v'}{2}B^{-1}\right)(t)$. We note that $R(t)$ is not symmetric in general. Integrating (2.1) from a to t, we have

$$W(t) = W(a) + \int_a^t D(s)ds + \int_a^t (\frac{1}{v}R^*BR)(s)ds.$$

Hence we have

$$R(t) = W(a) + \int_a^t (\frac{1}{v}R^*BR)(s)ds + J_0(t). \tag{2.2}$$

Since $J_0(t)$ is not symmetric, we define $J(t) = \frac{J_0(t)+J_0^*(t)}{2}$, which is usually known as the *hermitian part* of $J_0(t)$. Now, all the eigenvalues of $J(t)$ are real and $\mathrm{tr}J_0(t) = \mathrm{tr}J(t)$.

Lemma 2.1. *Assume that there exists a positive function $v \in \mathrm{C}^1([a,\infty),(0,\infty))$ satisfying $v(t) \leq \lambda_n(B(t))$. Then*

$$\lim_{t\to\infty} \int_a^t (\frac{1}{v}R^*BR)(s)ds < \infty \tag{2.3}$$

if and only if

$$\liminf_{t\to\infty} \frac{1}{t}\int_a^t \mathrm{tr}J(s)ds > -\infty. \tag{2.4}$$

Remark 2.1. Lemma 2.1 extends Lemma 5.1 in[1].

We introduce the following concepts from [2]. For any subset E of the real line $\mathbb{R}$, $\mu(E)$ denotes the Lebesgue measure of E. If $f : [a,\infty) \to \mathbb{R}$ is continuous and if l, m satisfy $-\infty \leq l, m \leq \infty$, then $\lim \mathrm{approx}\inf_{t\to\infty} f(t) = l$ if and only if $\mu\{t \in [a,\infty) : f(t) \leq l_1\} < \infty$ for all $l_1 < l$ and $\mu\{t \in [a,\infty) : f(t) \leq l_2\} = \infty$ for all $l_2 > l$. Similarly, $\lim \mathrm{approx}\sup_{t\to\infty} f(t) = m$ if and only if $\mu\{t \in [a,\infty) : f(t) \geq m_1\} = \infty$ for all $m_1 < m$ and $\mu\{t \in [a,\infty) : f(t) \geq m_2\} < \infty$ for all $m_2 > m$. We define $\lim \mathrm{approx}_{t\to\infty} f(t) = \lambda$ in case

$$\lim \mathrm{approx}\sup_{t\to\infty} f(t) = \lim \mathrm{approx}\inf_{t\to\infty} f(t) = \lambda.$$

Theorem 2.2. *Assume that there exists a positive function $v \in \mathrm{C}^1([a,\infty),(0,\infty))$ satisfying $v(t) \leq \lambda_n(B(t))$. Moreover,* (2.4) *is fulfilled. Then system* (1.1) *is oscillatory provided one of the following conditions holds.*

$$\text{(i)} \ \limsup_{t\to\infty} \frac{1}{t}\int_a^t \lambda_1(J(s))ds = \infty; \tag{2.5}$$

$$\text{(ii)} \ \limsup_{t\to\infty} \frac{1}{t}\int_a^t [\lambda_1(J(s))]^2 ds = \infty; \tag{2.6}$$

$$\text{(iii)} \ \lim\operatorname{approx}\sup_{t\to\infty}\lambda_1(J(t)) = \infty; \tag{2.7}$$

$$\text{(iv)} \ \lim\operatorname{approx}\inf_{t\to\infty}\lambda_1(J(t)) = -\infty. \tag{2.8}$$

Theorem 2.3. *Assume that there exists a positive function $v \in \mathrm{C}^1([a,\infty),(0,\infty))$ satisfying $v(t) \leq \lambda_n(B(t))$. Moreover,*

$$\liminf_{t\to\infty} \frac{1}{t}\int_a^t \mathrm{tr}J(s)ds = -\infty. \tag{2.9}$$

Then system (1.1) *is oscillatory if*

$$\lim\operatorname{approx}\sup_{t\to\infty}\lambda_n(J(t)) > -\infty; \tag{2.10}$$

Remark 2.2. Theorem 2.3 extends Theorem 2.2 in [1] to the case of (1.1).

3. PROOFS OF THE MAIN RESULTS

Lemma 3.1. *(See* [14]*) Suppose A, B, and C are $n\times n$-matrices, B is semi-positive definite, C is symmetric. Then*

(1) $\mathrm{tr}(A^*BA) \geq \lambda_n(B)\mathrm{tr}(A^*A)$;

(2) $(\mathrm{tr}A)^2 \leq n\mathrm{tr}(A^*A)$;

(3) $\mathrm{tr}(A^*A) \geq \mathrm{tr}(A^2)$;

(4) $\lambda_1^2(C) \leq \lambda_1(C^2) \leq \mathrm{tr}(C^2)$.

Proof of Lemma 2.1. Suppose (2.3) holds. Then we rewrite (2.2) and take the trace of both sides to get

$$\mathrm{tr}R(t) + \int_t^\infty \frac{1}{v(s)}\mathrm{tr}(R^*BR)(s)ds = \mathrm{tr}J(t) - L, \tag{3.1}$$

where $L = -\mathrm{tr}W(a) - \int_a^\infty \frac{1}{v(s)}\mathrm{tr}(R^*BR)(s)ds$ is a constant symmetric matrix. Since $v(t) \leq \lambda_n B(t)$, by Lemma 3.1, we have

$$\begin{aligned}
&\frac{1}{t}\int_a^t \left[\mathrm{tr}R(s) + \int_s^\infty \frac{1}{v(\tau)}\mathrm{tr}(R^*BR)(\tau)d\tau\right]^2 ds \\
&\leq \frac{2}{t}\int_a^t [\mathrm{tr}R(s)]^2 ds + \frac{2}{t}\int_a^t \left[\int_s^\infty \frac{1}{v(\tau)}\mathrm{tr}(R^*BR)(\tau)d\tau\right]^2 ds \\
&\leq \frac{2n}{t}\int_a^t \mathrm{tr}(R^*R)(s)ds + \frac{2}{t}\int_a^t \left[\int_s^\infty \frac{1}{v(\tau)}\mathrm{tr}(R^*BR)(\tau)d\tau\right]^2 ds
\end{aligned}$$

$$\leq \frac{2n}{t}\int_a^t \frac{\lambda_n(B(s))}{v(s)}\mathrm{tr}(R^*R)(s)ds + \frac{2}{t}\int_a^t \left[\int_s^\infty \frac{1}{v(\tau)}\mathrm{tr}(R^*BR)(\tau)d\tau\right]^2 ds$$

$$\leq \frac{2n}{t}\int_a^t \frac{1}{v(s)}\mathrm{tr}(R^*BR)(s)ds + \frac{2}{t}\int_a^t \left[\int_s^\infty \frac{1}{v(\tau)}\mathrm{tr}(R^*BR)(\tau)d\tau\right]^2 ds \tag{3.2}$$

By (2.3), we get the first term in (3.2) tends to 0 as $t \to \infty$. Since $L < \infty$, we obtain that the second term in (3.2) tends to 0 as $t \to \infty$. So (3.1) and (3.2) imply

$$\frac{1}{t}\int_a^t [\mathrm{tr}J(s) - L]^2\, ds \to 0 \text{ as } t \to \infty.$$

By Cauchy-Schwarz inequality, we get

$$\frac{1}{t}\int_a^t (\mathrm{tr}J(s) - L)\, ds \leq \left(\frac{1}{t}\int_a^t [\mathrm{tr}J(s) - L]^2\, ds\right)^{1/2} \to 0 \text{ as } t \to \infty.$$

It follows that

$$\lim_{t\to\infty}\frac{1}{t}\int_a^t \mathrm{tr}J(s)ds = L.$$

Conversely, suppose that there exists a prepared solution $(X(t), U(t))$ satisfying $\det X(t) \neq 0$ for $t \geq a$, such that (2.4) holds, then there exists $M_1 > 0$ such that $\frac{1}{t}\int_a^t \mathrm{tr}J(s)ds > -M_1$. From (2.2), we get

$$\begin{aligned}\frac{1}{t}\int_a^t\int_a^s \frac{1}{v(\tau)}\mathrm{tr}(R^*BR)(\tau)d\tau ds &= \frac{1}{t}\int_a^t \mathrm{tr}R(s)ds - \frac{1}{t}\int_a^t \mathrm{tr}J(s)ds - \frac{t-a}{t}\mathrm{tr}W(a)\\ &< \frac{1}{t}\int_a^t \mathrm{tr}R(s)ds + M_1 - \frac{t-a}{t}\mathrm{tr}W(a)\\ &\leq \frac{1}{t}\int_a^t \mathrm{tr}R(s)ds + M, \qquad (3.3)\end{aligned}$$

where M is a constant. Since $\mathrm{tr}(R^*BR)(t) \geq 0$ and $v(t) > 0$ for $t \geq a$, it follows that

$$\lim_{t\to\infty}\int_a^t \frac{1}{v(s)}\mathrm{tr}(R^*BR)(s)ds = \mu, \text{ where } 0 < \mu \leq \infty.$$

Suppose that $\mu = \infty$. Considering (3.3), we obtain

$$\lim_{t\to\infty}\frac{1}{t}\int_a^t\int_a^s \frac{1}{v(\tau)}\mathrm{tr}(R^*BR)(\tau)d\tau ds = \infty.$$

This and (3.3) yields

$$\lim_{t\to\infty}\frac{1}{t}\int_a^t \mathrm{tr}R(s)ds = \infty.$$

So there exist $T \geq a$ such that $\frac{1}{t}\int_a^t \mathrm{tr}R(s)ds > M$ for $t \geq T$. Again, this and (3.3) yields

$$\frac{1}{t}\int_a^t\int_a^s \frac{1}{v(\tau)}\mathrm{tr}(R^*BR)(\tau)d\tau ds < \frac{2}{t}\int_a^t \mathrm{tr}R(s)ds,\ t \geq T. \tag{3.4}$$

Using Cauchy-Schwarz inequality and Lemma 3.1, we get

$$\begin{aligned} \left|\frac{1}{t}\int_a^t \mathrm{tr}R(s)ds\right| &\le \left(\frac{1}{t}\int_a^t [\mathrm{tr}R(s)]^2\, ds\right)^{1/2} \\ &\le \left(\frac{n}{t}\int_a^t \mathrm{tr}(R^*R)(s)ds\right)^{1/2} \\ &\le \left(\frac{n}{t}\int_a^t \frac{\lambda_n(B(s))}{v(s)}\mathrm{tr}(R^*R)(s)ds\right)^{1/2} \\ &\le \left(\frac{n}{t}\int_a^t \frac{1}{v(s)}\mathrm{tr}(R^*BR)(s)ds\right)^{1/2},\ t \ge T. \end{aligned} \tag{3.5}$$

From (3.4) and (3.5), we get for $t \ge T$,

$$\frac{1}{t^2}\left[\int_a^t \int_a^s \frac{1}{v(\tau)}\mathrm{tr}(R^*BR)(\tau)d\tau ds\right]^2 \le \frac{4n}{t}\int_a^t \frac{1}{v(s)}\mathrm{tr}(R^*BR)(s)ds. \tag{3.6}$$

Now, set $H(t) = \int_a^t \int_a^s \frac{1}{v(\tau)}\mathrm{tr}(R^*BR)(\tau)d\tau ds$, then (3.6) yields

$$\frac{1}{t^2}H^2(t) \le \frac{4n}{t}H'(t), \quad t \ge T$$

and so we get

$$\frac{1}{4nt} \le \frac{H'(t)}{H^2(t)}, \quad t \ge T.$$

An integration from T to ∞ leads to an obvious contradiction, so we get $\mu < \infty$.

Let the operator norm of a matrix A be denoted by $|A|$. For $a \le s \le t$, define $\mathcal{A}(s,t) = \int_s^t (\frac{1}{v}R^*BR)(s)ds$. Then $\mathcal{A}(s,t)$ is a non-negative definite matrix and $|\mathcal{A}(s,t)| = \lambda_1(\mathcal{A}(s,t)) \le \mathrm{tr}(\mathcal{A}(s,t)) = \int_s^t \frac{1}{v(s)}\mathrm{tr}(R^*BR)(s)ds$. The last integrand converges to 0 as $s,t \to \infty$ and so we have $|\mathcal{A}(s,t)| \to 0$ as $s,t \to \infty$. So we obtain the existence of $\int_a^t (\frac{1}{v}R^*BR)(s)ds$ as asserted. This completes the proof of Lemma 2.1.

Proof of Theorem 2.2. (i) Suppose to the contrary that there is a nontrivial prepared solution $(X(t), U(t))$ of system (1) such that $X(t)$ is nonsingular for all sufficiently large t. Without loss of generality, we may suppose that $X(t)$ is nonsingular for $t \ge a$. Hence we have (2.2) for $t \ge a$. It follows that

$$\overline{R}(t) := \frac{1}{2}\left(R(t) + R^*(t)\right) = W(a) + \int_a^t (\frac{1}{v}R^*BR)(s)ds + J(t).$$

That is

$$\overline{R}(t) - W(a) = \int_a^t (\frac{1}{v}R^*BR)(s)ds + J(t). \tag{3.7}$$

Taking the maximum eigenvalue on both sides, we get

$$\lambda_1\left[\overline{R}(t) - W(a)\right] = \lambda_1\left[\int_a^t (\frac{1}{v}R^*BR)(s)ds + J(t)\right]. \tag{3.8}$$

Considering the convexity of $\lambda_1(\cdot)$, the fact that $\int_a^t(\frac{1}{v}R^*BR)(s)ds \geq 0$ and $\lambda_n(B(t)) \geq v(t)$, (3.8) implies

$$\lambda_1\left[\overline{R}(t)\right] + \lambda_1\left[-W(a)\right] \geq \lambda_1\left[J(t)\right].$$

Thus

$$\frac{1}{t}\int_a^t \lambda_1\left[\overline{R}(s)\right] ds + \frac{t-a}{t}\lambda_1\left[-W(a)\right] \geq \frac{1}{t}\int_a^t \lambda_1\left[J(s)\right] ds.$$

This and (2.5) yields

$$\limsup_{t\to\infty}\frac{1}{t}\int_a^t \lambda_1\left[\overline{R}(s)\right] ds = \infty.$$

Hence there exists a sequence $\{t_m\}$ such that $t_m \to \infty$ as $m \to \infty$ and

$$\lim_{m\to\infty}\frac{1}{t_m}\int_a^{t_m} \lambda_1\left[\overline{R}(s)\right] ds = \infty. \tag{3.9}$$

By Lemma 3.1 ((4), (3)), we have

$$\lambda_1^2\left[\overline{R}(t)\right] \leq \operatorname{tr}\left[\overline{R}(t)\right]^2 = \operatorname{tr}\left[\frac{1}{2}(R^2 + R^*R)(t)\right] \leq \operatorname{tr}(R^*R)(t). \tag{3.10}$$

We obtain by Cauchy-Schwarz inequality

$$\begin{aligned}\left|\frac{1}{t_m}\int_a^{t_m} \lambda_1\left[\overline{R}(s)\right] ds\right| &\leq \left(\frac{1}{t_m}\int_a^{t_m} \lambda_1^2\left[\overline{R}(s)\right] ds\right)^{1/2}\left(\frac{t_m - a}{t_m}\right)^{1/2} \\ &\leq \left(\frac{1}{t_m}\int_a^{t_m} \operatorname{tr}(R^*R)(s)ds\right)^{1/2} \\ &\leq \left(\frac{1}{t_m}\int_a^{t_m} \frac{\lambda_n(B(s))}{v(s)}\operatorname{tr}(R^*R)(s)ds\right)^{1/2} \\ &\leq \left(\frac{1}{t_m}\int_a^{t_m} \frac{1}{v(s)}\operatorname{tr}(R^*BR)(s)ds\right)^{1/2}.\end{aligned}$$

This and (3.9) implies

$$\lim_{m\to\infty}\frac{1}{t_m}\int_a^{t_m} \frac{1}{v(s)}\operatorname{tr}(R^*BR)(s)ds = \infty.$$

So

$$\lim_{m\to\infty}\int_a^{t_m} \frac{1}{v(s)}\operatorname{tr}(R^*BR)(s)ds = \infty. \tag{3.11}$$

On the other hand, by Lemma 2.1, we get

$$\lim_{t\to\infty}\int_a^{t} \frac{1}{v(s)}\operatorname{tr}(R^*BR)(s)ds < \infty. \tag{3.12}$$

This contradiction completes the first part of Theorem 2.2.

(ii) This result is clear as (2.5) implies (2.6) by Schwarzs inequality applied to Lebesgue measure divided by t.

(iii) Following the proof of the second part, we get (**??**). So

$$\lambda_1\left[\overline{R}(t)\right]+\lambda_1\left[\int_t^\infty(\frac{1}{v}R^*BR)(s)ds\right]+\lambda_1\left[L\right]\geq\lambda_1\left[J(t)\right]. \tag{3.13}$$

Since $\lambda_i\left[\int_t^\infty(\frac{1}{v}R^*BR)(s)ds\right]\geq 0$ and $\int_t^\infty \frac{1}{v(s)}\mathrm{tr}[(R^*BR)(s)]ds\to 0$, we have

$$\lambda_1\left[\int_t^\infty(\frac{1}{v}R^*BR)(s)ds\right]\to 0. \tag{3.14}$$

Now for any $k\geq 1$,

$$\mu\left\{t:\lambda_1(J(t))\geq k\right\}=\infty.$$

So that if $k\geq|\lambda_1(L)|+1$,

$$\mu(E_1)=\mu\left\{t:\lambda_1\left[\overline{R}(t)\right]+\lambda_1\left[\int_t^\infty(\frac{1}{v}R^*BR)(s)ds\right]>1\right\}=\infty.$$

By (3.14), we see that $\exists\ T\geq a$, such that $\lambda_1\left[\int_t^\infty(\frac{1}{v}R^*BR)(s)ds\right]<\frac{1}{2}$ for $t\geq T$. So

$$\mu(E_2)=\mu\left\{t:\lambda_1\left[\overline{R}(t)\right]>\frac{1}{2}\right\}=\infty.$$

Since $\lambda_1^2(A)\leq\lambda_1(A^2)$ for any symmetric matrix A, we have

$$\mu(E_3)=\mu\left\{t:\lambda_1\left[\overline{R}^2(t)\right]>\frac{1}{4}\right\}=\infty.$$

Since $\mathrm{tr}\overline{R}^2(t)\leq\mathrm{tr}(R^*R)(t)$, we deduce that

$$\mu(E_4)=\mu\left\{t:\mathrm{tr}(R^*R)(t)>\frac{n}{4}\right\}=\infty.$$

So

$$\infty=\int_{E_4}\mathrm{tr}(R^*R)(s)ds\leq\int_{E_4}\frac{1}{v(s)}\mathrm{tr}(R^*BR)(s)ds,$$

which contradicts Lemma 2.1.

(iv) Similar to the second part, we get (3.13). Since for any $M>0$,

$$\mu\left\{t:\lambda_1(J(t))\leq -M\right\}=\infty$$

and since

$$\lambda_1\left[\overline{R}(t)\right]\leq\lambda_1\left[\overline{R}(t)+\int_t^\infty(\frac{1}{v}R^*BR)(s)ds\right]\leq\lambda_1\left[J(t)\right]+\lambda_1\left[-L\right],$$

it follows that if $M\geq 1+|\lambda_1(-L)|$,

$$\mu(E_5)=\mu\left\{t:\lambda_1\left[\overline{R}(t)\right]\leq -1\right\}=\infty$$

and hence

$$\mu(E_6)=\mu\left\{t:\mathrm{tr}(R^*R)(t)>n\right\}=\infty.$$

So

$$\infty=\int_{E_6}\mathrm{tr}[(R^*R)(s)]ds\leq\int_{E_6}\frac{1}{v(s)}\mathrm{tr}(R^*BR)(s)ds,$$

which contradicts Lemma 2.1. This completes the proofs of Theorem 2.2.

Proof of Theorem 2.3. By (2.10), we may suppose that

$$\lim \operatorname{approx} \sup_{t\to\infty} \lambda_n(J(t)) = m > -\infty.$$

Following the proof of Theorem 2.2, we get (3.7). Since $\liminf_{t\to\infty} \frac{1}{t}\int_a^t \mathrm{tr} J(s)ds = -\infty$, it follows that $\int_a^t (\frac{1}{v}R^*BR)(s)ds \to \infty$, so $\lambda_1\{\int^t (\frac{1}{v}R^*BR)(s)ds\} \to \infty$ as $t \to \infty$. Since (2.10) implies for any $\epsilon > 0$, $\mu\{t : \lambda_n[J(t)] \geq m - \epsilon\} = \infty$, we have by (3.7) that

$$\begin{aligned}
\frac{1}{n}\mathrm{tr}W(a) + \frac{1}{n}\int_a^t \frac{1}{v(s)}\mathrm{tr}(R^*BR)(s)ds &= \frac{1}{n}\mathrm{tr}\left[\overline{R}(t) - J(t)\right] \\
&\leq \lambda_1\left[\overline{R}(t) - J(t)\right] \\
&\leq \lambda_1\left[\overline{R}(t)\right] - \lambda_n\left[J(t)\right]. \qquad (3.15)
\end{aligned}$$

Since

$$\begin{aligned}
\frac{1}{n}\int_a^t \frac{1}{v(s)}\mathrm{tr}(R^*BR)(s)ds &\geq \frac{1}{n}\int_a^t \frac{\lambda_n(B(s))}{v(s)}\mathrm{tr}(R^*R)(s)ds \\
&\geq \frac{1}{n}\int_a^t \mathrm{tr}(R^*R)(s)ds \\
&\geq \frac{1}{n}\int_a^t \mathrm{tr}\overline{R}^2(s)ds \\
&\geq \frac{1}{n}\int_a^t \lambda_1\left[\overline{R}^2(s)\right]ds. \qquad (3.16)
\end{aligned}$$

(3.15) and (3.16) imply

$$\frac{1}{n}\int_a^t \lambda_1\left[\overline{R}^2(s)\right]ds + \frac{1}{n}\mathrm{tr}W(a) \leq \lambda_1\left[\overline{R}(t)\right] - \lambda_n\left[J(t)\right].$$

Hence we have for $\epsilon > 0$,

$$\mu\left\{t : \frac{1}{n}\int_a^t \lambda_1\left[\overline{R}^2(s)\right]ds + \frac{1}{n}\mathrm{tr}W(a) \leq \lambda_1\left[\overline{R}(t)\right] - m + \epsilon\right\} = \infty.$$

So

$$\mu\left\{t : \frac{1}{n}\int_a^t \lambda_1\left[\overline{R}^2(s)\right]ds \leq \lambda_1\left[\overline{R}(t)\right]\right\} = \infty,$$

and

$$\mu(E_7) = \mu\left\{t : \frac{1}{n}\int_a^t \lambda_1\left[\overline{R}^2(s)\right]ds \leq \lambda_1\left[\overline{R}(t)\right]\right\} \cap [a+1, \infty) = \infty.$$

Set $F(t) = \int_a^t \lambda_1\left[\overline{R}^2(s)\right]ds$, we see that $0 < F(t) \leq \infty$ for $t \in E_7$, and $F'(t) = \lambda_1\left[\overline{R}^2(t)\right] \geq \left[\lambda_1(\overline{R}(t))\right]^2$. So $F'(t) \geq \frac{F^2(t)}{n^2}$, $t \in E_7$. Now $\int_{E_7} \frac{F'(s)}{F^2(s)}ds \geq \frac{\mu(E_7)}{n^2} = \infty$, but the integrand of the left-hand side is $\leq \frac{1}{F(a+1)}$, which is a contradiction. This completes the proof of Theorem 2.3.

4. EXAMPLE AND REMARKS

Example 4.1. Consider the four-dimensional system (1.1) where

$$A(t) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, B(t) = \frac{1}{t+1}E_2, C(t) = \begin{bmatrix} -(t+1) + \frac{1}{4(t+1)} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{4(t+1)} \end{bmatrix},$$

and X, U are 2×2 matrix functions of t on $[0, \infty)$. Let $v(t) = \frac{1}{t+1}$, then $\lambda_n(B(t)) = v(t)$. By direct computation, we have

$$J(t) = \begin{bmatrix} t & -\frac{1}{2} \\ -\frac{1}{2} & -t \end{bmatrix}.$$

Hence (2.4) is fulfilled and we note that $\mathrm{tr}J(t) = 0$, so Theorem 1.2 can not be used to verify the oscillation of this system. However, we find that

$$\limsup_{t\to\infty} \frac{1}{t} \int_a^t \lambda_1(J(s))ds = \infty.$$

So by Theorem 2.2, we deduce the system is oscillatory.

Remark 4.1. Similarly, we can deduce oscillation criteria via a more generalize Riccati transformation

$$W(t) = v(t)\left[U(t)X^{-1}(t) + \phi(t)B^{-1}(t)\right], \tag{4.1}$$

where $\phi(t) \in C^1[t_0, \infty)$ is a carefully chosen function and $v(t) = \exp\{-2\int^t \phi(s)ds\}$. Here we omit the details.

Remark 4.2. In paper [13], the authors obtain oscillation criteria for system (1.1) by using generalized Riccati transformation similar to (4.1) and fundamental matrix solution of $Y' = A(t)Y$. The results we obtained here are different to those mentioned above.

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