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GENERALIZED QUASILINEARIZATION FOR NONLINEAR IMPULSIVE THREE-POINT BOUNDARY VALUE PROBLEMS

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ABSTRACT. We apply the generalized quasilinearization technique to obtain a monotone sequence of iterates converging monotonically and quadratically to a unique solution of an impulsive three-point general nonlinear second order boundary value problem. The nth order ($n \geq 2$) convergence of the sequence of iterates has also been accomplished.

Key words: Quasilinearization, boundary value problems with impulse, rapid convergence

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1. INTRODUCTION

Impulsive hybrid systems are composed of some continuous variable dynamic systems along with certain reset maps that define impulsive switching among them. The switching perform resets to the modes and change the continuous state of the system. There are three classes of impulsive hybrid systems, namely, impulsive differential systems, sample data control systems and impulsive switched systems. In recent years, a number of research papers has dealt with dynamical systems with impulse effect as a class of general hybrid systems. Examples include the pulse frequency modulation, optimization of drug distribution in the human body and control systems with changing reference signal. Impulsive dynamical systems are characterized by the occurrence of abrupt change in the state of the system which occur at certain time instants over a period of negligible duration. The dynamical behavior of such systems is much more complex than the behavior of dynamical systems without impulse effects. The presence of impulse means that the state trajectory does not preserve the basic properties which are associated with non impulsive dynamical systems. Thus, the theory of impulsive differential equations is quite interesting and has attracted the attention of many scientists, for instance, see [1–5].

The method of quasilinearization initiated by Bellman and Kalaba [6], and generalized by Lakshmikantham [7–8] has been studied and extended in several diverse disciplines [9–16]. The convergence of the sequence of iterates converging to the solution of the problem has also been improved, see for example, [17–19].

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Multi-point nonlinear boundary value problems, which take into account the boundary data at intermediate points of the interval under consideration, have been receiving considerable attention [20–23]. P. Eloe and Y. Gao [24] discussed the quasilinearization method for a three-point boundary value problem. B. Ahmad, R. A. Khan and P. Eloe [25] developed the generalized quasilinearization method for a three-point problem with nonlinear boundary conditions.

The purpose of this paper is to develop the generalized quasilinearization method for a general impulsive hybrid nonlinear three-point boundary value problem. In fact, a monotone sequence of iterates converging uniformly and quadratically to a unique solution of the problem is obtained. Further, the rate of convergence has also been improved by establishing a convergence of order $n (n \geq 2)$.

2. TERMINOLOGY AND BASIC RESULTS

Let $PC[0,1]$ denote the piecewise continuous functions on [0,1] and let $PC^1[0,1]$ denote the functions, x such that $x \in PC[0,1]$ and $x' \in PC[0,1]$. Define an appropriate Banach space B by

$$B = \{x \in PC^1[0,1] : x^i|_{t_k, t_{k+1}]} \in C^i[t_k, t_{k+1}],\ k = 0,1,\ldots,m,\ i = 0,1\},$$

with

$$\|x\|_B = \max_{k=0,1,\ldots,m} \|x\|_k,\ \|x\|_k = \max_{i=0,1} \sup_{t_k \leq t \leq t_{k+1}} |x^i(t)|.$$

We consider the three-point problem with impulse

$$x''(t) = f(t, x(t), x'(t)),\ t_k < t < t_{k+1},\ k = 0,1,2,\ldots,m, \tag{2.1}$$

$$x(0) = a, \qquad x(1) = g(x(\frac{1}{2})),$$

and for $k = 1,2,\ldots,m$,

$$\triangle x(t_k) = u_k, \qquad \triangle x'(t_k) = v_k(x(t_k), x'(t_k)), \tag{2.2}$$

where $f : [0,1] \times R^2 \to R$ is continuous, $g : R \to R$ is continuous and bounded, $u_k \in R$, $v_k : R^2 \to R$ is continuous with the convention $x(t_k) = x(t_{k^-})$ and the impulse is defined by $\triangle x(t_k) = x(t_k^+) - x(t_k^-)$ for $0 = t_o < t_1 < t_2 \cdots < t_m < t_{m+1} = 1$.

We say that $\alpha_0 \in B$ is a lower solution of (2.1) and (2.2) if

$$\alpha_0''(t) \geq f(t, \alpha_0(t), \alpha_0'(t)),\ t_k < t < t_{k+1},\ k = 0,1,\ldots,m,$$

$$\alpha_0(0) \leq a, \qquad \alpha_0(1) \leq g(x(\frac{1}{2})),$$

and for $k = 1,\ldots,m$,

$$\triangle \alpha_0(t_k) = u_k, \qquad \triangle \alpha_0'(t_k) \geq v_k(\alpha_0(t_k), \alpha_0'(t_k)).$$

Similarly, $\beta_0 \in B$ is an upper solution of (2.1) and (2.2) if the inequalities are reversed.

For any $x \in B$, we define an operator T on x by

$$Tx(t) = a(1-t) + g(x(\frac{1}{2}))t + \int_0^1 H(t,s)f(s,x(s),x'(s))ds + I(t,x), \tag{2.3}$$

where

$$H(t,s) = \begin{cases} t(s-1), & \text{if } 0 \le t \le s \le 1, \\ (t-1)s, & \text{if } 0 \le s \le t \le 1, \end{cases}$$

is the Green's function satisfying the boundary value problem

$$x''(t) = \delta(t-s),\ 0 \le t \le 1,$$

$$x(0) = 0, \qquad\qquad x(1) = 0,$$

($\delta(t-s)$ is the Dirac delta function) and $I(t,x) = \sum_{k=1}^{m} I_k(t,x)$, where

$$I_k(t,x) = \begin{cases} t(-u_k - (1-t_k)v_k(x(t_k), x'(t_k)), & \text{if } 0 \le t \le t_k, \\ (1-t)(u_k - t_k v_k(x(t_k), x'(t_k))), & \text{if } t_k \le t \le 1. \end{cases}$$

As argued in reference [3], x is a solution of (2.1) and (2.2) if and only if $x \in B$ and $T(x) = x$. Finally, a partial order on B is defined as follows: for $\alpha_0, \beta_0 \in B$, we say that $\alpha_0 \le \beta_0$ if and only if

$$\alpha_0|_{[t_k,t_{k+1}]}(t) \le \beta_0|_{[t_k,t_{k+1}]}(t),\ t_k \le t \le t_{k+1},\ k = 0,1,\dots,m.$$

We need the following theorems to prove the main results. We do not provide the proof of these theorem as the method of proof is similar to the one employed in reference [3].

Theorem 2.1. Let $f, f_x \in C([0,1] \times R^2)$ be such that $f_x(t,x,y) > 0$; $g \in C(R)$ with $0 \le g' \le 1$ and each $v_k \in C^1(R^2)$, $k = 1,2,\dots,m$, satisfies $v_{kx}(x,y) > 0$, $v_{ky}(x,y) > 0$, $(x,y) \in R^2$. Assume that α_0, β_0 are lower and upper solutions of (2.1) and (2.2) respectively. Then $\alpha_0(t) \le \beta_0(t)$.

Theorem 2.2. Assume that $f \in C([0,1] \times R^2)$, $g \in C(R)$, $v_k \in C(R^2)$, $k = 1,2,\dots,m$ and each $v_k(x,y)$ is monotone increasing in y for fixed x. Assume that each solution of $x''(t) = f(t,x(t),x'(t))$ extends to $[0,1]$ or becomes unbounded on its maximal interval of convergence. Let α_0, β_0 be lower and upper solutions of (2.1) and (2.2) respectively such that $\alpha_0(t) \le \beta_0(t)$. Then there exists a solution, $x(t)$ of (2.1) and (2.2) such that $\alpha_0(t) \le x(t) \le \beta_0(t)$.

In passing we remark that the simplified version of the condition that each solution of $x''(t) = f(t,x(t),x'(t))$ extends to $[0,1]$ or becomes unbounded on its maximal interval of convergence is that f satisfies a Nagumo condition [9,12], that is, for each $M > 0$, there exists a positive continuous function h_M on $[0,\infty]$ such that $|f(t,x,x')| \le h_M(|x'|)$ for all $(t,x,x') \in [0,1] \times [-M,M] \times R$ and

$$\int_0^\infty s[h_M(s)]^{-1} ds = \infty.$$

3. MAIN RESULTS

Theorem 3.1 Assume that

($\mathbf{A_1}$) α_0, β_0 are lower and upper solutions of (2.1) and (2.2) respectively;

($\mathbf{A_2}$) $f(t,x,y) \in C([0,1]\times R^2)$ be such that $\frac{\partial f}{\partial x}(t,x,y) > 0$, $\frac{\partial^2}{\partial x^2}(f(t,x,y)+\phi(t,x,y)) \leq 0$, where $\frac{\partial^2}{\partial x^2}\phi(t,x,y) \leq 0$ for $\phi \in C^2[J\times R^2, R]$. Moreover, f satisfies a Nagumo condition in y;

($\mathbf{A_3}$) $v_k \in C^1[R^2, R]$ such that $v_{kx}(x,y) > 0$, $v_{ky}(x,y) > 0$, $(x,y) \in R^2$ and $\frac{\partial^2}{\partial x^2}v_k(x,y) \leq 0$;

($\mathbf{A_4}$) g, g' are continuous on R and g'' exists with $0 \leq g' < 1$, $g'' \geq 0$.

Then there exists a monotone sequence of solutions converging quadratically to the unique solution, $x(t)$ of (2.1) and (2.2).

Proof. Motivated by Eloe and Zhang [11], we define

$$f(t,x,y) = F(t,x) - \phi(t,x,y),\ t \in [0,1], \tag{3.1}$$

where $F(t,x) : [0,1] \to R$ is such that F, F_x, F_{xx} are continuous on $[0,1]\times R$ and in view of (A_2), it follows that $F_{xx}(t,x) \leq 0$. Applying the generalized mean value theorem on $F(t,x)$ gives

$$F(t,x) \leq F(t,x_1) + F_x(t,x_1)(x-x_1), \tag{3.2}$$

which together with (3.1) takes the form

$$f(t,x,y) \leq f(t,x_1,y) + F_x(t,x_1)(x-x_1) - (\phi(t,x,y) - \phi(t,x_1,y). \tag{3.3}$$

Define

$$G(t,x,x_1,y) = f(t,x_1,y) + F_x(t,x_1)(x-x_1) - (\phi(t,x,y) - \phi(t,x_1,y)). \tag{3.4}$$

Observe that

$$G(t,x,x_1,y) \geq f(t,x,y), \qquad G(t,x,x,y) = f(t,x,y). \tag{3.5}$$

Moreover, using (3.4) together with (A_2) yields

$$G_x(t,x,x_1,y) \geq F_x(t,x) - \phi_x(t,x) = f_x(t,x) > 0, \tag{3.6}$$

which implies that $G(t,x,x_1,y)$ is increasing in x for each fixed $(t,x_1,y) \in J\times R^2$.

For each $k = 1,2,3\ldots,m$, let $V_k(x) : R \to R$ be such that V_k, V_k', V_k'' are continuous on R with $V_k'' \leq 0$. Let us set

$$\psi(x,y) = V_k(x) - v_k(x,y) \text{ on } R^2.$$

Thus it follows that

$$V_k(x) \leq V_k(x_1) + V_k'(x_1)(x-x_1). \tag{3.7}$$

Now, using the generalized mean value theorem together with (3.7) and (A_3), we obtain

$$v_k(x,y) \le v_k(x_1,y) + V_k'(x_1)(x-x_1) - (\psi(x,y) - \psi(x_1,y)).$$

Define

$$h_k(x,x_1,y) = v_k(x_1,y) + V_k'(x_1)(x-x_1) - (\psi(x,y) - \psi(x_1,y)),$$

and observe that

$$v_k(x,y) \le h_k(x,x_1,y), \qquad v_k(x,y) = h_k(x,x,y). \tag{3.8}$$

Further it is easy to check that

$$h_{kx}(x,x_1,y) > 0, \qquad h_{ky}(x,x_1,y) > 0, \tag{3.9}$$

which imply that h_k is increasing in x and y respectively. In view of (A_4), we get

$$g(x) \ge g(y) + g'(y)(x-y).$$

Letting

$$g^*(x,y) = g(y) + g'(y)(x-y),$$

we notice that

$$g(x) = \max_y g^*(x,y), \qquad g(x) = g^*(x,x), \ (0 \le g^*_x(x,y) = g'(y) < 1). \tag{3.10}$$

Now, we set $x_1 = \alpha_0$ and consider the BVP

$$x''(t) = G(t,x(t),\alpha_0(t),x'(t)), \ t_k < t < t_{k+1}, \ k = 0,1,2,\ldots,m, \tag{3.11}$$

$$x(0) = a, \qquad x(1) = g^*(x(\frac{1}{2}),\alpha_0(\frac{1}{2})),$$

and for $k = 1,2,\ldots,m$,

$$\triangle x(t_k) = u_k, \qquad \triangle x'(t_k) = h_k(x(t_k),\alpha_0(t_k),x'(t_k)). \tag{3.12}$$

In view of (A_1), (3.5), (3.8) and (3.10), we have

$$\begin{aligned} \alpha_0''(t) &\ge f(t,\alpha_0(t),\alpha_0'(t)), \ t_k < t < t_{k+1}, \ k = 0,1,\ldots,m, \\ &= G(t,\alpha_0(t),\alpha_0(t),\alpha_0'(t)), \end{aligned}$$

$$\alpha_0(0) \le a, \qquad \alpha_0(1) \le g^*(\alpha_0(\frac{1}{2}),\alpha_0(\frac{1}{2})),$$

and for $k = 1,\ldots,m$,

$$\triangle\alpha_0(t_k) = u_k, \qquad \triangle\alpha_0'(t_k) \ge v_k(\alpha_0(t_k),\alpha_0'(t_k)) = h_k(\alpha_0(t_k),\alpha_0(t_k),\alpha_0'(t_k)),$$

and

$$\begin{aligned} \beta_0''(t) &\le f(t,\beta_0(t),\beta_0'(t)), \ t_k < t < t_{k+1}, \ k = 0,1,\ldots,m, \\ &\le G(t,\beta_0(t),\alpha_0(t),\beta_0'(t)) \end{aligned}$$

$$\beta_0(0) \ge a, \qquad \beta_0(1) \ge g^*(\beta_0(\frac{1}{2}),\alpha_0(\frac{1}{2})),$$

and for $k = 1, \ldots, m$,

$$\triangle\beta_0(t_k) = u_k, \qquad \triangle\beta_0'(t_k) \leq v_k(\beta_0(t_k), \beta_0'(t_k)) \leq h_k(\beta_0(t_k), \alpha_0(t_k), \beta_0'(t_k)),$$

which imply that α_0 and β_0 are lower and upper solutions of (3.11) and (3.12). In view of (3.6), (3.9) and (3.10), it follows by Theorem 2.1 that $\alpha_0(t) \leq \beta_0(t)$. Hence, by Theorem 2.2, there exists a unique solution α_1 of (3.11) and (3.12) such that

$$\alpha_0 \leq \alpha_1 \leq \beta_0.$$

Next, we consider the following problem with impulse

$$x''(t) = G(t, x(t), \alpha_1(t), x'(t)), \;\; t_k < t < t_{k+1}, \;\; k = 0, 1, 2, \ldots, m, \tag{3.13}$$

$$x(0) = a, \qquad x(1) = g^*(x(\frac{1}{2}), \alpha_1(\frac{1}{2})),$$

and for $k = 1, 2, \ldots, m$,

$$\triangle x(t_k) = u_k, \qquad \triangle x'(t_k) = v_k(x(t_k), \alpha_1(t), x'(t_k)), \tag{3.14}$$

Employing the earlier arguments, we find that

$$\begin{aligned} \alpha_1''(t) &= G(t, \alpha_1(t), \alpha_0(t), \alpha_1'(t)), \;\; t_k < t < t_{k+1}, \;\; k = 0, 1, \ldots, m, \\ &\geq G(t, \alpha_1(t), \alpha_1(t), \alpha_1'(t)), \end{aligned}$$

$$\alpha_1(0) \leq a, \qquad \alpha_1(1) = g^*(\alpha_1(\frac{1}{2}), \alpha_0(\frac{1}{2})) \leq g^*(\alpha_1(\frac{1}{2}), \alpha_1(\frac{1}{2})),$$

and for $k = 1, \ldots, m$,

$$\triangle\alpha_1(t_k) = u_k, \qquad \triangle\alpha_1'(t_k) = h_k(\alpha_1(t_k), \alpha_0(t_k), \alpha_1'(t_k)) \geq h_k(\alpha_1(t_k), \alpha_1(t_k), \alpha_1'(t_k)),$$

giving that α_1 is a lower solution of (3.13) and (3.14). Similarly, we can show that β_0 is an upper solution of (3.13) and (3.14), that is,

$$\begin{aligned} \beta_0''(t) &\leq f(t, \beta_0(t), \beta_0'(t)), \;\; t_k < t < t_{k+1}, \;\; k = 0, 1, \ldots, m, \\ &\leq G(t, \beta_0(t), \alpha_1(t), \beta_0'(t)) \end{aligned}$$

$$\beta_0(0) \geq a, \qquad \beta_0(1) \geq g^*(\beta_0(\frac{1}{2}), \alpha_1(\frac{1}{2})),$$

and for $k = 1, \ldots, m$,

$$\triangle\beta_0(t_k) = u_k, \qquad \triangle\beta_0'(t_k) \leq v_k(\beta_0(t_k), \beta_0'(t_k)) \leq h_k(\beta_0(t_k), \alpha_1(t_k), \beta_0'(t_k)).$$

Again by Theorem 2.1, we obtain $\alpha_1 \leq \beta_0$. Hence, by Theorem 2.2, there exists a unique solution α_2 of (3.13) and (3.14) such that

$$\alpha_1 \leq \alpha_2 \leq \beta_0.$$

Continuing this process successively, we obtain a monotone sequence $\{\alpha_j\}$ satisfying

$$\alpha_0 \leq \alpha_1 \leq \alpha_2 \leq \cdots, \leq \alpha_j \leq \beta_0,$$

where the element α_j of the sequence is a solution of the problem

$$x''(t) = G(t, x(t), \alpha_{j-1}(t), x'(t)),\ t_k < t < t_{k+1},\ k = 0, 1, 2, \ldots, m,$$

$$x(0) = a, \qquad x(1) = g^*(x(\frac{1}{2}), \alpha_{j-1}(\frac{1}{2})),$$

and for $k = 1, 2, \ldots, m$,

$$\triangle x(t_k) = u_k, \qquad \triangle x'(t_k) = h_k(x(t_k), \alpha_{j-1}(t_k), x'(t_k)).$$

Using the standard arguments [1, 3], it follows that $\{\alpha_j\}$ converges in B to x, the unique solution of (2.1) and (2.2).

Now, we prove the quadratic convergence. For that we set $e_j(t) = x(t) - \alpha_j(t)$, $a_j = \alpha_j(t) - \alpha_{j-1}(t)$ and note that $e_j(0) = 0$, $e_j(1) = g(x(\frac{1}{2})) - g^*(\alpha_j(\frac{1}{2}), \alpha_{j-1}(\frac{1}{2}))$ and for $k = 1, 2, \ldots, m$,

$$\triangle e_j(t_k) = 0, \qquad \triangle e_j'(t_k) = v_k(x(t_k), x'(t_k)) - h_k(\alpha_j(t_k), \alpha_{j-1}(t_k), \alpha_j'(t_k)).$$

Using the generalized mean value theorem together with (A_2), (3.1) and (3.4), we have

$$\begin{aligned}
e_j''(t) &= x''(t) - \alpha_j''(t),\ t_k < t < t_{k+1},\ k = 0, 1, 2, \ldots, m,\\
&= F(t, x) - \phi(t, x, x') - G(t, \alpha_j(t), \alpha_{j-1}(t), \alpha_j'(t))\\
&= F(t, x) - \phi(t, x, x') - \{F(t, \alpha_{j-1}) + F_x(t, \alpha_{j-1})(\alpha_j - \alpha_{j-1}) - \phi(t, \alpha_j, \alpha_j')\}\\
&= F_x(t, c_1)(x - \alpha_{j-1}) - F_x(t, \alpha_{j-1})(\alpha_j - \alpha_{j-1}) - (\phi(t, x, x') - \phi(t, \alpha_j, \alpha_j'))\\
&= [F_x(t, c_1) - F_x(t, \alpha_{j-1})]e_{j-1}(t) + F_x(t, \alpha_{j-1})]e_j(t)\\
&- \phi_x(t, c_2, c_3)e_j(t) - (\phi_{x'}(t, c_2, c_3)e_j'(t)\\
&= F_{xx}(t, c_4)(c_1 - \alpha_{j-1})e_{j-1}(t) + [F_x(t, \alpha_{j-1}) - \phi_x(t, c_2, c_3)]e_j(t)\\
&- \phi_{x'}(t, c_2, c_3)e_j'(t).\\
&\geq F_{xx}(t, c_4)e_{j-1}^2(t) + f_{x'}(t, c_2, c_3)e_j'(t),
\end{aligned}$$

where $\alpha_{j-1} \leq c_1 \leq x$, $\alpha_j \leq c_2 \leq x$, $\alpha_j' \leq c_3 \leq x'$, $\alpha_{j-1} \leq c_4 \leq c_1$. In particular, there exists $M_1 > 0$ such that

$$e_j''(t) - f_{x'}(t, c_2, c_3)e_j'(t) \geq -M_1 e_{j-1}^2(t), \tag{3.15}$$

where $M_1 > \max_i \max_{(t,x)\in D_i} F_{xx}(t, x)$ for $i = 0, 1, \ldots, m$ and

$$D_i = \{(t, x) : t_i < t < t_{i+1},\ \alpha_0 \leq x \leq \beta_0\}.$$

Let $\mu(t) = \exp\{-\int_0^t f_{x'}(s, c_2(s), c_3(s))ds\}$ be the integrating factor associated with (3.15) with $0 \leq t \leq t_1$. Then

$$e_j'(t)\mu(t) - e_j'(0) \geq -M_1 e_{j-1}^2 \int_0^t \mu(s)ds, \tag{3.16}$$

where M_1 is a bound on $F_{xx}(t,x)$. Since $e_j'(0) \geq 0$, therefore, there exists $M > 0$ such that

$$e_j''(t) \geq -M\|e_{j-1}\|^2.$$

Now, for $k = 1, 2, \ldots, m$, $\triangle e_j(t_k) = 0$, and

$$\begin{aligned}
\triangle e_j'(t_k) &= v_k(x(t_k), x'(t_k)) - h_k(\alpha_j(t_k), \alpha_{j-1}(t_k), \alpha_j'(t_k)) \\
&= V_k(x(t_k)) - \psi(x(t_k), x'(t_k)) - V_k(\alpha_{j-1}(t_k)) \\
&- V_k'(\alpha_{j-1}(t_k))(\alpha_j(t_k) - \alpha_{j-1}(t_k)) + \psi(\alpha_j(t_k), \alpha_j'(t_k)) \\
&= [V_k'(c_5(t_k)) - V_k'(\alpha_{j-1}(t_k))]e_{j-1}(t_k) + V_k'(\alpha_{j-1}(t_k))e_j(t_k) \\
&- [\psi(x(t_k), x'(t_k)) - \psi(\alpha_j(t_k), \alpha_j'(t_k)) \\
&= V_k''(c_6(t_k))(c_1(t_k) - \alpha_{j-1}(t_k))e_{j-1}(t_k) \\
&+ [V_k'(\alpha_{j-1}(t_k)) - \psi_x(c_7(t_k), c_8(t_k))]e_j(t_k) \\
&- \psi_{x'}(c_7(t_k), c_8(t_k))e_j'(t_k),
\end{aligned}$$

where $\alpha_{j-1}(t_k) \leq c_5(t_k) \leq x(t_k)$, $\alpha_{j-1}(t_k) \leq c_6(t_k) \leq c_5(t_k)$, $\alpha_j(t_k) \leq c_7(t_k) \leq x(t_k)$, $\alpha_j'(t_k) \leq c_8(t_k) \leq x'(t_k)$. Following the earlier procedure together with (A_3), we find that there exists $N > 0$ such that

$$\triangle e_j'(t_k) \geq -N e_{j-1}^2(t_k). \tag{3.17}$$

From(2.3), we have

$$e_j(t) = [g(x(\frac{1}{2})) - g^*(\alpha_j(\frac{1}{2}), \alpha_{j-1}(\frac{1}{2}))]t + \int_0^1 H(t,s)e_j''(s)ds + I(t,x), \tag{3.18}$$

where $I(t,x) = \sum_{k=1}^m I_k(t,x)$ and

$$I_k(t,x) = \begin{cases} -t(1-t_k)\triangle e_j'(t_k), & \text{if } 0 \leq t \leq t_k, \\ -(1-t)t_k\triangle e_j'(t_k), & \text{if } t_k \leq t \leq 1, \end{cases}$$

which, in view of (3.17), becomes

$$I_k(t,x) \leq \begin{cases} t(1-t_k)N e_{j-1}^2(t_k), & \text{if } 0 \leq t \leq t_k, \\ (1-t)t_k N e_{j-1}^2(t_k), & \text{if } t_k \leq t \leq 1. \end{cases} \tag{3.19}$$

Observe that

$$\begin{aligned}
&g(x(\frac{1}{2})) - g^*(\alpha_j(\frac{1}{2}), \alpha_{j-1}(\frac{1}{2})) \\
&\quad = g(x(\frac{1}{2})) - g(\alpha_{j-1}(\frac{1}{2})) - g'(\alpha_{j-1}(\frac{1}{2}))(\alpha_j(\frac{1}{2}) - \alpha_{j-1}(\frac{1}{2})) \\
&\quad = g'(c_o)e_{j-1}(\frac{1}{2}) - g'(\alpha_{j-1}(\frac{1}{2}))(e_{j-1}(\frac{1}{2}) - e_j(\frac{1}{2})) \\
&\quad = g''(c_1)e_{j-1}^2(\frac{1}{2}) + g'(\alpha_{j-1})e_j(\frac{1}{2}).
\end{aligned} \tag{3.20}$$

Using (3.19) and (3.20) in (3.18) and taking the maximum over the interval $[0,1]$, we obtain

$$\|e_j\| \leq M_2\|e_{j-1}\|^2 + \lambda\|e_j\| + M_1\|e_{j-1}\|^2 + N_2\|e_{j-1}\|^2, \tag{3.21}$$

where M_2 provides a bound for $\|g''\|$ on $[\alpha_{j-1}(\frac{1}{2}), x(\frac{1}{2})]$, $\|g'\| \leq \lambda < 1$, N_2 gives a bound on $I(t,x)$ and $M_1 = \max\int_0^1 M|H(t,s)|ds$. Solving (3.21) algebraically, we get

$$\|e_j\| \leq \delta\|e_{j-1}\|^2,$$

where $\delta = (M_2 + M_1 + N_2)/(1-\lambda)$ and $\|e_j\| = \max\{|e_j(t)| : t \in [0,1]\}$ is the usual uniform norm. This establishes the quadratic convergence.

Theorem 3.2 Assume that

($\mathbf{B_1}$) α_0, β_0 are lower and upper solutions of (2.1) and (2.2) respectively.

($\mathbf{B_2}$) $\frac{\partial^i}{\partial x^i}f(t,x,y) \in C([0,1]\times R^2)$ for $i = 0,1,2,\ldots,n$ such that $\frac{\partial^i}{\partial x^i}f(t,x,y) > 0$ for $i = 1,2,\ldots,n-1$, $\frac{\partial}{\partial y}(\frac{\partial^i}{\partial x^i}f(t,x,y)) \geq 0$, $\frac{\partial^n}{\partial x^n}(f(t,x,y) + \phi(t,x,y)) < 0$, where $\phi \in C^{0,n}[J\times R^2, R]$ such that $\frac{\partial^n}{\partial x^n}\phi(t,x,y) \leq 0$. Moreover, f satisfies a Nagumo condition in y.

($\mathbf{B_3}$) $v_k \in C^{0,n}(R^2)$ such that $\frac{\partial^i}{\partial x^i}v_{ky}(x,y) > 0$, $\frac{\partial^i}{\partial x^i}v_k(x,y) > 0$, $i = 1,2,\ldots,n-1$ and $\frac{\partial^n}{\partial x^n}v_k(x,x') \leq 0$.

($\mathbf{B_4}$) $\frac{d^i}{dx^i}g(x) \in C(R)$ for $i = 0,1,2,\ldots,n$ satisfying $0 \leq \frac{d^i}{dx^i}g(x) < \frac{M}{(\beta_0-\alpha_0)^{i-1}}$ for $i = 1,2,\ldots,n-1$ with $0 < M < \frac{1}{3}$ and $\frac{d^n}{dx^n}g(x) \geq 0$.

Then there exists a monotone sequence of solutions converging monotonically to the unique solution of (2.1) and (2.2) with the order of convergence $n \geq 2$.

Proof. Let us define

$$f(t,x,y) = F(t,x) - \phi(t,x,y),\ t \in [0,1]. \tag{3.22}$$

In view of (B_2), we note that $F \in C^{0,n}(J\times R)$ and $\frac{\partial^n}{\partial x^n}F(t,x) \leq 0$. Applying the generalized mean value theorem on $F(t,x)$ gives

$$F(t,x) \leq \sum_{i=0}^{n-1}\frac{\partial^i}{\partial x^i}F(t,x_1)\frac{(x-x_1)^i}{i!},$$

which together with (3.22) takes the form

$$f(t,x,y) \leq \sum_{i=0}^{n-1}\frac{\partial^i}{\partial x^i}f(t,x_1,y)\frac{(x-x_1)^i}{i!} - \frac{\partial^n}{\partial x^n}\phi(t,\xi,y)\frac{(x-x_1)^n}{n!}, \tag{3.23}$$

where $x_1 \leq \xi \leq x$. Define

$$G^*(t,x,x_1,y) = \sum_{i=0}^{n-1}\frac{\partial^i}{\partial x^i}f(t,x_1,y)\frac{(x-x_1)^i}{i!} - \frac{\partial^n}{\partial x^n}\phi(t,\xi,y)\frac{(x-x_1)^n}{n!}. \tag{3.24}$$

Observe that

$$G^*(t,x,x_1,y) \geq f(t,x,y), \qquad G^*(t,x,x,y) = f(t,x,y) \tag{3.25}$$

Moreover, using (B_2), we find that

$$G_x^*(t,x,x_1,y) \geq \sum_{i=1}^{n-1} \frac{\partial^i}{\partial x^i} f(t,x_1,y) \frac{(x-x_1)^{i-1}}{(i-1)!} > 0,$$

which implies that $G^*(t,x,x_1,y)$ is increasing in x for each fixed $(t,x_1,y) \in J \times R^2$. Further, using the generalized mean value theorem together with (B_3), we obtain

$$v_k(x,y) \leq \sum_{i=0}^{n-1} \frac{\partial^i}{\partial x^i} v_k(x_1,y) \frac{(x-x_1)^i}{i!}.$$

Now, we define

$$h_k^*(x,x_1,y) = \sum_{i=0}^{n-1} \frac{\partial^i}{\partial x^i} v_k(x_1,y) \frac{(x-x_1)^i}{i!},$$

and observe that

$$v_k(x,y) \leq h_k^*(x,x_1,y), \qquad v_k(x,y) = h_k^*(x,x,y). \tag{3.26}$$

In view of (B_3), we further conclude that

$$h_{kx}^*(x,x_1,y) = \sum_{i=1}^{n-1} \frac{\partial^i}{\partial x^i} v_k(x_1,y) \frac{(x-x_1)^{i-1}}{(i-1)!} > 0,$$

and

$$h_{ky}^*(x,x_1,y) = \sum_{i=0}^{n-1} \frac{\partial^i}{\partial x^i} v_{ky}(x_1,y) \frac{(x-x_1)^i}{i!} > 0,$$

which imply that h_k^* is increasing in x and y respectively. In view of (B_4), we get

$$g(x) \geq \sum_{i=0}^{n-1} \frac{d^i}{dx^i} g(y) \frac{(x-y)^i}{i!}.$$

Letting

$$g^{**}(x,y) = \sum_{i=0}^{n-1} \frac{d^i}{dx^i} g(y) \frac{(x-y)^i}{i!},$$

we notice that

$$g(x) = \max_y g^{**}(x,y), \qquad g(x) = g^{**}(x,x). \tag{3.27}$$

Clearly $g_x^{**}(x,y) \geq 0$ and

$$\begin{aligned} g_x^{**}(x,y) &= \sum_{i=1}^{n-1} \frac{d^i}{dx^i} g(y) \frac{(x-y)^{i-1}}{(i-1)!} \\ &\leq \sum_{i=1}^{n-1} \frac{d^i}{dx^i} g(y) \frac{(\beta_0-\alpha_0)^{i-1}}{(i-1)!} \\ &\leq \sum_{i=1}^{n-1} \frac{M}{(i-1)!} < M(3 - \frac{1}{2^{n-3}}) < 3M < 1. \end{aligned}$$

Now, we set $x_1 = \alpha_0$ and consider the BVP

$$x''(t) = G^*(t, x(t), \alpha_0(t), x'(t)),\ t_k < t < t_{k+1},\ k = 0, 1, 2, \ldots, m, \tag{3.28}$$

$$x(0) = a, \qquad x(1) = g^{**}(x(\frac{1}{2}), \alpha_0(\frac{1}{2})),$$

and for $k = 1, 2, \ldots, m$,

$$\triangle x(t_k) = u_k, \qquad \triangle x'(t_k) = h_k^*(x(t_k), \alpha_0(t_k), x'(t_k)). \tag{3.29}$$

In view of (B_1), (3.25), (3.26) and (3.27), we have

$$\begin{aligned} \alpha_0''(t) &\geq f(t, \alpha_0(t), \alpha_0'(t)),\ t_k < t < t_{k+1},\ k = 0, 1, \ldots, m, \\ &= G^*(t, \alpha_0(t), \alpha_0(t), \alpha_0'(t)), \end{aligned}$$

$$\alpha_0(0) \leq a, \qquad \alpha_0(1) \leq g^{**}(\alpha_0(\frac{1}{2}), \alpha_0(\frac{1}{2})),$$

and for $k = 1, \ldots, m$,

$$\triangle \alpha_0(t_k) = u_k, \qquad \triangle \alpha_0'(t_k) \geq v_k(\alpha_0(t_k), \alpha_0'(t_k)) = h_k^*(\alpha_0(t_k), \alpha_0(t_k), \alpha_0'(t_k)),$$

and

$$\begin{aligned} \beta_0''(t) &\leq f(t, \beta_0(t), \beta_0'(t)),\ t_k < t < t_{k+1},\ k = 0, 1, \ldots, m, \\ &\leq G^*(t, \beta_0(t), \alpha_0(t), \beta_0'(t)), \end{aligned}$$

$$\beta_0(0) \geq a, \qquad \beta_0(1) \geq g^{**}(\beta_0(\frac{1}{2}), \alpha_0(\frac{1}{2})),$$

and for $k = 1, \ldots, m$,

$$\triangle \beta_0(t_k) = u_k, \qquad \triangle \beta_0'(t_k) \leq v_k(\beta_0(t_k), \beta_0'(t_k)) \leq h_k^*(\beta_0(t_k), \alpha_0(t_k), \beta_0'(t_k)),$$

which imply that α_0 and β_0 are lower and upper solutions of (3.28) and (3.29). Thus, by Theorem 2.1 and Theorem 2.2 (as in the proof of Theorem 3.1), there exists a unique solution α_1 of (3.28) and (3.29) such that

$$\alpha_0 \leq \alpha_1 \leq \beta_0.$$

Continuing this process successively, we obtain a monotone sequence $\{\alpha_j\}$ satisfying

$$\alpha_0 \leq \alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_j \leq \beta_0,$$

where the element α_j of the sequence is a solution of the problem

$$x''(t) = G^*(t, x(t), \alpha_{j-1}(t), x'(t)),\ t_k < t < t_{k+1},\ k = 0, 1, 2, \ldots, m,$$

$$x(0) = a, \qquad x(1) = g^{**}(x(\frac{1}{2}), \alpha_{j-1}(\frac{1}{2})),$$

and for $k = 1, 2, \ldots, m$,

$$\triangle x(t_k) = u_k, \qquad \triangle x'(t_k) = h_k^*(x(t_k), \alpha_{j-1}(t_k), x'(t_k)).$$

Employing the arguments used in the proof of Theorem 3.1, it follows that $\{\alpha_j\}$ converges in B to x, the unique solution of (2.1) and (2.2).

Now, we prove the convergence of order $n \geq 2$. For that we set $e_j(t) = x(t) - \alpha_j(t)$, $a_{j-1} = \alpha_j(t) - \alpha_{j-1}(t)$, and note that $e_j(0) = 0,\ e_j(1) = g(x(\frac{1}{2})) - g^{**}(\alpha_j(\frac{1}{2}), \alpha_{j-1}(\frac{1}{2}))$ and for $k = 1, 2, \ldots, m$,

$$\triangle e_j(t_k) = 0, \qquad \triangle e_j'(t_k) = v_k(x(t_k), x'(t_k)) - h_k^*(\alpha_j(t_k), \alpha_{j-1}(t_k), \alpha_j'(t_k)).$$

Using the generalized mean value theorem, (B_2), (3.22) and (3.24), we can find $\alpha_{j-1} \leq \xi \leq x$ such that

$$\begin{aligned}
e_j''(t) &= x''(t) - \alpha_j''(t),\ t_k < t < t_{k+1},\ k = 0, 1, 2, \ldots, m \\
&= \sum_{i=0}^{n-1} \frac{\partial^i}{\partial x^i} f(t, \alpha_{j-1}, x') \frac{(x - \alpha_{j-1})^i}{i!} + \frac{\partial^n}{\partial x^n} f(t, \xi, x') \frac{(x - \alpha_{j-1})^n}{n!} \\
&- \sum_{i=0}^{n-1} \frac{\partial^i}{\partial x^i} f(t, \alpha_{j-1}, \alpha_j') \frac{(\alpha_j - \alpha_{j-1})^i}{i!} + \frac{\partial^n}{\partial x^n} \phi(t, \xi, \alpha_j') \frac{(\alpha_j - \alpha_{j-1})^n}{n!} \\
&\geq \sum_{i=0}^{n-1} \frac{\partial^i}{\partial x^i} f(t, \alpha_{j-1}, \alpha_j') \frac{e_{j-1}^i}{i!} - \sum_{i=0}^{n-1} \frac{\partial^i}{\partial x^i} f(t, \alpha_{j-1}, \alpha_j') \frac{a_{j-1}^i}{i!} \\
&+ [\frac{\partial^n}{\partial x^n} f(t, \xi, x) + \frac{\partial^n}{\partial x^n} \phi(t, \xi, \alpha_j')] \frac{e_{j-1}^n}{n!} \\
&\geq \sum_{i=1}^{n-1} \frac{\partial^i}{\partial x^i} f(t, \alpha_{j-1}, \alpha_j') \frac{1}{i!} [e_{j-1}^i - a_{j-1}^i] + [\frac{\partial^n}{\partial x^n} f(t, \xi, \alpha_j') \\
&+ \frac{\partial^n}{\partial x^n} \phi(t, \xi, \alpha_j')] \frac{e_{j-1}^n}{n!} \\
&= \sum_{i=1}^{n-1} \frac{\partial^i}{\partial x^i} f(t, \alpha_{j-1}, \alpha_j') \frac{1}{i!} \sum_{r=0}^{i-1} e_{j-1}^{i-r-1} a_{j-1}^r (e_{j-1} - a_{j-1}) \\
&+ \frac{\partial^n}{\partial x^n} F(t, \xi, \alpha_j') \frac{e_{j-1}^n}{n!} \\
&= [\sum_{i=1}^{n-1} \frac{\partial^i}{\partial x^i} f(t, \alpha_{j-1}, \alpha_j') \frac{1}{i!} \sum_{r=0}^{i-1} e_{j-1}^{i-r-1} a_{j-1}^r] e_j + \frac{\partial^n}{\partial x^n} F(t, \xi, \alpha_j') \frac{e_{j-1}^n}{n!} \\
&= \omega(t) e_j + \frac{\partial^n}{\partial x^n} F(t, \xi, \alpha_j') \frac{e_{j-1}^n}{n!} \geq g(t) e_j - \epsilon_1 e_{j-1}^n,
\end{aligned}$$

where $\omega(t) = \sum_{i=1}^{n-1} \frac{\partial^i}{\partial x^i} f(t, \alpha_{j-1}, \alpha_j') \frac{1}{i!} \sum_{r=0}^{i-1} e_{j-1}^{i-r-1} a_j^r > 0$ and $\frac{1}{n!} \frac{\partial^n}{\partial x^n} F(t, \xi, \alpha_j') \geq -\epsilon_1$ for some $\epsilon_1 > 0$. Thus, for each j, we have

$$e_j''(t) \geq -\epsilon_1 e_{j-1}^n\ , t_k < t < t_{k+1},\ k = 0, 1, 2, \ldots, m, \tag{3.30}$$

$$e_j(0) = 0,\ e_j(1) = g(x(\frac{1}{2})) - g^*(\alpha_j(\frac{1}{2}), \alpha_{j-1}(\frac{1}{2})),$$

and for $k = 1, 2, \ldots, m$,

$$\triangle e_j(t_k) = 0,$$

and

$$\triangle e_j{}'(t_k) = v_k(x(t_k), x'(t_k)) - h_k^*(\alpha_j(t_k), \alpha_j'(t_k), \alpha_{j-1}(t_k))$$

$$\begin{aligned}
&= \sum_{i=0}^{n-1}\frac{\partial^i}{\partial x^i}v_k(\alpha_{j-1},x')\frac{(x-\alpha_{j-1})^i}{i!}+\frac{\partial^n}{\partial x^n}v_k(c_1,x')\frac{(x-\alpha_{j-1})^n}{n!}\\
&- \sum_{i=0}^{n-1}\frac{\partial^i}{\partial x^i}v_k(\alpha_{j-1},\alpha_j')\frac{(\alpha_j-\alpha_{j-1})^i}{i!}\\
&\geq \sum_{i=1}^{n-1}\frac{\partial^i}{\partial x^i}v_k(\alpha_{j-1},\alpha_j')\frac{1}{i!}[e_{j-1}^i-a_{j-1}^i]+\frac{\partial^n}{\partial x^n}v_k(c_1,x')\frac{e_{j-1}^n}{n!}\\
&\geq [\sum_{i=1}^{n-1}\frac{\partial^i}{\partial x^i}v_k(\alpha_{j-1},\alpha_j')\frac{1}{i!}\sum_{r=0}^{i-1}e_{j-1}^{i-r-1}a_{j-1}^r]e_j-\epsilon_2 e_{j-1}^n\\
&= \omega_1(t)e_j(t)-\epsilon_2 e_{j-1}^n,
\end{aligned}$$

where $\omega_1(t)=\sum_{i=1}^{n-1}\frac{\partial^i}{\partial x^i}v_k(\alpha_{j-1},\alpha_j')\frac{1}{i!}\sum_{r=0}^{i-1}e_{j-1}^{i-r-1}a_j^r>0$ and $\frac{1}{n!}\frac{\partial^n}{\partial x^n}v_k(c_1,x')\geq -\epsilon_2$ for some $\epsilon_2>0$. Hence, for each j, it follows that

$$\triangle e_j'(t_k) > -\epsilon_2 e_{j-1}^n. \tag{3.31}$$

Now from(2.3), we have

$$e_j(t)=[g(x(\frac{1}{2}))-g^{**}(\alpha_j(\frac{1}{2}),\alpha_{j-1}(\frac{1}{2}))]t+\epsilon_1\int_0^1|H(t,s)|e_{j-1}^n(s)ds+I(t,x), \tag{3.32}$$

where $I(t,x)=\sum_{k=1}^m I_k(t,x)$ and

$$I_k(t,x)=\begin{cases} -t(1-t_k)\triangle e_j'(t_k), & \text{if } 0\leq t\leq t_k,\\ -(1-t)t_k\triangle e_j'(t_k), & \text{if } t_k\leq t\leq 1,\end{cases}$$

which, in view of (3.31), becomes

$$I_k(t,x)\leq\begin{cases} t(1-t_k)\epsilon_2 e_{j-1}^n(t_k), & \text{if } 0\leq t\leq t_k,\\ (1-t)t_k\epsilon_2 e_{j-1}^n(t_k), & \text{if } t_k\leq t\leq 1.\end{cases} \tag{3.33}$$

Further, we find that

$$\begin{aligned}
&g(x(\frac{1}{2}))-g^{**}(\alpha_j(\frac{1}{2}),\alpha_{j-1}(\frac{1}{2})) \qquad (3.34)\\
&=\sum_{i=0}^{n-1}\frac{d^i}{dx^i}g(\alpha_{j-1}(\frac{1}{2}))\frac{(x(\frac{1}{2})-\alpha_{j-1}(\frac{1}{2}))^i}{i!}+\frac{d^n}{dx^n}g(\xi(\frac{1}{2}))\frac{(x(\frac{1}{2})-\alpha_{j-1}(\frac{1}{2}))^n}{n!}\\
&-\sum_{i=0}^{n-1}\frac{d^i}{dx^i}g(\alpha_{j-1}(\frac{1}{2}))\frac{(\alpha_j(\frac{1}{2})-\alpha_{j-1}(\frac{1}{2}))^i}{i!}\\
&=\sum_{i=1}^{n-1}\frac{d^i}{dx^i}g(\alpha_{j-1}(\frac{1}{2}))\frac{(e_{j-1}^i(\frac{1}{2})-a_{j-1}^i(\frac{1}{2}))}{i!}+\frac{d^n}{dx^n}g(\xi(\frac{1}{2}))\frac{(e_{j-1}(\frac{1}{2}))^n}{n!}\\
&=\sum_{i=1}^{n-1}\frac{d^i}{dx^i}g(\alpha_{j-1}(\frac{1}{2}))\frac{1}{i!}\sum_{r=0}^{n-1}e_{j-1}^r(\frac{1}{2})a_{j-1}^{i-1-r}(\frac{1}{2})e_j(\frac{1}{2})\\
&+\frac{d^n}{dx^n}g(\xi(\frac{1}{2}))\frac{(e_{j-1}(\frac{1}{2}))^n}{n!}
\end{aligned}$$

$$\le [\sum_{i=0}^{n-1} \frac{M}{(\beta_0-\alpha_0)^{i-1}} \frac{1}{i!} \sum_{r=0}^{i-1} e_{j-1}^{i-1-r}(\frac{1}{2}) a_{j-1}^r(\frac{1}{2})] e_j(\frac{1}{2}) + \epsilon_3 e_{j-1}^n,$$

where ϵ_3 provides a bound for $\frac{1}{n!}\frac{d^n}{dx^n} g(\xi(\frac{1}{2}))$. Letting

$$P_j(t) = \sum_{i=0}^{n-1} \frac{M}{(\beta_0-\alpha_0)^{i-1}} \frac{1}{i!} \sum_{r=0}^{i-1} e_{j-1}^{i-1-r}(\frac{1}{2}) a_{j-1}^r(\frac{1}{2}),$$

we observe that

$$\lim_{j\to\infty} P_j(t) = \lim_{j\to\infty} \sum_{i=0}^{n-1} \frac{M}{(\beta_0-\alpha_0)^{i-1}} \frac{1}{i!} \sum_{r=0}^{i-1} e_{j-1}^{i-1-r}(\frac{1}{2}) a_{j-1}^r(\frac{1}{2}) = M < \frac{1}{3}.$$

Therefore, we can choose $\lambda_1 < \frac{1}{3}$ and $j_0 \in N$ such that for $n \ge j_0$, we have $P_j(t) < \lambda_1$. Thus, using (3.33) and (3.34) in (3.32) and taking the maximum over the interval $[0,1]$, we obtain

$$\|e_j\| \le \epsilon_3 \|e_{j-1}\|^n + \lambda_1 \|e_j\| + \epsilon_4 \|e_{j-1}\|^n + \epsilon_5 \|e_{j-1}\|^n \tag{3.35}$$

where ϵ_5 gives a bound on $I(t,x)$ and $\epsilon_4 = \max \int_0^1 \epsilon_1 |H(t,s)| ds$. Solving (3.35) algebraically, we get

$$\|e_j\| \le \epsilon \|e_{j-1}\|^n,$$

where $\delta = (\epsilon_3 + \epsilon_4 + \epsilon_5)/(1-\lambda_1)$ and $\|e_j\| = \max\{|e_j(t)| : t \in [0,1]\}$ is the usual uniform norm. This establishes the convergence of order $n \ge 2$.

Remark. It is clear that Theorem 3.2 remains valid if we replace the condition $\frac{\partial^i}{\partial x^i} f(t,x,x') > 0$ for $i = 1,2,\ldots,n-1$ in (B_2) by that of $\Gamma f(t,x,x') > 0$ with $\frac{\partial}{\partial x} f(t,x,x') > 0$, where $\Gamma = \sum_{i=1}^{n-1} \frac{\partial^i(.)}{\partial x^i} \frac{(x-y)^{i-1}}{(i-1)}$.

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