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EXISTENCE THEORY FOR SINGLE AND MULTIPLE SOLUTIONS TO SINGULAR DISCRETE BOUNDARY VALUE PROBLEMS OF SECOND-ORDER DIFFERENTIAL SYSTEMS

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ABSTRACT. In this paper we establish the existence of single and multiple solutions to the singular discrete boundary value problem

$$\begin{cases} \Delta^2 x(i-1) + q_1(i) f_1(i, x(i), y(i)) = 0, \ i \in \{1, 2, \dots, T\}, \\ \Delta^2 y(i-1) + q_2(i) f_2(i, x(i), y(i)) = 0, \\ x(0) = x(T+1) = y(0) = y(T+1) = 0, \end{cases}$$

where nonlinear term $f_k(i, x, y)$ may be singular at $(x, y) = (0, 0)$, $k = 1, 2$.

Key Words. Multiple solutions, singular, existence, discrete boundary value problem, fixed point theorem in cones, Lerary-Schauder alternative theorem.

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1. INTRODUCTION

In this paper we establish the existence of single and multiple solutions to the singular discrete boundary value problem

$$\begin{cases} \Delta^2 x(i-1) + q_1(i) f_1(i, x(i), y(i)) = 0, \ i \in \{1, 2, \dots, T\}, \\ \Delta^2 y(i-1) + q_2(i) f_2(i, x(i), y(i)) = 0, \\ x(0) = x(T+1) = y(0) = y(T+1) = 0, \end{cases} \tag{1.1}$$

where nonlinearity term $f_k(i, x, y)$ may be singular at $(0, 0)$, $k = 1, 2$. $T \in \{1, 2, \dots\}$, $N = \{1, \dots, T\}$, $N^+ = \{0, 1, \dots, T+1\}$ and $x(i) : N^+ \to (0, \infty)$, $y(i) : N^+ \to (0, \infty)$.

Throughout this paper we will assume $f_k : N \times ([0, \infty)^2 \backslash \{O\}) \to (0, \infty)$ is continuous, $k = 1, 2$. $(O = (0, 0))$.

Remark 1.1. Recall a map $f_k : N \times ([0, \infty)^2 \backslash \{O\}) \to (0, \infty)$ is continuous if it is continuous as a map of the topological space $N \times ([0, \infty)^2 \backslash \{O\})$ into the topological

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space $(0, \infty)$, $k = 1, 2$. Throughout this paper the topology on N will be the discrete topology.

We will let $C(N^+, [0, \infty)^2 \backslash \{O\})$ denote the class of maps (x, y) continuous on N^+ (discrete topology), with norm $\|x\| = \max_{i \in N^+} |x(i)|$, $\|y\| = \max_{i \in N^+} |y(i)|$. By a solution to (1.1) we mean a $(x(i), y(i)) \in C(N^+, [0, \infty)^2 \backslash \{O\})$ such that (x, y) satisfies (1.1) for $i \in N$ and (x, y) satisfies the boundary conditions.

Recently, the singular boundary value problems have been studied extensively. For details, see, for instance, papers [1,4] and the references therein. However, there are only few works on singular boundary value problems for differential systems [9]. Agarwal and Regan [2] considered the singular boundary value problem

$$\begin{cases} y''(t) + q(t)[g(y(t)) + h(y(t))] = 0 \ t \in (0, 1) \\ y(0) = y(1) = 0, \end{cases}$$

where $q(t)$ may be singular at $t = 0$ or $t = 1$, nonlinearity g may be singular at $y = 0$, h may be superlinear at $y = \infty$. They showed that this problem has twin positive solutions by using a Lerary-Shauder alternative and a fixed point theorem in cones. Jiang and Xu [9] studied the singular continuous boundary value problem. They also showed the existence of single and multiple positive solutions to the singular continuous boundary value problems.

However, for the discrete case, the works on the existence of single and multiple solutions to singular discrete boundary value problems for differential systems are quite rarely seen.

In this paper we only consider existence theorem of single and multiple positive solutions to singular discrete boundary value problems for differential systems (1.1).

2. PRELIMINARY LEMMAS

In this section, we give out some results which will be need in section 3.

Lemma 2.1.[1] Assume Ω is a relatively subset of a convex set K in a normal space E. Let $A : \bar{\Omega} \to K$ be a compact map with $p \in \Omega$. Then either

(2.1) (A_1) A has a fixed point in $\bar{\Omega}$; or

(2.2) (A_2) there is an $x \in \partial\Omega$ with $x = \lambda A(x) + (1 - \lambda)p$ for some $0 < \lambda < 1$.

Remark 2.1. By a map being compact we mean it is continuous with relatively compact range.

Lemma 2.2. [2] Let $E = (E, \|\cdot\|)$ be a Banach space and let $K \subset E$ be a cone in E, and let $\|\cdot\|$ be increasing with respect to K. Also, r, R are constants with $0 < r < R$. Suppose $\Phi : \bar{\Omega}_R \cap K \to K$(here $\Omega_R = \{x \in E, \|x\| < R\}$) is a continuous, compact map and assume the conditions

$$x \neq \lambda\Phi(x), \ \text{for } \lambda \in [0, 1) \text{ and } x \in \partial\Omega_r \cap K \tag{2.3}$$

and

$$\|\Phi x\| > \|x\|, \text{ for } x \in \partial\Omega_R \cap K \tag{2.4}$$

hold. Then Φ has a fixed point in $K \cap \{x \in E : r \leq \|x\| \leq R\}$.

Remark 2.2. In Lemma 2.2 if (2.3) and (2.4) are replaced by

$$x \neq \lambda\Phi(x), \text{ for } \lambda \in [0,1) \text{ and } x \in \partial\Omega_R \cap K \tag{2.3*}$$

and

$$\|\Phi x\| > \|x\|, \text{ for } x \in \partial\Omega_r \cap K \tag{2.4*}$$

then Φ has a fixed point in $K \cap \{x \in E : r \leq \|x\| \leq R\}$.

Lemma 2.3 [6] Let $y \in C(N^+, \mathbf{R})$ satisfy $y(i) \geq 0$ for $i \in N^+$. If $u \in C(N^+, \mathbf{R})$ satisfies

$$\begin{cases} \Delta^2 u(i-1) + y(i) = 0, \ i \in N, \\ u(0) = u(T+1) = 0, \end{cases} \tag{2.5}$$

then

$$u(i) \geq \mu(i)\|u\| \text{ for } i \in N^+; \tag{2.6}$$

here

$$\mu(i) = \min\{\frac{T+1-i}{T+1}, \frac{i}{T}\}. \tag{2.7}$$

In this paper, let $\|u\| = \max_{i\in N^+} |u(i)|$, $u(i) \in C(N^+, R)$, then $E_1 = (C(N,R), \|\cdot\|)$ is a Banach space. Let

$$K_1 = \{u \in C(N^+, [0,+\infty)) : u(i) \geq \mu(i)\|u\|, i \in N^+\}. \tag{2.8}$$

Let $E = E_1 \times E_1$, $K = K_1 \times K_1$, and $\|z\| = \|(x,y)\| = \max\{\|x\|, \|y\|\}$, $\forall z = (x,y) \in E$. Then $(E, \|\cdot\|)$ is a Banach space and K is a cone in E.

3. EXISTENCE PRINCIPLES

Throughout this paper, we make the following hypotheses:

(H_1): $q_k(i) \in C(N, (0,+\infty)), k = 1,2$.

(H_2): Let $f_k(i,x,y) \leq g_k(x,y) + h_k(x,y)$ on $N \times ([0,+\infty)^2 \backslash \{O\})$, with $g_k > 0$ continuous and nonincreasing on $[0,\infty)^2 \backslash \{O\}$, $h_k \geq 0$ continuous on $[0,\infty)^2$, and $\frac{h_k}{g_k}$ nondecreasing on $[0,\infty)^2 \backslash \{O\}$, $k = 1,2$.

(H_3): There exists a constant $r > 0$ such that

$$\int_0^r \frac{du}{g_1(u,0)} > \{1 + \frac{h_1(r,r)}{g_1(r,r)}\}b_{10}, \ \int_0^r \frac{dv}{g_2(0,v)} > \{1 + \frac{h_2(r,r)}{g_2(r,r)}\}b_{20},$$

where $b_{k0} = \max_{i\in N}\{\sum_{j=1}^i jq_k(j), \sum_{j=i}^T (T+1-j)q_k(j)\}$, $k = 1,2$.

(H_4): For each constant $H > 0$ there exists a function ψ_H continuous on N^+ and

positive on N such that $f_k(i,x,y) \ge \psi_H^{(k)}(i)$ for $i \in N$, $O \le (x,y) \le H = (H_1, H_2)$, $k = 1, 2$.
(H_5): Let $f_k(i,x,y) \ge \bar{g}_k(x,y) + \bar{h}_k(x,y)$ on $N \times ([0,+\infty)^2 \backslash \{O\})$,
with $\bar{g}_k > 0$ continuous and nonincreasing on $[0,\infty)^2 \backslash \{O\}$, $\bar{h}_k \ge 0$ continuous on $[0,\infty)^2$, and $\frac{\bar{h}_k}{\bar{g}_k}$ nondecreasing on $[0,\infty)^2 \backslash \{O\}$, $k = 1, 2$.
(H_6): There exists a constant $R > r$ such that

$$\frac{R}{\bar{g}_1(R,R)\{1 + \frac{\bar{h}_1(\frac{R}{T+1},0)}{\bar{g}_1(\frac{R}{T+1},0)}\}} < \sum_{j=1}^{T} G(\sigma,j) q_1(j),$$

$$\frac{R}{\bar{g}_2(R,R)\{1 + \frac{\bar{h}_2(0,\frac{R}{T+1})}{\bar{g}_2(0,\frac{R}{T+1})}\}} < \sum_{j=1}^{T} G(\sigma,j) q_2(j),$$

where $\sum_{j=1}^T G(\sigma,j) = \max_{i \in N^+} \sum_{j=1}^T G(i,j)$, and

$$G(i,j) = \begin{cases} \frac{j(T+1-i)}{T+1}, & 0 \le j \le i-1, \\ \frac{i(T+1-j)}{T+1}, & i \le j \le T+1, \end{cases}$$

is following boundary value problem's Green function

$$\begin{cases} -\Delta^2 u(i-1) = 0, \ i \in N, \\ u(0) = u(T+1) = 0. \end{cases}$$

Here and henceforth, we denote $(x_1, y_1) > (x_2, y_2)$ $((x_1,y_1) \ge (x_2,y_2))$ if $(x_1 - x_2, y_1 - y_2) \in \bar{R}_+^2 ((x_1 - x_2, y_1 - y_2) \in R_+^2)$, $(\bar{R}_+^2 = [0,+\infty)^2 \backslash \{O\}, R_+^2 = [0,+\infty)^2)$.

Further, we say that a vector (x,y) is positive (nonnegative) if $(x,y) > (0,0)((x,y) \ge (0,0))$. The hypothesis (H_2) allows $f_k(i,x,y)$ to have singularity at $O = (0,0)$.

For example,

$$f_k(i,x,y) = [\sqrt{x^2(i) + y^2(i)}]^{-\alpha_k} + \gamma[\sqrt{x^2(i) + y^2(i)}]^{\beta_k}$$

satisfy (H_2), where $0 < \alpha_k < +\infty$, $\beta_k > 1$, $\gamma > 0$, $k = 1, 2$.

We have the following main result:

Theorem 3.1 Let (H_1)–(H_3) hold. Then the problem (1.1) has one positive solutions (x,y) with $\|(x,y)\| < r$.

Proof. Choose $\epsilon > 0$, $\epsilon < r$ with

$$\int_\epsilon^r \frac{du}{g_1(u,0)} > b_{10}\{1 + \frac{h_1(r,r)}{g_1(r,r)}\}, \int_\epsilon^r \frac{dv}{g_2(0,v)} > b_{20}\{1 + \frac{h_2(r,r)}{g_2(r,r)}\}. \tag{3.1}$$

Let $m_0 \in \{1, 2, \dots\}$ be chosen so that $\frac{1}{m_0} < \epsilon$, $\frac{1}{m_0} < \frac{r}{T+1}$, and let $N_0 = \{m_0, m_0 + 1, \dots\}$.

We first show that the following boundary value problem

$$(3.2)^m \qquad \begin{cases} \Delta^2 x(i-1) + q_1(i) f_1(i, x(i), y(i)) = 0, \ i \in \{1, 2, \ldots, T\}, \\ \Delta^2 y(i-1) + q_2(i) f_2(i, x(i), y(i)) = 0, \\ x(0) = x(T+1) = y(0) = y(T+1) = \frac{1}{m}, \ m \in N_0, \end{cases}$$

has a solution $(x_m(i), y_m(i))$ for $m \in N_0$, $(x_m(i), y_m(i)) > (\frac{1}{m}, \frac{1}{m})$ on N and $\|(x_m, y_m)\| < r$.

To show that $(3.2)^m$ has a solution for $m \in N^+$, we will deal with the modified boundary value problem

$$(3.3)^m \qquad \begin{cases} \Delta^2 x(i-1) + q_1(i) F_1(i, x(i), y(i)) = 0, \ i \in \{1, 2, \ldots, T\}, \\ \Delta^2 y(i-1) + q_2(i) F_2(i, x(i), y(i)) = 0, \\ x(0) = x(T+1) = y(0) = y(T+1) = \frac{1}{m}, \ m \in N_0, \end{cases}$$

where

$$F_1(i, x, y) = f_1(i, \max\{x, \frac{1}{m}\}, \ \max\{0, y\}), \ F_2(i, x, y) = f_2(i, \max\{x, 0\}, \ \max\{y, \frac{1}{m}\}).$$

Let $\Omega_1 = \Omega_r \times \Omega_r$ where $\Omega_r = \{x \in E_1 : \|x\| < r\}$. Let $A : \bar{\Omega}_1 \to E$ be defined by

$$(3.4) \qquad \begin{aligned} A(x(i), y(i)) =& (\sum_{j=1}^{T} G(i,j) q_1(j) F_1(j, x(j), y(j)) + \frac{1}{m}, \\ & \sum_{j=1}^{T} G(i,j) q_2(j) F_2(j, x(j), y(j)) + \frac{1}{m}). \end{aligned}$$

From the definition of A, we know that

$$(3.5) \qquad \begin{cases} (Ax)(i) = \sum\limits_{j=1}^{T} G(i,j) q_1(j) F_1(j, x(j), y(j)) + \frac{1}{m}, \\ (Ay)(i) = \sum\limits_{j=1}^{T} G(i,j) q_2(j) F_2(j, x(j), y(j)) + \frac{1}{m}, \end{cases}$$

then $A : \bar{\Omega}_1 \to E$ is continuous and completely continuous.

We first show

$$(3.6) \qquad (x, y) \neq \lambda A(x, y) + (1-\lambda)\frac{1}{m} \text{ for } \lambda \in (0, 1), \ (x, y) \in \partial\Omega_1 \cap K.$$

Suppose this is false, namely that there exist a $\lambda \in (0, 1)$ and $(x, y) \in \partial\Omega_1 \cap K$ with $(x, y) = \lambda A(x, y) + (1-\lambda)\frac{1}{m}$. Then we have

$$(3.7) \qquad \begin{cases} x(i) = \lambda (Ax)(i) + (1-\lambda)\frac{1}{m}, \\ y(i) = \lambda (Ay)(i) + (1-\lambda)\frac{1}{m}, \end{cases}$$

that is

$$(3.8) \qquad \begin{cases} -\Delta^2 x(i-1) = \lambda q_1(i) F_1(i, x(i), y(i)), \ i \in \{1, 2, \ldots, T\} \\ -\Delta^2 y(i-1) = \lambda q_2(i) F_2(i, x(i), y(i)), \\ x(0) = x(T+1) = y(0) = y(T+1) = \frac{1}{m}, \ m \in N_0. \end{cases}$$

Since $\|(x,y)\| = \max\{\|x\|, \|y\|\} = r$, without loss of generality, we assume that $\|x\| = r$.

Since $\Delta^2 x(i-1) \le 0$ on N and $x(i) \ge \frac{1}{m}$ on N^+, there exists $i_0 \in N$ with $\Delta x(i) \ge 0$ on $[0, i_0) = \{0, 1, \ldots, i_0 - 1\}$, $\Delta x(i) \le 0$ on $[i_0, T+1) = \{i_0, i_0+1, \ldots, T\}$ and $x(i_0) = \|x\| = r$. Therefore

$$x(i) - \frac{1}{m} \ge \mu(i)\|x - \frac{1}{m}\| \ge \mu(i)\|x - \frac{1}{m}\|,$$

then

$$x(i) \ge \mu(i)\|x\| = \mu(i)r \ge \frac{r}{T+1} > \frac{1}{m_0}.$$

Also notice that

$$F_1(i, x(i), y(i)) = f_1(i, x(i), y(i)) \le g_1(x(i), y(i)) + h_1(x(i), y(i)),\ i \in N,$$

then for $z \in N$, we have

$$-\Delta^2 x(z-1) \le g_1(x(z), y(z))\{1 + \frac{h_1(x(z), y(z))}{g_1(x(z), y(z))}\}q_1(z). \tag{3.9}$$

We sum the inequation (3.9) from $i+1 (i \le i_0)$ to i_0 to obtain

$$\Delta x(i) \le \Delta x(i_0) + \{1 + \frac{h_1(r,r)}{g_1(r,r)}\} \sum_{z=i+1}^{i_0} g_1(x(z), y(z)) q_1(z), \tag{3.10}$$

since $\Delta x(i_0) \le 0$, then we have

$$\Delta x(i) \le \{1 + \frac{h_1(r,r)}{g_1(r,r)}\} g_1(x(i+1), 0) \sum_{z=i+1}^{i_0} q_1(z),\ i < i_0,$$

i.e.,

$$\frac{\Delta x(i)}{g_1(x(i+1), 0)} \le \{1 + \frac{h_1(r,r)}{g_1(r,r)}\} \sum_{z=i+1}^{i_0} q_1(z),\ i < i_0.$$

Since $g_1(x(i+1), 0) \le g_1(u, 0) \le g_1(x(i), 0)$ for $(x(i+1), 0) \ge (u, 0) \ge (x(i), 0)$ when $i < i_0$, then we have

$$\int_{x(i)}^{x(i+1)} \frac{du}{g_1(u,0)} \le \{1 + \frac{h_1(r,r)}{g_1(r,r)}\} \sum_{z=i+1}^{i_0} q_1(z),\ i < i_0, \tag{3.11}$$

and then we sum the above from 0 to $i_0 - 1$ to obtain

$$\int_{\frac{1}{m}}^{r} \frac{du}{g_1(u,0)} \le \{1 + \frac{h_1(r,r)}{g_1(r,r)}\} \sum_{i=0}^{i_0-1} \sum_{z=i+1}^{i_0} q_1(z) = \{1 + \frac{h_1(r,r)}{g_1(r,r)}\} \sum_{i=1}^{i_0} i q_1(i). \tag{3.12}$$

Similarly, if we sum the inequation (3.9) from i_0 to $i (i \ge i_0)$ to obtain

$$-\Delta x(i) \le -\Delta x(i_0 - 1) + \{1 + \frac{h_1(r,r)}{g_1(r,r)}\} \sum_{i=i_0}^{i} g_1(x(z), y(z)) q_1(z),\ i \ge i_0,$$

since $\Delta x(i_0-1)\geq 0$, then we have

$$-\Delta x(i)\leq \{1+\frac{h_1(r,r)}{g_1(r,r)}\}g_1(x(i),0)\sum_{i=i_0}^{i}q_1(z),\ i\geq i_0,$$

i.e.,

$$\frac{-\Delta x(i)}{g_1(x(i),0)}\leq \{1+\frac{h_1(r,r)}{g_1(r,r)}\}\sum_{z=i_0}^{i}q_1(z),\ i\geq i_0,$$

so we have

$$\int_{x(i+1)}^{x(i)}\frac{du}{g_1(u,0)}\leq \{1+\frac{h_1(r,r)}{g_1(r,r)}\}\sum_{z=i_0}^{i}q_1(z),\ i\geq i_0, \tag{3.13}$$

and then we sum the above from i_0 to T to obtain

(3.14)
$$\int_{\frac{1}{m}}^{r}\frac{du}{g_1(u,0)}\leq \{1+\frac{h_1(r,r)}{g_1(r,r)}\}\sum_{i=i_0}^{T}\sum_{z=i_0}^{i}q_1(z)=\{1+\frac{h_1(r,r)}{g_1(r,r)}\}\sum_{i=i_0}^{T}(T+1-i)q_1(i).$$

Now (3.12), (3.14) imply

$$\int_{\epsilon}^{r}\frac{du}{g_1(u,0)}\leq \{1+\frac{h_1(r,r)}{g_1(r,r)}\}b_{10}. \tag{3.15}$$

If we assume that $\|y\|=r$, we have also

$$\int_{\epsilon}^{r}\frac{dv}{g_2(0,v)}\leq \{1+\frac{h_2(r,r)}{g_2(r,r)}\}b_{20}. \tag{3.16}$$

This contradicts (3.1) and consequently (3.6) is true.

Now Lemma 2.1 imply A has a fixed point $(x_m(i),y_m(i))\in\bar{\Omega}_1$, i.e. $\frac{1}{m}\leq\|(x_m,y_m\|<r$ (note if $\|(x_m,y_m)\|=r$, then following essentially the same argument from (3.9)-(3.16) will yield a contradiction). It follows from the fact $(x_m,y_m)\geq(\frac{1}{m},\frac{1}{m})$, we can obtain $(x_m(i),y_m(i))$ is a solution of $(3.2)^m$ too.

Next we obtain a sharper lower bound on x_m, namely we will show that there exist constant $C_k>0$ independent on m $(k=1,2)$, with $x_m(i)\geq C_1\mu(i), y_m(i)\geq C_2\mu(i)$ for $i\in N^+$, where $\mu(i)$ is as in Lemma 2.3.

To see this, notice (H_4) guarantees the existence of a function $\psi_r^{(k)}(i)$ continuous on $N^+$and positive on N with $f_k(i,x,y)\geq\psi_r^{(k)}(i)$ $(k=1,2)$ for $(i,x,y)\in N\times(0,r]^2$.

Since $G(j,j)\geq G(i,j)\geq\mu(i)G(j,j)$, $i,j\in N^+$, then for $i\in N^+$

$$\begin{aligned}x_m(i)&\geq\frac{1}{m}+\sum_{j=1}^{T}G(i,j)q_1(j)\psi_r^{(1)}(j)\\&\geq\frac{1}{m}+\mu(i)\sum_{j=1}^{T}G(j,j)q_1(j)\psi_r^{(1)}(j)\geq\mu(i)C_1\geq\frac{C_1}{T+1},\end{aligned} \tag{3.17}$$

$$
\begin{aligned}
(3.18) \qquad y_m(i) &\geq \frac{1}{m} + \sum_{j=1}^{T} G(i,j)q_2(j)\psi_r^{(2)}(j) \\
&\geq \frac{1}{m} + \mu(i)\sum_{j=1}^{T} G(j,j)q_1(j)\psi_r^{(2)}(j) \geq \mu(i)C_2 \geq \frac{C_2}{T+1},
\end{aligned}
$$

where $C_1 = \sum_{j=1}^{T} G(j,j)q_1(j)\psi_r^{(1)}(j)$, $C_2 = \sum_{j=1}^{T} G(j,j)q_2(j)\psi_r^{(2)}(j)$.

The Arzela-Ascoli theorem guarantees the existence of subsequence $N_1 \subset N_0$, and $x(i) \in C(N^+, [0, +\infty))$ with $x_m \to x$ in $C(N^+, [0, +\infty))$ as $m \to \infty$ through N_1, $y(i) \in C(N^+, [0, +\infty))$ with $y_m \to y$ in $C(N^+, [0, +\infty))$ as $m \to +\infty$ through N_1. Then (x_m, y_m) is converging uniformly on N^+ to (x, y) as $m \to +\infty$, for $m \in N_1$. Also, we have $x(0) = x(T+1) = y(0) = y(T+1) = \lim_{m\to+\infty} \frac{1}{m} = 0$ and $\|(x, y)\| \leq r$ for $i \in N^+$. In particular $x(i) \geq \mu(i)C_1 \geq \frac{C_1}{T+1}$, $y(i) \geq \mu(i)C_2 \geq \frac{C_2}{T+1}$ on N.

Fix $i \in N$ and we obtain $-\Delta^2 x_m(i-1) = \Delta(x_m(i) - x_m(i-1)) = \Delta x_m(i) - \Delta x_m(i-1) = x_m(i+1) - 2x_m(i) + x_m(i-1) \to x(i+1) - 2x(i) + x(i-1) = \Delta^2 x(i-1)$ for $i \in N$, $m \in N_1$, $m \to \infty$, $k = 1, 2$, and $f_k(i, x_m(i), y_m(i)) \to f_k(i, x(i), y(i))$ for $i \in N$, $m \in N_1$, $m \to \infty$, $k = 1, 2$.

Thus

$$
\begin{cases}
\Delta^2 x(i-1) + q_1(i)f_1(i, x(i), y(i)) = 0, \ i \in \{1, 2, \ldots, T\}, \\
\Delta^2 y(i-1) + q_2(i)f_2(i, x(i), y(i)) = 0, \\
x(0) = x(T+1) = y(0) = y(T+1) = 0.
\end{cases}
$$

Finally it is easy to see that $\max\{\|x\|, \|y\|\} < r$ (note if $\|x\| = r$ or $\|y\| = r$, then following essentially the same argument from (3.9)–(3.16) will yield a contradiction again). Thus we have proved that problem (1.1) has one solution $(x(i), y(i)) \in C(N^+, [0, +\infty)^2 \setminus \{O\})$ and $0 \leq \|(x(i), y(i))\| \leq r$.

Theorem 3.2. Let $(H_1) - (H_3)$and $(H_5), (H_6)$ hold, then (1.1) has a solution $(x, y) \in C(N^+, [0, +\infty)^2 \setminus \{O\})$ with $r < \|(x, y)\| \leq R$ on N.

Proof. Choose $\epsilon > 0$, $\epsilon < r$ such that (3.1) hold. Let $m_0 \in \{1, 2, \cdots\}$ be chosen so that $\frac{1}{m_0} < \epsilon, \frac{1}{m_0} < \frac{r}{T+1}$, and let $N_0 = \{m_0, m_0 + 1, \ldots\}$. First we will show that the following boundary value problem

$$
(3.2)^m \qquad
\begin{cases}
\Delta^2 x(i-1) + q_1(i)f_1(i, x(i), y(i)) = 0, \ i \in \{1, 2, \ldots, T\}, \\
\Delta^2 y(i-1) + q_2(i)f_2(i, x(i), y(i)) = 0, \\
x(0) = x(T+1) = y(0) = y(T+1) = \frac{1}{m}, \ m \in N_0,
\end{cases}
$$

has a solution $(x_m(i), y_m(i))$ for each $m \in N_0$ with $(x_m(i), y_m(i)) > (\frac{1}{m}, \frac{1}{m})$ on N and $r < \|(x_m, y_m)\| \leq R$.

To show that $(3.2)^m$ has a solution for each $m \in N_0$, we will deal with the mondified boundary value problem

$$(3.3)^m \qquad \begin{cases} \Delta^2 x(i-1) + q_1(i)F_1(i, x(i), y(i)) = 0, \ i \in \{1, 2, \dots, T\}, \\ \Delta^2 y(i-1) + q_2(i)F_2(i, x(i), y(i)) = 0, \\ x(0) = x(T+1) = y(0) = y(T+1) = \frac{1}{m}, \ m \in N_0, \end{cases}$$

where F_1 and F_2 are as in Theorem 3.1.

Fixed $m \in N_0$, Let $A : K \to E$ be defined by

$$(3.19) \qquad \begin{aligned} A(x(i), y(i)) = (&\sum_{j=1}^{T} G(i,j)q_1(j)F_1(j, x(j), y(j)) + \frac{1}{m}, \\ &\sum_{j=1}^{T} G(i,j)q_2(j)F_2(j, x(j), y(j)) + \frac{1}{m}), \end{aligned}$$

then $A : K \to E$ is continuous and completely continuous.

Moreover, we have

$$\begin{cases} \Delta^2 (Ax)(i-1) + q_1(i)F_1(i, x(i), y(i)) = 0, \ i \in \{1, 2, \dots, T\}, \\ \Delta^2 (Ay)(i-1) + q_2(i)F_2(i, x(i), y(i)) = 0, \\ (Ax)(0) = (Ax)(T+1) = (Ay)(0) = (Ay)(T+1) = \frac{1}{m}, \ m \in N_0. \end{cases}$$

This imples that $\Delta^2(Ax)(i-1) \le 0$, $\Delta^2(Ay)(i-1) \le 0$, $i \in N$, and $(Ax)(i) \ge \frac{1}{m}$, $(Ay)(i) \ge \frac{1}{m}$.

Consequently, we have (from Lemma 2.3)

$$(Ax)(i) - \frac{1}{m} \ge \mu(i)\|Ax - \frac{1}{m}\|,$$

thus

$$(Ax)(i) \ge \frac{1}{m} + \mu(i)(\|Ax\| - \frac{1}{m}) \ge \mu(i)\|Ax\|, \ i \in N^+.$$

Similarly, we have

$$(Ay)(i) \ge \mu(i)\|Ay\|, \ i \in N^+$$

and so $A : K \to K$. Let $\Omega_2 = \Omega_R \times \Omega_R$ where $\Omega_R = \{x \in E_1 : \|x\| < R\}$.

We first show

$$(3.20) \qquad (x, y) \ne \lambda A(x, y) \text{ for } \lambda \in (0, 1), \ (x, y) \in \partial\Omega_1 \cap K,$$

where Ω_1 is defined above. Suppose this is false, namely that there exist a $\lambda \in [0, 1)$ and $(x, y) \in \partial\Omega_1 \cap K$ with $(x, y) = \lambda A(x, y)$. Then we have

$$(3.21) \qquad \begin{cases} x(i) = \lambda(Ax)(i), \\ y(i) = \lambda(Ay)(i), \end{cases}$$

that is

$$\left\{\begin{array}{l} -\Delta^2 x(i-1) = \lambda q_1(i) F_1(i, x(i), y(i)),\ i \in N, \\ -\Delta^2 y(i-1) = \lambda q_2(i) F_2(i, x(i), y(i)), \\ x(0) = x(T+1) = y(0) = y(T+1) = \frac{\lambda}{m},\ m \in N_0. \end{array}\right. \tag{3.22}$$

Since $\|(x,y)\| = \max\{\|x\|, \|y\|\} = r$, without loss of generality, we assume that $\|x\| = r$. Since $\Delta^2 x(i-1) \le 0$ on N and $x(i) \ge \frac{\lambda}{m}$ on N^+, there exists $i_0 \in N$ with $\Delta x(i) \ge 0$ on $[0, i_0) = \{0, 1, \dots, i_0 - 1\}$, $\Delta x(i) \le 0$ on $[i_0, T+1) = \{i_0, i_0+1, \dots, T\}$ and $x(i_0) = \|x\| = r$.

Therefore

$$x(i) - \frac{\lambda}{m} \ge \mu(i)\|x - \frac{\lambda}{m}\| \ge \mu(i)\|x - \frac{\lambda}{m}\|,$$

then

$$x(i) \ge \mu(i)\|x\| = \mu(i) r \ge \frac{r}{T+1} > \frac{1}{m_0}.$$

Also notice that

$$F_k(i, x(i), y(i)) \le g_k(x(i), y(i)) + h_k(x(i), y(i)),\ \text{for } i \in N,\ k = 1, 2.$$

then for $z \in N$, we have

$$-\Delta^2 x(z-1) \le g_1(x(z), y(z))\{1 + \frac{h_1(x(z), y(z))}{g_1(x(z), y(z))}\} q_1(z), \tag{3.23}$$

we sum the inequation (3.23) from $i+1 (i \le i_0)$ to i_0 to obtain

$$\Delta x(i) \le \Delta x(i_0) + \{1 + \frac{h_1(r,r)}{g_1(r,r)}\} \sum_{z=i+1}^{i_0} g_1(x(z), y(z)) q_1(z). \tag{3.24}$$

Since $\Delta x(i_0) \le 0$, then we have

$$\Delta x(i) \le \{1 + \frac{h_1(r,r)}{g_1(r,r)}\} g_1(x(i+1), 0) \sum_{z=i+1}^{i_0} q_1(z),\ i < i_0,$$

i.e.

$$\frac{\Delta x(i)}{g_1(x(i+1), 0)} \le \{1 + \frac{h_1(r,r)}{g_1(r,r)}\} \sum_{z=i+1}^{i_0} q_1(z),\ i < i_0,$$

since $g_1(x(i+1), 0) \le g_1(u, 0) \le g_1(x(i), 0)$ for $(x(i+1), 0) \ge (u, 0) \ge (x(i), 0)$ when $i < i_0$, then we have

$$\int_{x(i)}^{x(i+1)} \frac{du}{g_1(u,0)} \le \{1 + \frac{h_1(r,r)}{g_1(r,r)}\} \sum_{z=i+1}^{i_0} q_1(z),\ i < i_0 \tag{3.25}$$

and then we sum the above from 0 to $i_0 - 1$ to obtain

$$\textstyle\int_{\frac{\lambda}{m}}^{r} \frac{du}{g_1(u,0)} \le \{1 + \frac{h_1(r,r)}{g_1(r,r)}\} \sum_{i=0}^{i_0-1} \sum_{z=i+1}^{i_0} q_1(z) = \{1 + \frac{h_1(r,r)}{g_1(r,r)}\} \sum_{i=1}^{i_0} i q_1(i). \tag{3.26}$$

Similarly, if we sum the inequation (3.23) from i_0 to $i (i \geq i_0)$ to obtain

$$-\Delta x(i) \leq -\Delta x(i_0 - 1) + \{1 + \frac{h_1(r,r)}{g_1(r,r)}\} \sum_{i=i_0}^{i} g_1(x(z), y(z)) q_1(z), \ i \geq i_0,$$

since $\Delta x(i_0 - 1) \geq 0$, then we have

$$-\Delta x(i) \leq \{1 + \frac{h_1(r,r)}{g_1(r,r)}\} g_1(x(i), 0) \sum_{i=i_0}^{i} q_1(z), \ i \geq i_0,$$

i.e.,

$$\frac{-\Delta x(i)}{g_1(x(i), 0)} \leq \{1 + \frac{h_1(r,r)}{g_1(r,r)}\} \sum_{z=i_0}^{i} q_1(z), \ i \geq i_0,$$

so we have

$$\int_{x(i+1)}^{x(i)} \frac{du}{g_1(u, 0)} \leq \{1 + \frac{h_1(r,r)}{g_1(r,r)}\} \sum_{z=i_0}^{i} q_1(z), \ i \geq i_0, \tag{3.27}$$

and then we sum the above from i_0 to T to obtain

$$\int_{\frac{\lambda}{m}}^{r} \frac{du}{g_1(u, 0)} \leq \{1 + \frac{h_1(r,r)}{g_1(r,r)}\} \sum_{i=i_0}^{T} \sum_{z=i_0}^{i} q_1(z) = \{1 + \frac{h_1(r,r)}{g_1(r,r)}\} \sum_{i=i_0}^{T} (T + 1 - i) q_1(i). \tag{3.28}$$

Now (3.26), (3.28) imply

$$\int_{\epsilon}^{r} \frac{du}{g_1(u, 0)} \leq \{1 + \frac{h_1(r,r)}{g_1(r,r)}\} b_{10}. \tag{3.29}$$

If we assume that $\|y\| = r$, we have also

$$\int_{\epsilon}^{r} \frac{dv}{g_2(0, v)} \leq \{1 + \frac{h_2(r,r)}{g_2(r,r)}\} b_{20}. \tag{3.30}$$

This contradicts (3.1) and consequently (3.20) is true.

Next we will show

$$\|A(x,y)\| \geq \|(x,y)\| \text{ for } (x,y) \in \partial\Omega_2 \cap K. \tag{3.31}$$

To see this let $(x,y) \in \partial\Omega_2 \bigcap K$ such that $\|(x,y)\| = R$. Without loss of generality, we assume that $\|x\| = R$. Also since $(x,y) \in \partial\Omega_2 \bigcap K$ then

$$x(i) \geq \mu(i)\|x(i)\| = \mu(i)R \geq \frac{R}{T+1} > \frac{1}{m_0},$$
$$y(i) \geq \mu(i)\|y(i)\| = \mu(i)R \geq \frac{R}{T+1} > \frac{1}{m_0} \ \forall \ i \in N.$$

Thus

$$F_k(i, x(i), y(i)) = f_k(i, x(i), y(i)) \geq \bar{g}_k(x(i), y(i)) + \bar{h}_k(x(i), y(i)) \ \forall \ i \in N, \ k = 1, 2,$$

so we have (from (H_5))

$$\begin{aligned}
(Ax)(\sigma) &= \sum_{j=1}^{T} G(\sigma,j)q_1(j)F_1(j,x(j),y(j)) + \frac{\lambda}{m} \\
&\geq \sum_{j=1}^{T} G(\sigma,j)q_1(j)\bar{g}_1(x(j),y(j))\{1+\frac{\bar{h}_1(x(j),y(j))}{\bar{g}_1(x(j),y(j))}\} \\
&\geq \bar{g}_1(R,R)\{1+\frac{\bar{h}_1(\frac{R}{T+1},\ 0)}{\bar{g}_1(\frac{R}{T+1},\ 0)}\}\sum_{j=1}^{T} G(\sigma,j)q_1(j)\ > R = \|x\|,
\end{aligned}$$

if we assume that $\|y\| = R$, we have also $(Ay)(\sigma) > R = \|y\|$. Thus $\|(A(x,y))\| > R = \|(x,y)\|$ for $(x,y) \in \partial\Omega_2 \bigcap K$.

Now Lemma 2.2 implies A has a fixed point $(x_m(i),y_m(i)) \in K \cap (\bar{\Omega}_2 \setminus \Omega_1)$, i.e., $r < \|(x_m(i),y_m(i))\| \leq R$. Clearly, $\|(x_m(i),y_m(i))\| \neq r$.

Consequently $(3.3)^m$ has solution $(x_m(i),y_m(i))$ with

$$(x_m(i),y_m(i)) \geq (\frac{1}{m},\frac{1}{m}) \text{ for } i \in N^+ \text{ and } r < \|(x_m,y_m)\| \leq R, \tag{3.32}$$

which shows that $(3.2)^m$ has a positive solution $(x_m(i),y_m(i))$.

By the same way above, $(x_m(i),y_m(i))$ have subsequences $N_1 of N_0$, with $(x_m(i), y_m(i))$ converging uniformly on N^+ to $(x(i),y(i))$ as $m \to +\infty$ through N_1.

Also, $(x(0),\ y(0)) = (x(T+1),\ y(T+1)) = (0,0)$. It is easy to show that $(x(i),y(i)) \in C(N^+,[0,+\infty)^2 \setminus \{O\})$ is apositive solution of (1.1) and $r < \|(x,y)\| \leq R$. Thus the proof Theorem 3.2 is complete.

Theorem 3.3. Let (H_1)–(H_6) hold, then (1.1) have two positive solutions $(x_k(i),y_k(i)) \in C(N^+,[0,+\infty)^2 \setminus \{O\})$ with $(x_k(i),y_k(i)) > (0,0)$ for each $i \in N$, $k = 1,2$, and $0 < \|(x_1(i),y_1(i))\| < r < \|(x_2(i),y_2(i))\| \leq R$.

Proof. The existence of (x_1,y_1) follows from Theorem 3.1 and the existence of (x_2,y_2) follows from Theorem 3.2.

4. AN EXAMPLE

Example 4.1. Consider the singular discrete boundary value problem

$$\begin{cases}
\Delta^2 x(i-1) + \delta[(\sqrt{x^2(i)+y^2(i)})^{-\alpha} + \gamma(\sqrt{x^2(i)+y^2(i)})^{\beta}] = 0, \ i \in N \\
\Delta^2 y(i-1) + \delta[(\sqrt{x^2(i)+y^2(i)})^{-\alpha} + \gamma(\sqrt{x^2(i)+y^2(i)})^{\beta}] = 0, \\
x(0) = x(T+1) = y(0) = y(T+1) = 0,
\end{cases} \tag{4.1}$$

with $\alpha > 0$, $\beta > 1$, $\gamma = (\frac{1}{\sqrt{2}})^{\alpha+\beta}$ is such that

$$\delta < [\frac{2}{T(T+1)(\alpha+1)}] \sup_{c\in(0,+\infty)} (\frac{c^{\alpha+1}}{1+c^{\alpha+\beta}}) \tag{4.2}$$

hold, then (4.1) have two positive solutions $(x_k(i), y_k(i))$ $(k = 1, 2)$ with

$$0 < \|(x_1(i), y_1(i))\| < r < \|(x_2(i), y_2(i))\| \le R \ \forall\ i \in N.$$

To see this, we will apply Theorem 3.3 with $q_k(i) = \delta$,

$$g_k(x(i), y(i)) = \bar{g}_k(x(i), y(i)) = (\sqrt{x^2(i) + y^2(i)})^{-\alpha},$$

$$h_k(x(i), y(i)) = \bar{h}_k(x(i), y(i)) = \gamma(\sqrt{x^2(i) + y^2(i)})^{\beta}\ (k = 1, 2).$$

Clearly (H_1), (H_2), (H_4) and (H_5) hold. Also note,

$$b_{k0} = \max\{\sum_{i=1}^{i_0} i\delta, \sum_{i=i_0}^{T}(T + 1 - i)\delta\} = \frac{T(T + 1)}{2}\delta,\ \ k = 1, 2,$$

$$\frac{1}{\{1 + \frac{h_1(r,r)}{g_1(r,r)}\}}\int_0^r \frac{du}{g_1(u, 0)} = \frac{1}{\{1 + \frac{h_2(r,r)}{g_2(r,r)}\}}\int_0^r \frac{dv}{g_2(0, v)} = \frac{1}{\alpha + 1}\frac{r^{\alpha+1}}{1 + r^{\alpha+\beta}}.$$

Since (4.2) implies there exists $r > 0$ such that

$$\delta < [\frac{2}{T(T + 1)(\alpha + 1)}](\frac{r^{\alpha+1}}{1 + r^{\alpha+\beta}}).$$

Consequently (H_3) holds.

Finally, notice that (since $\beta > 1$)

$$\lim_{R\to+\infty}\frac{R}{\bar{g}_1(R, R)(1 + \frac{\bar{h}_1(\frac{R}{T+1},\ 0)}{\bar{g}_1(\frac{R}{T+1},\ 0)})} = \lim_{R\to+\infty}\frac{(\sqrt{2})^{\alpha}R^{\alpha+1}}{1 + (\frac{R}{\sqrt{2}(T+1)})^{\alpha+\beta}} = 0,$$

$$\lim_{R\to+\infty}\frac{R}{\bar{g}_2(R, R)(1 + \frac{\bar{h}_2(0, \frac{R}{T+1})}{\bar{g}_2(0, \frac{R}{T+1})})} = \lim_{R\to+\infty}\frac{(\sqrt{2})^{\alpha}R^{\alpha+1}}{1 + (\frac{R}{\sqrt{2}(T+1)})^{\alpha+\beta}} = 0,$$

so there exists $R > r$ with (H_6) holding.

Thus all the conditions of Theorem 3.3 are satisfied so existence is guaranteed.

Remark 4.1: If $\beta < 1$, since $\sup_{c\in(0,+\infty)}(\frac{c^{\alpha+1}}{1+c^{\alpha+\beta}}) = \infty$, then 4.2 is automatically satisfied.

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