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STABILITY OF A POSITIVE POINT OF EQUILIBRIUM OF ONE NONLINEAR SYSTEM WITH AFTEREFFECT AND STOCHASTIC PERTURBATIONS

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ABSTRACT. The aim of the paper is to show one useful way for stability investigation of the positive point of equilibrium of some nonlinear system with aftereffect and stochastic perturbations. Obtained results are applied for stability investigation of some mathematical predator-prey models.

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1. INTRODUCTION

The study of realistic mathematical models in ecology, especially the study of relations between species and their environment has become a very popular topic that interested both mathematicians and biologists. Investigations on various population models reflect their use in helping to understand the dynamic processes involved in such areas as predator-prey and competition, renewable resource management, evolution of pesticide resistant strains, ecological control of pests, multispecies societies, plant-herbivore systems, and so on. Well known Lotka-Volterra predator-prey mathematical model after its appearance got specially wide development in many different directions, in particular, for systems with delays and stochastic perturbations [1, 3, 10, 11, 12, 16, 17, 18, 27, 28, 33, 34, 35, 36].

In this paper some general nonlinear mathematical model is considered that is destined to unify different known models, in particular, the models type of predator-prey. The following method for stability investigation of the positive point of equilibrium is proposed. The system under consideration is exposed to stochastic perturbations and is linearized in the neighborhood of the positive point of equilibrium. Asymptotic mean square stability conditions are obtained for the constructed linear system. In the case if the order of nonlinearity more than 1 these conditions are sufficient ones [2, 30, 31, 32] for stability in probability of the initial nonlinear system by stochastic perturbations.

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This method can be successfully used for stability investigation of many other types of known biological systems: stage-structure predator-prey models [9, 15, 37], SIR epidemic models [2, 6], chemostat models [4, 5] and other [7, 8, 14, 25, 26].

2. SYSTEM UNDER CONSIDERATION INTRODUCTION

Consider the system of two nonlinear differential equations

$$\begin{aligned} \dot{x}_1(t) &= x_1(t)\,(a - F_0(x_{1t}, x_{2t})) - F_1(x_{1t}, x_{2t}), \\ \dot{x}_2(t) &= -x_2(t)\,(b + G_0(x_{1t}, x_{2t})) + G_1(x_{1t}, x_{2t}), \\ x_i(s) &= \phi_i(s), \quad s \le 0, \quad i = 1, 2. \end{aligned} \tag{2.1}$$

Here $x_i(t)$, $i = 1, 2$, is a value of process x_i in the point of time t and $x_{it} = x_i(t + s)$, $s \le 0$, is a trajectory of the process x_i to the point of t.

Put, for example,

$$\begin{aligned} F_0(x_{1t}, x_{2t}) &= \int_0^\infty f_0(x_1(t-s))dK_0(s), \\ F_1(x_{1t}, x_{2t}) &= \prod_{i=1}^2 \int_0^\infty f_i(x_i(t-s))dK_i(s), \\ G_0(x_{1t}, x_{2t}) &= \int_0^\infty g_0(x_1(t-s))dR_0(s), \\ G_1(x_{1t}, x_{2t}) &= \prod_{i=1}^2 \int_0^\infty g_i(x_i(t-s))dR_i(s), \end{aligned} \tag{2.2}$$

where $K_i(s)$ and $R_i(s)$, $i = 0, 1, 2$, are nondecreasing functions, such that

$$\begin{aligned} K_i &= \int_0^\infty dK_i(s) < \infty, \qquad R_i = \int_0^\infty dR_i(s) < \infty, \\ \hat{K}_i &= \int_0^\infty s dK_i(s) < \infty, \qquad \hat{R}_i = \int_0^\infty s dR_i(s) < \infty, \end{aligned} \tag{2.3}$$

and all integrals are understanding in Stieltjes sense. In this case system (2.1) takes the form

$$\begin{aligned} \dot{x}_1(t) &= x_1(t)\left(a - \int_0^\infty f_0(x_1(t-s))dK_0(s)\right) - \prod_{i=1}^2 \int_0^\infty f_i(x_i(t-s))dK_i(s), \\ \dot{x}_2(t) &= -x_2(t)\left(b + \int_0^\infty g_0(x_1(t-s))dR_0(s)\right) + \prod_{i=1}^2 \int_0^\infty g_i(x_i(t-s))dR_i(s). \end{aligned} \tag{2.4}$$

Systems type of (2.1) are investigated in some biological problems. Put here, for example,

$$\begin{aligned} & f_0(x) = f_1(x) = f_2(x) = g_1(x) = g_2(x) = x, \\ & g_0(x) = 0, \quad dK_1(s) = \delta(s)ds, \quad dR_0(s) = 0, \end{aligned} \tag{2.5}$$

($\delta(s)$ is Dirac's function). If a and b are positive constants, $x_1(t)$ and $x_2(t)$ are respectively the densities of prey and predator populations then (2.4) is transformed to the mathematical predator-prey model [32]

$$\begin{aligned} \dot{x}_1(t) &= x_1(t)\left(a - \int_0^\infty x_1(t-s)dK_0(s) - \int_0^\infty x_2(t-s)dK_2(s)\right), \\ \dot{x}_2(t) &= -bx_2(t) + \int_0^\infty x_1(t-s)dR_1(s) \int_0^\infty x_2(t-s)dR_2(s). \end{aligned} \tag{2.6}$$

Putting in (2.6)

$$\begin{aligned} & dK_0(s) = a_1\delta(s)ds, \quad dK_2(s) = a_2\delta(s)ds, \\ & dR_i(s) = b_i\delta(s - h_i)ds, \quad i = 1, 2, \end{aligned} \tag{2.7}$$

we obtain the known predator-prey mathematical model with delays

$$\begin{aligned} \dot{x}_1(t) &= x_1(t)(a - a_1x_1(t) - a_2x_2(t)), \\ \dot{x}_2(t) &= -bx_2(t) + b_1b_2x_1(t-h_1)x_2(t-h_2). \end{aligned} \tag{2.8}$$

If here $h_1 = h_2 = 0$ we have the classical Lotka-Volterra model

$$\begin{aligned} \dot{x}_1(t) &= x_1(t)(a - a_1x_1(t) - a_2x_2(t)), \\ \dot{x}_2(t) &= x_2(t)(-b + b_1b_2x_1(t)). \end{aligned} \tag{2.9}$$

Many authors [1, 3, 9, 11, 12, 34, 35] consider ratio-dependent predator-prey models with delays type of

$$\begin{aligned} \dot{x}_1(t) &= x_1(t)\left(a - \int_0^\infty x_1(t-s)dK_0(s)\right) - \int_0^\infty \tfrac{x_1^k(t-s)x_2(t)}{x_1^k(t-s)+a_2x_2^k(t-s)}dK_1(s), \\ \dot{x}_2(t) &= -bx_2(t) + \int_0^\infty \tfrac{x_1^m(t-s)x_2(t)}{x_1^m(t-s)+b_2x_2^m(t-s)}dR_1(s). \end{aligned} \tag{2.10}$$

Here it is supposed that m and k are positive constants.

System (2.10) follows from (2.1) if

$$\begin{aligned} &F_0(x_{1t}, x_{2t}) = \int_0^\infty x_1(t-s)dK_0(s), \\ &F_1(x_{1t}, x_{2t}) = \int_0^\infty f(x_1(t-s), x_2(t-s))x_2(t)dK_1(s), \\ &G_1(x_{1t}, x_{2t}) = \int_0^\infty g(x_1(t-s), x_2(t-s))x_2(t)dR_1(s), \\ &f(x_1, x_2) = \tfrac{x_1^k}{x_1^k+a_2x_2^k}, \qquad g(x_1, x_2) = \tfrac{x_1^m}{x_1^m+b_2x_2^m}. \end{aligned} \tag{2.11}$$

Putting in (2.10), for example,

$$\begin{aligned} &dK_0(s) = a_0\delta(s)ds, \qquad dK_1(s) = a_1\delta(s)ds, \qquad k = 1, \\ &dR_1(s) = b_1\delta(s-h)ds, \qquad m = 1, \end{aligned} \tag{2.12}$$

we obtain the system

$$\begin{aligned} \dot{x}_1(t) &= x_1(t)\left(a - a_0x_1(t) - \tfrac{a_1x_2(t)}{x_1(t)+a_2x_2(t)}\right), \\ \dot{x}_2(t) &= x_2(t)\left(-b + \tfrac{b_1x_1(t-h)}{x_1(t-h)+b_2x_2(t-h)}\right), \end{aligned} \tag{2.13}$$

that was considered in [3, 9].

3. POSITIVE POINT OF EQUILIBRIUM, STOCHASTIC PERTURBATIONS, CENTERING AND LINEARIZATION

3.1. Let in system (2.1) $F_i = F_i(\phi, \psi)$ and $G_i = G_i(\phi, \psi)$, $i = 0, 1$, be functionals defined on $H \times H$, where H be a set of functions $\phi = \phi(s)$, $s \le 0$, with the norm $\|\phi\| = \sup_{s\le 0} |\phi(s)|$, the functionals F_i and G_i be nonnegative ones for nonnegative functions ϕ and ψ. Let us suppose also that system (2.1) has a positive point (x_1^*, x_2^*) of equilibrium. This point is obtained from the conditions $\dot{x}_1(t) \equiv 0$, $\dot{x}_2(t) \equiv 0$ and is defined by the system of algebraic equations

$$\begin{aligned} x_1^*(a - F_0(x_1^*, x_2^*)) &= F_1(x_1^*, x_2^*), \\ x_2^*(b + G_0(x_1^*, x_2^*)) &= G_1(x_1^*, x_2^*). \end{aligned} \tag{3.1}$$

Note that system (3.1) has a positive solution only by the condition

$$a > F_0(x_1^*, x_2^*). \tag{3.2}$$

For example, for system (2.4) a positive point of equilibrium is defined by the equations

$$\begin{aligned} x_1^*(a - K_0 f_0(x_1^*)) &= K_1 K_2 f_1(x_1^*) f_2(x_2^*), \\ x_2^*(b + R_0 g_0(x_1^*)) &= R_1 R_2 g_1(x_1^*) g_2(x_2^*), \end{aligned} \tag{3.3}$$

if $a > K_0 f_0(x_1^*)$. In particular, in the case $f_0(x) = f_1(x) = f_2(x) = g_0(x) = g_1(x) = g_2(x) = x$ system (3.3) has the positive solution

$$x_1^* = \frac{b}{R_1R_2 - R_0}, \qquad x_2^* = \frac{a - K_0 x_1^*}{K_1K_2} = \frac{a - K_0(R_1R_2 - R_0)^{-1} b}{K_1 K_2}, \tag{3.4}$$

by the condition

$$a > \frac{K_0 b}{R_1 R_2 - R_0} > 0.$$

For system (2.10) the positive point of equilibrium is defined as follows

$$x_1^* = \frac{A}{K_0}, \quad x_2^* = \frac{A}{BK_0}, \quad A = a - \frac{K_1}{B + a_2 B^{1-k}} > 0, \quad B = \left(\frac{b_2 b}{R_1 - b}\right)^{\frac{1}{m}} > 0.$$

In particular, for system (2.13) it is

$$x_1^* = \frac{A}{a_0}, \qquad x_2^* = \frac{A}{Ba_0}, \qquad A = a - \frac{a_1}{B + a_2} > 0, \qquad B = \frac{bb_2}{b_1 - b} > 0. \tag{3.5}$$

3.2. As it was proposed in [2, 32] and used later in [1, 8] let us assume that system (2.1) is exposed to stochastic perturbations, which are of white noise type, are directly proportional to the deviations of $x_1(t)$ and $x_2(t)$ from the values of x_1^*, x_2^* and influence on $\dot{x}_1(t)$, $\dot{x}_2(t)$ respectively. In this way system (2.1) is transformed to the form

$$\begin{aligned} \dot{x}_1(t) &= x_1(t)\,(a - F_0(x_{1t}, x_{2t})) - F_1(x_{1t}, x_{2t}) + \sigma_1(x_1(t) - x_1^*)\dot{w}_1(t), \\ \dot{x}_2(t) &= -x_2(t)\,(b + G_0(x_{1t}, x_{2t})) + G_1(x_{1t}, x_{2t}) + \sigma_2(x_2(t) - x_2^*)\dot{w}_2(t). \end{aligned} \tag{3.6}$$

Here σ_1, σ_2 are constants, w_1, w_2 are independent of each other Wiener processes [13].

3.3. Centering system (3.6) on the positive point of equilibrium via new variables $y_1 = x_1 - x_1^*$, $y_2 = x_2 - x_2^*$, we obtain

$$\begin{aligned} \dot{y}_1(t) &= (y_1(t) + x_1^*)\,(a - F_0(y_{1t} + x_1^*, y_{2t} + x_2^*)) \\ &\quad - F_1(y_{1t} + x_1^*, y_{2t} + x_2^*) + \sigma_1 y_1(t)\dot{w}_1(t), \\ \dot{y}_2(t) &= -(y_2(t) + x_2^*)\,(b + G_0(y_{1t} + x_1^*, y_{2t} + x_2^*)) \\ &\quad + G_1(y_{1t} + x_1^*, y_{2t} + x_2^*) + \sigma_2 y_2(t)\dot{w}_2(t). \end{aligned} \tag{3.7}$$

It is clear that stability of system (3.6) equilibrium (x_1^*, x_2^*) is equivalent to stability of the trivial solution of system (3.7).

3.4. Along with system (3.6) we will consider the linear part of this system. Let us suppose that the functions $f_i(x)$, $g_i(x)$, $i=0,1,2$, in system (2.4) are differentiable ones. Using for all these functions the representation

$$f(z+x^*)=f_0+f_1z+o(z), \qquad f_0=f(x^*), \qquad f_1=\frac{df}{dx}(x^*),$$

and neglecting by $o(z)$, via (2.2), (3.7)) we obtain the linear part (process $(z_1(t),z_2(t))$) of system (2.4) after adding stochastic perturbations and centering

$$\begin{aligned}\dot z_1(t) &= (a-K_0f_{00})z_1(t)-\int_0^\infty z_1(t-s)dK(s)\\ &\quad -K_1f_{10}f_{21}\int_0^\infty z_2(t-s)dK_2(s)+\sigma_1z_1(t)\dot w_1(t),\\ \dot z_2(t) &= -(b+R_0g_{00})z_2(t)+\int_0^\infty z_1(t-s)dR(s)\\ &\quad +R_1g_{10}g_{21}\int_0^\infty z_2(t-s)dR_2(s)+\sigma_2z_2(t)\dot w_2(t),\end{aligned} \tag{3.8}$$

where

$$\begin{aligned}dK(s)&=K_2f_{20}f_{11}dK_1(s)+f_{01}x_1^*dK_0(s),\\ dR(s)&=R_2g_{20}g_{11}dR_1(s)-g_{01}x_2^*dR_0(s).\end{aligned} \tag{3.9}$$

Below we will say about system (3.8) also as about the linear part appropriate to system (2.4) or for brevity as about the linear part of system (2.4).

In particular, by conditions (2.5), (3.3) from (3.8), (3.9) we obtain the linear part of system (2.6)

$$\begin{aligned}\dot z_1(t) &= -x_1^*\left(\int_0^\infty z_1(t-s)dK_0(s)+\int_0^\infty z_2(t-s)dK_2(s)\right)\\ &\quad +\sigma_1z_1(t)\dot w_1(t),\\ \dot z_2(t) &= -bz_2(t)+R_2x_2^*\int_0^\infty z_1(t-s)dR_1(s)\\ &\quad +R_1x_1^*\int_0^\infty z_2(t-s)dR_2(s)+\sigma_2z_2(t)\dot w_2(t).\end{aligned} \tag{3.10}$$

Via (2.7) from (3.10) we have the linear part of system (2.8)

$$\begin{aligned}\dot z_1(t)&=-x_1^*(a_1z_1(t)+a_2z_2(t))+\sigma_1z_1(t)\dot w_1(t),\\ \dot z_2(t)&=-bz_2(t)+b_1b_2(x_2^*z_1(t-h_1)+x_1^*z_2(t-h_2))+\sigma_2z_2(t)\dot w_2(t).\end{aligned} \tag{3.11}$$

4. AUXILIARY STATEMENTS

We will use two definitions of stability.

Definition 4.1. The trivial solution of system (3.7) is called stable in probability if for any $\epsilon_1>0$ and $\epsilon_2>0$ there exists $\delta>0$ such that the solution $y(t)=y(t,\phi)$, where $y=(y_1,y_2)$, $\phi=(\phi_1,\phi_2)$, satisfies $\mathbf{P}\{\sup_{t\geq 0}|y(t,\phi)|>\epsilon_1\}<\epsilon_2$ for any initial function $\phi\in H$ satisfying $\mathbf{P}\{\|\phi\|\leq\delta\}=1$.

Definition 4.2. The trivial solution of system (3.8) is called mean square stable if for any $\epsilon > 0$ there exists $\delta > 0$ such that the solution $z(t) = z(t,\phi)$, where $z = (z_1, z_2)$, $\phi = (\phi_1, \phi_2)$, satisfies $\mathbf{E}|z(t,\phi)|^2 < \epsilon$ for any initial function $\phi \in H$ such that $\sup_{s\le 0} \mathbf{E}|\phi(s)|^2 < \delta$. If besides $\lim_{t\to\infty} \mathbf{E}|z(t,\phi)|^2 = 0$ for any initial function $\phi \in H$ then the trivial solution of equation (3.8) is called asymptotically mean square stable.

As it is shown in [28, 29] if the order of nonlinearity of the system under consideration is more than 1 then a sufficient condition for asymptotic mean square stability of the linear part of the initial nonlinear system is also a sufficient condition for stability in probability of the initial system. So, in this paper we will obtain sufficient conditions for asymptotic mean square stability of the linear part of considered nonlinear systems.

We will use also the following auxiliary statements.

Theorem 4.3. [19] *Let there exists a functional $V(t,\varphi)$, $t \ge 0$, $\varphi \in H$, such that*

$$c_1\mathbf{E}|\varphi(0)|^2 \le \mathbf{E}V(t,\varphi) \le c_2 \sup_{s\le 0} \mathbf{E}|\varphi(s)|^2,$$

$$\mathbf{E}LV(t,\varphi) \le -c_3\mathbf{E}|\varphi(0)|^2,$$

where L is the generator [13] *of equation* (3.7)*, $c_i > 0$, $i = 1,2,3$. Then the trivial solution of system* (3.7) *is asymptotically mean square stable.*

Consider system of stochastic differential equations without delays

$$\begin{aligned} \dot u_1(t) &= \alpha_1 u_1(t) + \alpha_2 u_2(t) + \sigma_1 u_1(t)\dot w_1(t),\\ \dot u_2(t) &= \beta_1 u_1(t) + \beta_2 u_2(t) + \sigma_2 u_2(t)\dot w_2(t), \end{aligned} \tag{4.1}$$

and put

$$\epsilon_i = \frac{1}{2}\sigma_i^2, \qquad i = 1,2. \tag{4.2}$$

Lemma 4.4. *Let there exist numbers μ and γ satisfying the conditions*

$$\gamma > \mu^2, \qquad \alpha_1 + \mu\beta_1 + \epsilon_1 < 0, \qquad \mu\alpha_2 + \gamma(\beta_2 + \epsilon_2) < 0, \tag{4.3}$$

$$4(\alpha_1 + \mu\beta_1 + \epsilon_1)(\mu\alpha_2 + \gamma(\beta_2 + \epsilon_2)) > (\mu(\alpha_1 + \beta_2) + \alpha_2 + \gamma\beta_1)^2. \tag{4.4}$$

Then the trivial solution of system (4.1) *is asymptotically mean square stable.*

Proof. Let L_0 be the generator [13] of system (4.1). Using the function

$$v(t) = u_1^2(t) + 2\mu u_1(t)u_2(t) + \gamma u_2^2(t) \tag{4.5}$$

and (4.2) for system (4.1) we have

$$\begin{aligned} L_0 v(t) \;=\; & 2(u_1(t) + \mu u_2(t))(\alpha_1 u_1(t) + \alpha_2 u_2(t))\\ & +2(\mu u_1(t) + \gamma u_2(t))(\beta_1 u_1(t) + \beta_2 u_2(t)) + \sigma_1^2 u_1^2(t) + \gamma\sigma_2^2 u_2^2(t) \end{aligned}$$

$$= 2(\alpha_1 + \mu\beta_1 + \epsilon_1)u_1^2(t) + 2(\mu\alpha_2 + \gamma(\beta_2 + \epsilon_2))u_2^2(t) + 2(\mu(\alpha_1 + \beta_2) + \alpha_2 + \gamma\beta_1)u_1(t)u_2(t). \tag{4.6}$$

By conditions (4.3), (4.4) from (4.5), (4.6) it follows that function (4.5) is a positive definite one and $L_0 v(t)$ is a negative definite one, i.e. function (4.5) satisfies Theorem 4.3. Thus, the trivial solution of system (4.1) is asymptotically mean square stable. Lemma is proven. □

Corollary 4.5. *Suppose that the parameters of system* (4.1) *satisfy the conditions*

$$\beta_1 > 0, \qquad (\alpha_1 + \beta_2)(\beta_2 + \epsilon_2) > \beta_1\alpha_2,$$

$$\frac{\alpha_2(\beta_2 + \epsilon_2)}{(\alpha_1 + \beta_2)(\beta_2 + \epsilon_2) - \beta_1\alpha_2} > \frac{\alpha_1 + \epsilon_1}{\beta_1}, \qquad (\alpha_1 + \beta_2)^2 > 4\beta_1\alpha_2, \tag{4.7}$$

and the intervals

$$\left(-\frac{\alpha_2(\beta_2 + \epsilon_2)}{(\alpha_1 + \beta_2)(\beta_2 + \epsilon_2) - \beta_1\alpha_2}, -\frac{\alpha_1 + \epsilon_1}{\beta_1}\right), \tag{4.8}$$

$$\left(\frac{-\sqrt{(\alpha_1 + \beta_2)^2 - 4\beta_1\alpha_2} - (\alpha_1 + \beta_2)}{2\beta_1}, \frac{\sqrt{(\alpha_1 + \beta_2)^2 - 4\beta_1\alpha_2} - (\alpha_1 + \beta_2)}{2\beta_1}\right) \tag{4.9}$$

have common points. Then the trivial solution of system (4.1) *is asymptotically mean square stable.*

Proof. Choose γ from the condition $\mu(\alpha_1 + \beta_2) + \alpha_2 + \gamma\beta_1 = 0$. To satisfy the conditions (4.3), (4.4) μ has belong to intervals (4.8), (4.9). □

Lemma 4.6. *For positive P_2, x, y and nonnegative P_1, Q_1, Q_2, such that $P_2 > Q_1x + Q_2y$ the following inequality holds*

$$\frac{P_1 + Q_1x^{-1} + Q_2y^{-1}}{P_2 - Q_1x - Q_2y} \geq \left(\frac{\sqrt{(Q_1 + Q_2)^2 + P_1P_2} + Q_1 + Q_2}{P_2}\right)^2.$$

Proof. It is enough to show that the function

$$f(x, y) = \frac{P_1 + Q_1x^{-1} + Q_2y^{-1}}{P_2 - Q_1x - Q_2y}$$

reaches its minimum in the point

$$x_0 = y_0 = \frac{P_2}{\sqrt{(Q_1 + Q_2)^2 + P_1P_2} + Q_1 + Q_2}.$$

Lemma is proven. □

5. STABILITY OF EQUILIBRIUM OF SYSTEM (2.4) WITH STOCHASTIC PERTURBATIONS

Consider system (3.8) as the linear part of system (2.4) with stochastic perturbations. Following the general method of Lyapunov functionals construction (GMLFC) [20, 21, 22, 23, 24, 29] rewrite system (3.8) in the form

$$\begin{aligned}\dot{Z}_1(t) &= \alpha_1 z_1(t) + \alpha_2 z_2(t) + \sigma_1 z_1(t)\dot{w}_1(t),\\ \dot{Z}_2(t) &= \beta_1 z_1(t) + \beta_2 z_2(t) + \sigma_2 z_2(t)\dot{w}_2(t),\end{aligned} \tag{5.1}$$

where

$$\begin{aligned}Z_1(t) &= z_1(t) - \int_0^\infty \int_{t-s}^t z_1(\theta)d\theta dK(s) - K_1 f_{10} f_{21} \int_0^\infty \int_{t-s}^t z_2(\theta)d\theta dK_2(s),\\ Z_2(t) &= z_2(t) + \int_0^\infty \int_{t-s}^t z_1(\theta)d\theta dR(s) + R_1 g_{10} g_{21} \int_0^\infty \int_{t-s}^t z_2(\theta)d\theta dR_2(s),\end{aligned} \tag{5.2}$$

and via (3.3), (3.9)

$$\begin{aligned}&\alpha_1 = a - K - K_0 f_{00} = K_1 K_2 f_{20}\left(\frac{f_{10}}{x_1^*} - f_{11}\right) - K_0 f_{01} x_1^*,\\ &\alpha_2 = -K_1 K_2 f_{10} f_{21}, \qquad \beta_1 = R = R_1 R_2 g_{20} g_{11} - R_0 g_{01} x_2^*,\\ &\beta_2 = R_1 R_2 g_{10} g_{21} - b - R_0 g_{00} = -R_1 R_2 g_{10}\left(\frac{g_{20}}{x_2^*} - g_{21}\right).\end{aligned} \tag{5.3}$$

We will suppose that the corresponding auxiliary system without delays (4.1) with α_i, β_i, $i = 1, 2$, defined by (5.3), is asymptotically mean square stable.

Following GMLFC, we have construct Lyapunov functional V for system (5.1) in the form $V = V_1 + V_2$, where in corresponding with (4.5)

$$V_1(t) = Z_1^2(t) + 2\mu Z_1(t) Z_2(t) + \gamma Z_2^2(t) \tag{5.4}$$

and V_2 has be chosen by some standard way after estimation of LV_1, where L is the generator of system (5.1). Via (5.1)

$$\begin{aligned}LV_1(t) &= 2(Z_1(t) + \mu Z_2(t))(\alpha_1 z_1(t) + \alpha_2 z_2(t))\\ &\quad +2(\mu Z_1(t) + \gamma Z_2(t))(\beta_1 z_1(t) + \beta_2 z_2(t)) + \sigma_1^2 z_1^2(t) + \gamma \sigma_2^2 z_2^2(t)\\ &= 2(\alpha_1 + \mu\beta_1) Z_1(t) z_1(t) + 2(\mu\alpha_1 + \gamma\beta_1) Z_2(t) z_1(t)\\ &\quad +2(\alpha_2 + \mu\beta_2) Z_1(t) z_2(t) + 2(\mu\alpha_2 + \gamma\beta_2) Z_2(t) z_2(t) + \sigma_1^2 z_1^2(t) + \gamma\sigma_2^2 z_2^2(t).\end{aligned} \tag{5.5}$$

Substituting (5.2) into (5.5) and using (4.2), we have

$$\begin{aligned}LV_1 &= 2(\alpha_1 + \mu\beta_1 + \epsilon_1) z_1^2(t) + 2(\mu\alpha_2 + \gamma(\beta_2 + \epsilon_2)) z_2^2(t)\\ &\quad +2(\mu(\alpha_1 + \beta_2) + \alpha_2 + \gamma\beta_1) z_1(t) z_2(t)\\ &\quad -2(\alpha_1 + \mu\beta_1) \int_0^\infty \int_{t-s}^t z_1(t) z_1(\theta) d\theta dK(s)\\ &\quad -2(\alpha_1 + \mu\beta_1) K_1 f_{10} f_{21} \int_0^\infty \int_{t-s}^t z_1(t) z_2(\theta) d\theta dK_2(s)\\ &\quad +2(\mu\alpha_1 + \gamma\beta_1) \int_0^\infty \int_{t-s}^t z_1(t) z_1(\theta) d\theta dR(s)\end{aligned}$$

$$
\begin{aligned}
&+2(\mu\alpha_1+\gamma\beta_1)R_1 g_{10} g_{21}\int_0^\infty\int_{t-s}^t z_1(t)z_2(\theta)d\theta dR_2(s)\\
&-2(\alpha_2+\mu\beta_2)\int_0^\infty\int_{t-s}^t z_2(t)z_1(\theta)d\theta dK(s)\\
&-2(\alpha_2+\mu\beta_2)K_1 f_{10} f_{21}\int_0^\infty\int_{t-s}^t z_2(t)z_2(\theta)d\theta dK_2(s)\\
&+2(\mu\alpha_2+\gamma\beta_2)\int_0^\infty\int_{t-s}^t z_2(t)z_1(\theta)d\theta dR(s)\\
&+2(\mu\alpha_2+\gamma\beta_2)R_1 g_{10} g_{21}\int_0^\infty\int_{t-s}^t z_2(t)z_2(\theta)d\theta dR_2(s).
\end{aligned}
$$

From here, using (2.3),

$$
|\hat{K}|=\int_0^\infty s|dK(s)|,\qquad |\hat{R}|=\int_0^\infty s|dR(s)|,
$$

and some positive numbers γ_i, $i=1,\dots,4$, we obtain

$$
\begin{aligned}
LV_1 \le\ & 2(\alpha_1+\mu\beta_1+\epsilon_1)z_1^2(t)+2(\mu\alpha_2+\gamma(\beta_2+\epsilon_2))z_2^2(t)\\
&+2(\mu(\alpha_1+\beta_2)+\alpha_2+\gamma\beta_1)z_1(t)z_2(t)\\
&+|\hat{K}||\alpha_1+\mu\beta_1|z_1^2(t)+|\alpha_1+\mu\beta_1|\int_0^\infty\int_{t-s}^t z_1^2(\theta)d\theta|dK(s)|\\
&+\gamma_1 K_1\hat{K}_2 f_{10}|f_{21}||\alpha_1+\mu\beta_1|z_1^2(t)\\
&+\gamma_1^{-1}K_1 f_{10}|f_{21}||\alpha_1+\mu\beta_1|\int_0^\infty\int_{t-s}^t z_2^2(\theta)d\theta dK_2(s)\\
&+|\hat{R}||\mu\alpha_1+\gamma\beta_1|z_1^2(t)+|\mu\alpha_1+\gamma\beta_1|\int_0^\infty\int_{t-s}^t z_1^2(\theta)d\theta|dR(s)|\\
&+\gamma_2 R_1\hat{R}_2 g_{10}|g_{21}||\mu\alpha_1+\gamma\beta_1|z_1^2(t)\\
&+\gamma_2^{-1}R_1 g_{10}|g_{21}||\mu\alpha_1+\gamma\beta_1|\int_0^\infty\int_{t-s}^t z_2^2(\theta)d\theta dR_2(s)\\
&+\gamma_3^{-1}|\hat{K}||\alpha_2+\mu\beta_2|z_2^2(t)+\gamma_3|\alpha_2+\mu\beta_2|\int_0^\infty\int_{t-s}^t z_1^2(\theta)d\theta|dK(s)|\\
&+K_1\hat{K}_2 f_{10}|f_{21}||\alpha_2+\mu\beta_2|z_2^2(t)+K_1 f_{10}|f_{21}||\alpha_2+\mu\beta_2|\int_0^\infty\int_{t-s}^t z_2^2(\theta)d\theta dK_2(s)\\
&+\gamma_4^{-1}|\hat{R}||\mu\alpha_2+\gamma\beta_2|z_2^2(t)+\gamma_4|\mu\alpha_2+\gamma\beta_2|\int_0^\infty\int_{t-s}^t z_1^2(\theta)d\theta|dR(s)|\\
&+R_1\hat{R}_2 g_{10}|g_{21}||\mu\alpha_2+\gamma\beta_2|z_2^2(t)+R_1 g_{10}|g_{21}||\mu\alpha_2+\gamma\beta_2|\int_0^\infty\int_{t-s}^t z_2^2(\theta)d\theta dR_2(s)
\end{aligned}
$$

or

$$
\begin{aligned}
LV_1 \le\ & [2(\alpha_1+\mu\beta_1+\epsilon_1)+|\hat{K}||\alpha_1+\mu\beta_1|+\gamma_1 K_1\hat{K}_2 f_{10}|f_{21}||\alpha_1+\mu\beta_1|\\
&+|\hat{R}||\mu\alpha_1+\gamma\beta_1|+\gamma_2 R_1\hat{R}_2 g_{10}|g_{21}||\mu\alpha_1+\gamma\beta_1|]z_1^2(t)
\end{aligned}
$$

$$+[2(\mu\alpha_2+\gamma(\beta_2+\epsilon_2))+\gamma_3^{-1}|\hat{K}||\alpha_2+\mu\beta_2|+K_1\hat{K}_2f_{10}|f_{21}||\alpha_2+\mu\beta_2|$$
$$(5.6)\qquad +\gamma_4^{-1}|\hat{R}||\mu\alpha_2+\gamma\beta_2|+R_1\hat{R}_2g_{10}|g_{21}||\mu\alpha_2+\gamma\beta_2|]z_2^2(t)$$
$$+2[\mu(\alpha_1+\beta_2)+\alpha_2+\gamma\beta_1]z_1(t)z_2(t)+\sum_{i=1}^{2}\int_0^\infty\int_{t-s}^t z_i^2(\theta)d\theta dF_i(s),$$

where

$$\begin{aligned} dF_1(s) &= (|\alpha_1+\mu\beta_1|+\gamma_3|\alpha_2+\mu\beta_2|)|dK(s)| \\ &\quad +(|\mu\alpha_1+\gamma\beta_1|+\gamma_4|\mu\alpha_2+\gamma\beta_2|)|dR(s)|, \\ dF_2(s) &= K_1f_{10}|f_{21}|(\gamma_1^{-1}|\alpha_1+\mu\beta_1|+|\alpha_2+\mu\beta_2|)dK_2(s) \\ &\quad +R_1g_{10}|g_{21}|(\gamma_2^{-1}|\mu\alpha_1+\gamma\beta_1|+|\mu\alpha_2+\gamma\beta_2|)dR_2(s). \end{aligned}$$

Following GMLFC, the additional functional V_2 we have to choose in the form

$$V_2(t)=\sum_{i=1}^{2}\int_0^\infty\int_{t-s}^t(\theta-t+s)z_i^2(\theta)d\theta dF_i(s).$$

Then

$$LV_2(t)=\hat{F}_1z_1^2(t)+\hat{F}_2z_2^2(t)-\sum_{i=1}^{2}\int_0^\infty\int_{t-s}^t z_i^2(\theta)d\theta dF_i(s), \tag{5.7}$$

where

$$\begin{aligned} \hat{F}_1 &= (|\alpha_1+\mu\beta_1|+\gamma_3|\alpha_2+\mu\beta_2|)|\hat{K}| \\ &\quad +(|\mu\alpha_1+\gamma\beta_1|+\gamma_4|\mu\alpha_2+\gamma\beta_2|)|\hat{R}|, \\ \hat{F}_2 &= K_1f_{10}|f_{21}|(\gamma_1^{-1}|\alpha_1+\mu\beta_1|+|\alpha_2+\mu\beta_2|)\hat{K}_2 \\ &\quad +R_1g_{10}|g_{21}|(\gamma_2^{-1}|\mu\alpha_1+\gamma\beta_1|+|\mu\alpha_2+\gamma\beta_2|)\hat{R}_2. \end{aligned}$$

Via (5.6), (5.7) the functional $V=V_1+V_2$ satisfies the condition $LV(t)\leq z'(t)Pz(t)$, where

$$z(t)=\begin{pmatrix} z_1(t) \\ z_2(t) \end{pmatrix}, \qquad P=\begin{pmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{pmatrix}, \tag{5.8}$$

$$\begin{aligned} p_{11} &= 2(\alpha_1+\mu\beta_1+\epsilon_1) \\ &\quad +(2|\hat{K}|+\gamma_1K_1\hat{K}_2f_{10}|f_{21}|)|\alpha_1+\mu\beta_1| \\ &\quad +(2|\hat{R}|+\gamma_2R_1\hat{R}_2g_{10}|g_{21}|)|\mu\alpha_1+\gamma\beta_1| \\ &\quad +\gamma_3|\hat{K}||\alpha_2+\mu\beta_2|+\gamma_4|\hat{R}||\mu\alpha_2+\gamma\beta_2|, \\ (5.9)\qquad p_{22} &= 2(\mu\alpha_2+\gamma(\beta_2+\epsilon_2)) \\ &\quad +\gamma_1^{-1}K_1\hat{K}_2f_{10}|f_{21}||\alpha_1+\mu\beta_1|+\gamma_2^{-1}R_1\hat{R}_2g_{10}|g_{21}||\mu\alpha_1+\gamma\beta_1| \\ &\quad +(\gamma_3^{-1}|\hat{K}|+2K_1\hat{K}_2f_{10}|f_{21}|)|\alpha_2+\mu\beta_2| \\ &\quad +(\gamma_4^{-1}|\hat{R}|+2R_1\hat{R}_2g_{10}|g_{21}|)|\mu\alpha_2+\gamma\beta_2|, \end{aligned}$$

$$p_{12} = \mu(\alpha_1+\beta_2)+\alpha_2+\gamma\beta_1.$$

Corollary 5.1. *If there exist numbers μ, $\gamma>\mu^2$, $\gamma_i>0$, $i=1,2,3,4$, such that the matrix P, defined by* (5.8), (5.9), *is negative definite one, i.e.*

$$p_{11}<0, \qquad p_{22}<0, \qquad p_{11}p_{22}>p_{12}^2,$$

then the trivial solution of system (3.8) *is asymptotically mean square stable and the positive point of equilibrium of system* (2.4) *with stochastic perturbations is stable in probability.*

To simplify the obtained stability condition consider some particular cases of system (3.8).

Via (2.5), (3.9), (5.3) for system (3.10) that is the linear part of system (2.6) with stochastic perturbations we have:

$$\begin{aligned}
&\alpha_1=-K_0x_1^*<0, \qquad \alpha_2=-K_2x_1^*<0,\\
&\beta_1=R_1R_2x_2^*>0, \qquad \beta_2=0,\\
&dK(s)=K_2x_2^*\delta(s)ds+x_1^*dK_0(s), \qquad dR(s)=R_2x_2^*dR_1(s),\\
&K_1=1, \qquad |\hat{K}|=\hat{K}_0x_1^*, \qquad |\hat{R}|=\hat{R}_1R_2x_2^*.
\end{aligned} \tag{5.10}$$

Choosing γ from the condition $p_{12}=0$, we obtain

$$\gamma=-\frac{\mu\alpha_1+\alpha_2}{\beta_1}=\frac{\mu|\alpha_1|+|\alpha_2|}{\beta_1}. \tag{5.11}$$

From the condition $p_{11}<0$ it follows $\alpha_1+\mu\beta_1<0$. So, using (5.9), (5.10), (5.11), we have

$$\begin{aligned}
p_{11} &= 2(-|\alpha_1|+\mu\beta_1+\epsilon_1)+(2\hat{K}_0+\gamma_1\hat{K}_2)x_1^*(|\alpha_1|-\mu\beta_1)\\
&\quad +(2\hat{R}_1R_2x_2^*+\gamma_2R_1\hat{R}_2x_1^*)|\alpha_2|+\gamma_3\hat{K}_0x_1^*|\alpha_2|+\mu\gamma_4\hat{R}_1R_2x_2^*|\alpha_2|,\\
p_{22} &= 2\left(-\mu|\alpha_2|+\epsilon_2\frac{\mu|\alpha_1|+|\alpha_2|}{\beta_1}\right)+\gamma_1^{-1}\hat{K}_2x_1^*(|\alpha_1|-\mu\beta_1)+\gamma_2^{-1}R_1\hat{R}_2x_1^*|\alpha_2|\\
&\quad +(\gamma_3^{-1}\hat{K}_0+2\hat{K}_2)x_1^*|\alpha_2|+\mu(\gamma_4^{-1}\hat{R}_1R_2x_2^*+2R_1\hat{R}_2x_1^*)|\alpha_2|.
\end{aligned}$$

From the conditions $p_{11}<0$, $p_{22}<0$ it follows

$$\frac{A_1+B_1\gamma_1^{-1}+B_2\gamma_2^{-1}+B_3\gamma_3^{-1}}{A_4+B_0\gamma_1^{-1}-B_4\gamma_4^{-1}}<\mu<\frac{A_2-B_1\gamma_1-B_2\gamma_2-B_3\gamma_3}{A_3-B_0\gamma_1+B_4\gamma_4}, \tag{5.12}$$

where

$$\begin{aligned}
&A_1=2|\alpha_2|(\hat{K}_2x_1^*+\epsilon_2\beta_1^{-1}), \qquad A_2=2[|\alpha_1|(1-\hat{K}_0x_1^*)-\hat{R}_1R_2|\alpha_2|x_2^*-\epsilon_1],\\
&A_3=2\beta_1(1-\hat{K}_0x_1^*), \qquad A_4=2[|\alpha_2|(1-R_1\hat{R}_2x_1^*)-\epsilon_2|\alpha_1|\beta_1^{-1}],\\
&B_0=\hat{K}_2\beta_1x_1^*, \qquad B_1=\hat{K}_2|\alpha_1|x_1^*, \qquad B_2=R_1\hat{R}_2|\alpha_2|x_1^*,\\
&B_3=\hat{K}_0|\alpha_2|x_1^*, \qquad B_4=\hat{R}_1R_2|\alpha_2|x_2^*.
\end{aligned} \tag{5.13}$$

Via (5.12) we can write

$$\left(\frac{A_1 + B_1\gamma_1^{-1} + B_2\gamma_2^{-1} + B_3\gamma_3^{-1}}{A_2 - B_1\gamma_1 - B_2\gamma_2 - B_3\gamma_3}\right)\left(\frac{A_3 - B_0\gamma_1 + B_4\gamma_4}{A_4 + B_0\gamma_1^{-1} - B_4\gamma_4^{-1}}\right) < 1 \tag{5.14}$$

and minimize the left part of inequality (5.14) with respect to γ_1, γ_2, γ_3, γ_4 by conditions

$$A_2 > B_1\gamma_1 + B_2\gamma_2 + B_3\gamma_3, \quad A_3 + B_4\gamma_4 > B_0\gamma_1, \quad A_4 + B_0\gamma_1^{-1} > B_4\gamma_4^{-1}. \tag{5.15}$$

Corollary 5.2. *If minimum of the left part of inequality* (5.14) *with respect to* γ_1, γ_2, γ_3, γ_4 *by conditions* (5.15) *less than 1 then the trivial solution of system* (3.10) *is asymptotically mean square stable and the positive equilibrium of system* (2.6) *with stochastic perturbations is stable in probability.*

Let us show that in some cases minimum of the left part of inequality (5.14) can be easy obtained. Suppose, for example, that $\hat{K}_2 = 0$.

From (5.13) it follows that $B_0 = B_1 = 0$ and $A_1 = 2|\alpha_2|\epsilon_2\beta_1^{-1}$. Inequality (5.14) takes the form

$$\left(\frac{A_1 + B_2\gamma_2^{-1} + B_3\gamma_3^{-1}}{A_2 - B_2\gamma_2 - B_3\gamma_3}\right)\left(\frac{A_3 + B_4\gamma_4}{A_4 - B_4\gamma_4^{-1}}\right) < 1$$

Note μ from inequalities (5.12) satisfies the condition $\gamma > \mu^2$ that via (5.10), (5.11) is equivalent to $\beta_1\mu^2 - \mu|\alpha_1| - |\alpha_2| < 0$ or

$$\frac{|\alpha_1| - \sqrt{\alpha_1^2 + 4\beta_1|\alpha_2|}}{2\beta_1} < \mu < \mu_0 = \frac{|\alpha_1| + \sqrt{\alpha_1^2 + 4\beta_1|\alpha_2|}}{2\beta_1}.$$

Really, from (5.12), (5.15) via $B_0 = B_1 = 0$ it follows

$$0 < \mu < \frac{A_2}{A_3} < \frac{|\alpha_1|}{\beta_1} < \mu_0.$$

Using Lemma 4.2 firstly for $P_1 = A_1$, $P_2 = A_2$, $Q_1 = B_2$, $Q_2 = B_3$, $x = \gamma_2$, $y = \gamma_3$ and secondly for $P_1 = A_3$, $P_2 = A_4$, $Q_1 = B_4$, $Q_2 = 0$, $x = \gamma_4^{-1}$, we obtain the following

Corollary 5.3. *Let* A_1, A_2, A_3, A_4, B_2, B_3, B_4 *are defined by* (5.13). *If* $\hat{K}_2 = 0$, $A_2 > 0$, $A_3 \geq 0$, $A_4 > 0$ *and*

$$\left(\sqrt{(B_2 + B_3)^2 + A_1A_2} + B_2 + B_3\right)\left(\sqrt{B_4^2 + A_3A_4} + B_4\right) < A_2A_4$$

then the trivial solution of equation (3.10) *is asymptotically mean square stable and the positive point of equilibrium of system* (2.6) *is stable in probability.*

Example 5.4. Suppose that the parameters of system (2.8) that is a particular case of system (2.6) satisfy the condition

$$A = a - \frac{a_1 b}{b_1 b_2} > 0. \tag{5.16}$$

From (3.4) via (2.3), (2.5), (2.7) it follows that the positive point of equilibrium of system (2.8) is

$$x_1^* = \frac{b}{b_1 b_2}, \qquad x_2^* = \frac{A}{a_2}.$$

The linear part of system (2.8) with stochastic perturbations is defined by (3.11). Via (2.7), (5.10), (5.13) we obtain

$$A_1 = \frac{b a_2^2 \sigma_2^2}{A b_1^2 b_2^2}, \qquad A_2 = 2b\left(\frac{a_1}{b_1 b_2} - A h_1\right) - \sigma_1^2, \qquad A_3 = \frac{2A b_1 b_2}{a_2},$$

$$A_4 = \frac{a_2 b}{b_1 b_2}\left(2(1 - b h_2) - \frac{a_1 \sigma_2^2}{A b_1 b_2}\right), \qquad B_2 = \frac{a_2 b^2}{b_1 b_2} h_2, \qquad B_4 = A b h_1.$$

Via Corollary 5.3 if

$$h_1 < \frac{1}{A}\left(\frac{a_1}{b_1 b_2} - \frac{\sigma_1^2}{2b}\right), \qquad h_2 < \frac{1}{b}\left(1 - \frac{a_1 \sigma_2^2}{2A b_1 b_2}\right)$$

and

$$\left(\sqrt{B_2^2 + A_1 A_2} + B_2\right)\left(\sqrt{B_4^2 + A_3 A_4} + B_4\right) < A_2 A_4 \tag{5.17}$$

then the trivial solution of equation (3.11) is asymptotically mean square stable and the positive point of equilibrium of system (2.8) is stable in probability.

Note that in the case $h_1 = h_2 = 0$ the obtained stability condition follows from Corollary 5.1 via $\alpha_1 = -a_1 x_1^*$, $\alpha_2 = -a_2 x_1^*$, $\beta_1 = b_1 b_2 x_2^*$, $\beta_2 = 0$. In the case $\sigma_1^2 = \sigma_2^2 = h_1 = h_2 = 0$ inequality (5.17) holds and condition (5.16) ensures asymptotic stability of the positive point of equilibrium of the classical Lotka-Volterra model (2.9).

Stability regions for the positive point of equilibrium of system (2.8), obtained by conditions (5.16), (5.17), are shown in the space (a, b) for $a_1 = 0.6$, $a_2 = 1$, $b_1 = 1$, $b_2 = 1$ and different values of the other parameters on Fig. 5.1 ($\sigma_1^2 = 0$, $\sigma_2^2 = 0$, $h_1 = 0$, $h_2 = 0$), Fig. 5.2 ($\sigma_1^2 = 0.4$, $\sigma_2^2 = 0.6$, $h_1 = 0$, $h_2 = 0$), Fig. 5.3 ($\sigma_1^2 = 0$, $\sigma_2^2 = 0$, $h_1 = 0.1$, $h_2 = 0.15$), Fig. 5.4 ($\sigma_1^2 = 0.1$, $\sigma_2^2 = 0.3$, $h_1 = 0.01$, $h_2 = 0.1$).

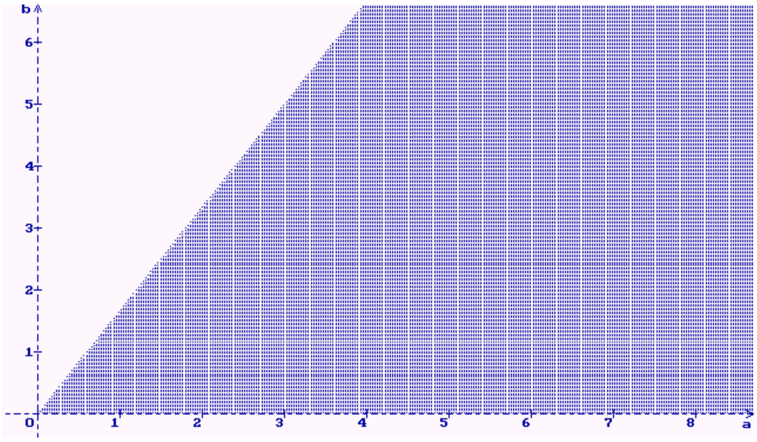

Fig.5.1

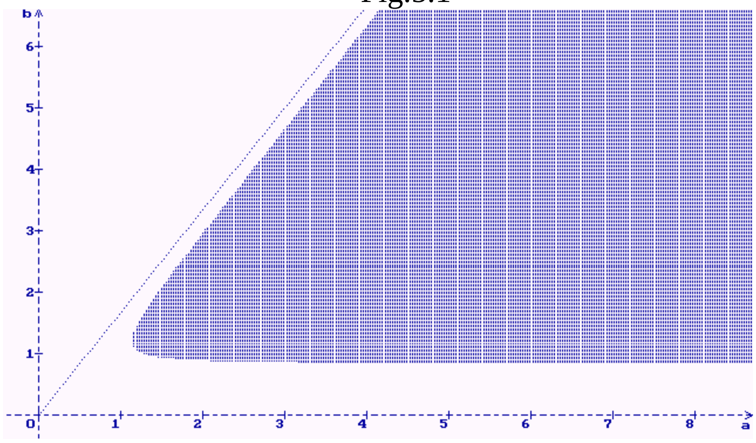

Fig.5.2

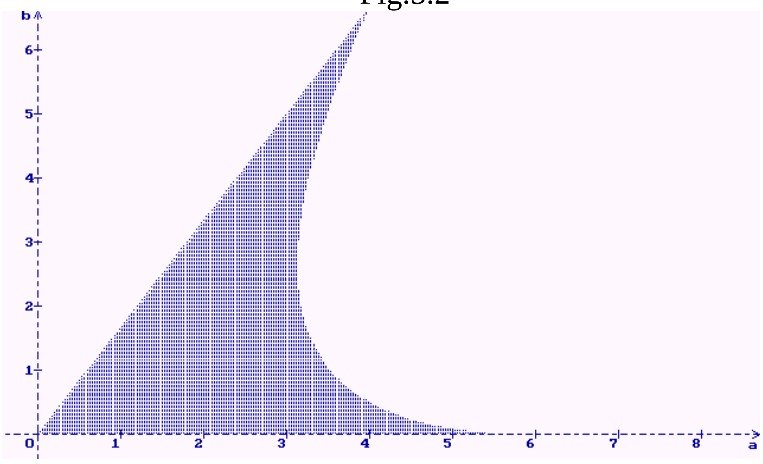

Fig.5.3

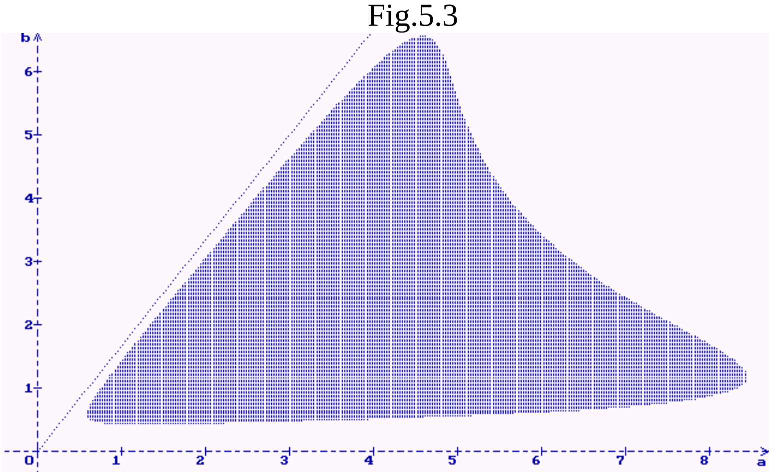

Fig.5.4

6. SHORT SKETCH OF RESEARCH OF SYSTEM (2.10) WITH STOCHASTIC PERTURBATIONS

Consider now system (2.10) that was obtained from (2.1) by conditions (2.11). Via (3.1), (3.2) the positive point of equilibrium (x_1^*, x_2^*) of system (2.10) is defined by the conditions

$$\begin{aligned} & x_1^*(a - K_0 x_1^*) = K_1 f(x_1^*, x_2^*) x_2^*, \\ & b = R_1 g(x_1^*, x_2^*), \qquad a > K_0 x_1^*. \end{aligned} \tag{6.1}$$

Suppose that the functions $f(x_1, x_2)$ and $g(x_1, x_2)$ in (2.11) are differentiable and can be represented in the form

$$\begin{aligned} f(y_1 + x_1^*, y_2 + x_2^*) &= f_0 + f_1 y_1 - f_2 y_2 + o(y_1, y_2), \\ g(y_1 + x_1^*, y_2 + x_2^*) &= g_0 + g_1 y_1 - g_2 y_2 + o(y_1, y_2), \end{aligned}$$

where $\lim_{|y| \to 0} \frac{o(y_1, y_2)}{|y|} = 0$ for $|y| = \sqrt{y_1^2 + y_2^2}$ and

$$\begin{aligned} & f_0 = f(x_1^*, x_2^*), \quad f_1 = x_2^* \hat{f}, \quad f_2 = x_1^* \hat{f}, \quad \hat{f} = \frac{k a_2 (x_1^* x_2^*)^{k-1}}{((x_1^*)^k + a_2 (x_2^*)^k)^2}, \\ & g_0 = g(x_1^*, x_2^*), \quad g_1 = x_2^* \hat{g}, \quad g_2 = x_1^* \hat{g}, \quad \hat{g} = \frac{m b_2 (x_1^* x_2^*)^{m-1}}{((x_1^*)^m + b_2 (x_2^*)^m)^2}. \end{aligned}$$

So, the functionals $F_0(x_{1t}, x_{2t})$, $F_1(x_{1t}, x_{2t})$, $G_1(x_{1t}, x_{2t})$ in (2.11) have representations

$$\begin{aligned} F_0(y_{1t} + x_1^*, y_{2t} + x_2^*) &= K_0 x_1^* + \int_0^\infty y_1(t-s) dK_0(s), \\ F_1(y_{1t} + x_1^*, y_{2t} + x_2^*) &= K_1 f_0 x_2^* + f_1 x_2^* \int_0^\infty y_1(t-s) dK_1(s) + \\ & \quad + K_1 f_0 y_2(t) - f_2 x_2^* \int_0^\infty y_2(t-s) dK_1(s) + o(y_1, y_2), \\ G_1(y_{1t} + x_1^*, y_{2t} + x_2^*) &= R_1 g_0 x_2^* + g_1 x_2^* \int_0^\infty y_1(t-s) dR_1(s) + \\ & \quad + R_1 g_0 y_2(t) - g_2 x_2^* \int_0^\infty y_2(t-s) dR_1(s) + o(y_1, y_2). \end{aligned} \tag{6.2}$$

Via (6.1), (6.2) the linear part of system (2.10) with stochastic perturbations takes the form

$$\begin{aligned} \dot{z}_1(t) &= (a - K_0 x_1^*) z_1(t) - K_1 f_0 z_2(t) \\ & \quad - \int_0^\infty z_1(t-s) dK(s) + f_2 x_2^* \int_0^\infty z_2(t-s) dK_1(s) + \sigma_1 z_1(t) \dot{w}_1(t), \\ \dot{z}_2(t) &= g_1 x_2^* \int_0^\infty z_1(t-s) dR_1(s) - g_2 x_2^* \int_0^\infty z_2(t-s) dR_1(s) + \sigma_2 z_2(t) \dot{w}_2(t), \end{aligned} \tag{6.3}$$

where $dK(s) = x_1^* dK_0(s) + f_1 x_2^* dK_1(s)$. Following GMLFC one can represent system (6.3) in form (5.1) with

$$Z_1(t) = z_1(t) - \int_0^\infty \int_{t-s}^t z_1(\theta) d\theta dK(s) + f_2 x_2^* \int_0^\infty \int_{t-s}^t z_2(\theta) d\theta dK_1(s),$$
$$Z_2(t) = z_2(t) + g_1 x_2^* \int_0^\infty \int_{t-s}^t z_1(\theta) d\theta dR_1(s) - g_2 x_2^* \int_0^\infty \int_{t-s}^t z_2(\theta) d\theta dR_1(s),$$
$$\alpha_1 = K_1 x_2^* \left(\frac{f_0}{x_1^*} - f_1 \right) - K_0 x_1^*, \qquad \alpha_2 = K_1 (f_2 x_2^* - f_0),$$
$$\beta_1 = R_1 g_1 x_2^*, \qquad \beta_2 = -R_1 g_2 x_2^*.$$

Further investigations are similar to Section 5.

For short consider system (2.13) that is a particular case of system (2.10). The point of equilibrium of system (2.13) is defined by (3.5). From (2.12), (3.5), (6.3) it follows that the linear part of system (2.13) with stochastic perturbations has the form

$$\begin{aligned} \dot z_1(t) &= \alpha_1 z_1(t) + \alpha_2 z_2(t) + \sigma_1 z_1(t) \dot w_1(t), \\ \dot z_2(t) &= \beta_1 z_1(t-h) + \beta_2 z_2(t-h) + \sigma_2 z_2(t) \dot w_2(t), \end{aligned} \tag{6.4}$$

where

$$\alpha_1 = a_1 \alpha (2B + a_2) - a, \quad \alpha_2 = -B^2 a_1 \alpha, \quad \alpha = \frac{1}{(B + a_2)^2},$$
$$\beta_1 = b_1 b_2 \beta, \quad \beta_2 = -B\beta_1, \quad B = \frac{bb_2}{b_1 - b}, \quad \beta = \frac{1}{(B + b_2)^2}.$$

System (6.4) can be represented in form (5.1) with

$$Z_1(t) = z_1(t), \qquad Z_2(t) = z_2(t) + \int_{t-h}^t (\beta_1 z_1(\theta) + \beta_2 z_2(\theta)) d\theta.$$

Suppose that all parameters of system (2.13) are positive and besides

$$b_1 > b, \qquad \alpha_1 + \epsilon_1 < 0, \qquad \beta_2 + \epsilon_2 < 0. \tag{6.5}$$

Then $\alpha_1 + \beta_2 < 0$. Since here $\beta_1 > 0$, $\alpha_2 < 0$ then conditions (4.7) hold. Let us show that intervals (4.8), (4.9) have common points. Really, it is easy to see that the left bounds of both intervals are nonpositive ones. Besides,

$$\frac{\sqrt{(\alpha_1 + \beta_2)^2 - 4\beta_1 \alpha_2} - (\alpha_1 + \beta_2)}{2\beta_1} \geq \frac{|\alpha_1 + \beta_2|}{\beta_1} \geq \frac{|\alpha_1|}{\beta_1} \geq -\frac{\alpha_1 + \epsilon_1}{\beta_1} > 0,$$

i.e., at least the positive part of interval (4.8) belongs to interval (4.9).

From Corollary 4.5 it follows that the trivial solution of system (6.4) with $h = 0$ is asymptotically mean square stable. It means that the trivial solution of system (6.4) can be asymptotically mean square stable and for enough small $h > 0$. A coarse estimate for h one can get using the functional V_1 defined in (5.4) for fixed values of μ and γ. Putting for example $\mu = 0$, $\gamma = -\alpha_2\beta_1^{-1} > 0$, via (6.4), (4.2) we obtain

$$\begin{aligned}
LV_1(t) &= 2Z_1(t)(\alpha_1 z_1(t) + \alpha_2 z_2(t)) \\
&\quad +2\gamma Z_2(t)(\beta_1 z_1(t) + \beta_2 z_2(t)) + \sigma_1^2 z_1^2(t) + \gamma\sigma_2^2 z_2^2(t) \\
&= 2z_1(t)(\alpha_1 z_1(t) + \alpha_2 z_2(t)) + \sigma_1^2 z_1^2(t) + \gamma\sigma_2^2 z_2^2(t) \\
&\quad +2\gamma(\beta_1 z_1(t) + \beta_2 z_2(t)) \left(z_2(t) + \int_{t-h}^{t} (\beta_1 z_1(\theta) + \beta_2 z_2(\theta)) d\theta \right) \\
&= 2(\alpha_1 + \epsilon_1) z_1^2(t) + 2\gamma(\beta_2 + \epsilon_2) z_2^2(t) \\
&\quad +2\gamma\beta_1 \int_{t-h}^{t} (\beta_1 z_1(t) z_1(\theta) + \beta_2 z_1(t) z_2(\theta)) d\theta \\
&\quad +2\gamma\beta_2 \int_{t-h}^{t} (\beta_1 z_2(t) z_1(\theta) + \beta_2 z_2(t) z_2(\theta)) d\theta \\
&\leq 2(\alpha_1 + \epsilon_1) z_1^2(t) + 2\gamma(\beta_2 + \epsilon_2) z_2^2(t) \\
&\quad +\gamma\beta_1 \left[\beta_1 \left(h z_1^2(t) + \int_{t-h}^{t} z_1^2(\theta) d\theta \right) + |\beta_2| \left(h z_1^2(t) + \int_{t-h}^{t} z_2^2(\theta) d\theta \right) \right] \\
&\quad +\gamma|\beta_2| \left[\beta_1 \left(h z_2^2(t) + \int_{t-h}^{t} z_1^2(\theta) d\theta \right) + |\beta_2| \left(h z_2^2(t) + \int_{t-h}^{t} z_2^2(\theta) d\theta \right) \right] \\
&= [2(\alpha_1 + \epsilon_1) + Q_1 h] z_1^2(t) + [2\gamma(\beta_2 + \epsilon_2) + Q_2 h] z_2^2(t) + \sum_{i=1}^{2} Q_i \int_{t-h}^{t} z_i^2(\theta) d\theta,
\end{aligned}$$

where $Q_1 = \gamma\beta_1(\beta_1 + |\beta_2|)$, $Q_2 = \gamma|\beta_2|(\beta_1 + |\beta_2|)$. Following GMLFC, the additional functional V_2 we have to choose by standard way

$$V_2(t) = \sum_{i=1}^{2} Q_i \int_{t-h}^{t} (\theta - t + h) z_i^2(\theta) d\theta.$$

Then for the functional $V = V_1 + V_2$ we have

$$\begin{aligned}
LV_1(t) &\leq 2[\alpha_1 + \epsilon_1 + |\alpha_2|(\beta_1 + |\beta_2|)h] z_1^2(t) \\
&\quad +2\gamma[\beta_2 + \epsilon_2 + |\beta_2|(\beta_1 + |\beta_2|)h] z_2^2(t).
\end{aligned}$$

As a result we obtain the following statement: if conditions (6.5) hold and

$$h < \min\left(\frac{|\alpha_1 + \epsilon_1|}{|\alpha_2|(\beta_1 + |\beta_2|)}, \frac{|\beta_2 + \epsilon_2|}{|\beta_2|(\beta_1 + |\beta_2|)} \right)$$

then the trivial solution of system (6.4) is asymptotically mean square stable and the positive point of equilibrium of system (2.13) with stochastic perturbations stable in probability.

Note that choosing optimal values of μ and γ in the functional V_1 (instead of the fixed $\mu = 0$ and $\gamma = |\alpha_2|\beta_1^{-1}$) one can loosen estimation for h.

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