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ASYMPTOTIC NONUNIFORM NONRESONANCE CONDITIONS FOR A NONLINEAR DISCRETE BOUNDARY VALUE PROBLEM

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ABSTRACT. Let $\mathbb{T} := \{a+1,\ \ldots,\ b+1\}$. We study the solvability of nonlinear discrete two-point boundary value problem

$$\begin{cases} \Delta^2 u(t-1) + g(t, u(t)) = h(t), \quad t \in \mathbb{T}, \\ u(a) = u(b+2) = 0 \end{cases}$$

where $h : \mathbb{T} \to \mathbb{R}$, $g : \mathbb{T} \times \mathbb{R} \to \mathbb{R}$ satisfies

$$\alpha(t) \le \liminf_{|x|\to\infty} x^{-1} g(t,x) \le \limsup_{|x|\to\infty} x^{-1} g(t,x) \le \beta(t)$$

uniformly on $\mathbb{T}$, and α and β satisfy some nonresonance conditions of nonuniform type with respect to two consecutive eigenvalues of the associated linear problem. The proof is based on the Leray-Schauder continuation theorem.

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1. PRELIMINARIES

Let $a, b \in \mathbb{N}$ with $b - a > 2$. Let $\mathbb{T} := \{a+1,\ \ldots,\ b+1\}$ and $\hat{\mathbb{T}} := \{a,\ a+1, \ldots, b+1,\ b+2\}$.

Definition 1.1 Suppose that a function $y : \hat{\mathbb{T}} \to \mathbb{R}$. If $y(t) = 0$ then t is a *zero* of y. If $y(t) = 0$ and $\Delta y(t) \neq 0$ then t is a *simple zero* of y. If $y(t)y(t+1) < 0$ then we say that y has a *node* at the point $s = \frac{ty(t+1)-(t+1)y(t)}{y(t+1)-y(t)} \in (t,\ t+1)$. The nodes and simple zeros of y are called the *simple generalized zeros* of y. □

Let μ be a real parameter. It is well-known that the linear eigenvalue problem

$$\begin{cases} \Delta^2 y(t-1) + \mu y(t) = 0, \quad t \in \mathbb{T}, \\ u(a) = u(b+2) = 0 \end{cases} \tag{1.1}$$

has exactly $N := b - a + 1$ eigenvalues

$$\mu_1 < \mu_2 < \cdots < \mu_N, \tag{1.2}$$

which are real and the eigenspace corresponding to any such eigenvalue is one dimensional. The following Lemma is crucial to the study of the nonlinear perturbations of

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the linear problem (1.1). The required results are somewhat scattered in [3, Ch. 6∼7], so we restate them here.

Lemma 1.1[3] Let (μ_i, ψ_i), $i \in \{1, \dots, N\}$, denote eigenvalue pairs of (1.1) with

$$\sum_{t=a+1}^{b+1} \psi_j(t)\psi_j(t) = 1, \qquad j \in \{1, \dots, N\}. \tag{1.3}$$

Then

(1) ψ_i has $i-1$ simple generalized zeros in $[a+1, b+1]$; also if $j \neq k$, then

$$\sum_{t=a+1}^{b+1} \psi_j(t)\psi_k(t) = 0. \tag{1.4}$$

(2) if $h : \{a+1, \dots, b+1\} \to \mathbb{R}$ is given, then the problem

$$\begin{cases} \Delta^2 u(t-1) + \mu_k u(t) = h(t), & t \in \mathbb{T}, \\ u(a) = u(b+2) = 0 \end{cases} \tag{1.5}$$

has solutions if and only if $\sum_{t=a+1}^{b+1} h(t)\psi_k(t) = 0$. □

In this paper, we study the existence of solutions of the nonlinear discrete boundary value problem

$$\begin{cases} \Delta^2 u(t-1) + g(t, u(t)) = h(t), & t \in \mathbb{T}, \\ u(a) = u(b+2) = 0 \end{cases} \tag{1.6}$$

where $g : \mathbb{T} \times \mathbb{R} \to \mathbb{R}$ is a function.

Definition 1.2 By a solution of (1.6) we mean a function $u : \{a, a+1, \dots, b+1, b+2\} \to \mathbb{R}$ which satisfies the difference equation and the boundary value conditions in (1.6). □

Theorem 1.1 Assume

$$\alpha(t) \le \liminf_{|x|\to\infty} x^{-1} g(t,x) \le \limsup_{|x|\to\infty} x^{-1} g(t,x) \le \beta(t) \tag{1.7}$$

uniformly on $\mathbb{T}$, where α, $\beta : \mathbb{T} \to \mathbb{R}$ are such that there exists a positive integer $m \in \{1, \dots, N-1\}$ with

$$\mu_m \le \alpha(t) \le \beta(t) \le \mu_{m+1}, \qquad t \in \mathbb{T}, \tag{1.8}$$

$$\mu_m < \alpha(\tau_\alpha), \qquad \text{for some } \ \tau_\alpha \in \mathbb{T} \setminus \{t \in \mathbb{T} \mid \psi_m(t) = 0\}, \tag{1.9}$$

$$\beta(\tau_\beta) < \mu_{m+1}, \qquad \text{for some } \ \tau_\beta \in \mathbb{T} \setminus \{t \in \mathbb{T} \mid \psi_{m+1}(t) = 0\}. \tag{1.10}$$

Then for each $h : \mathbb{T} \to \mathbb{R}$, (1.6) has at least one solution. □

Theorem 1.2 Let $g : \mathbb{T} \times \mathbb{R} \to \mathbb{R}$ be such that

$$\alpha(t) \le \frac{g(t,u) - g(t,v)}{u - v} \le \beta(t), \qquad t \in \mathbb{T}, u \neq v$$

with α and β as in Theorem 1.1. Then (1.6) has a unique solution. $\square$

Conditions (1.8)-(1.10) are known as the 'asymptotic nonuniform nonresonance condition', whereas the stronger condition

$$\mu_m + \delta \leq \alpha(t) \leq \beta(t) \leq \mu_{m+1} - \delta, \qquad t \in \mathbb{T},$$

(where $\delta \in (0, \infty)$ is fixed), is called the 'uniform nonresonance condition'. The analogue of Theorem 1.1 and 1.2 were obtained for periodic-Dirichlet problems of semilinear wave equations by Mawhin and Ward [5], and for boundary value problems of second order ordinary differential equations by Mawhin [6]. The main tool we use is the Leray-Schauder continuation theorem [7].

For results on the existence of solutions of discrete equations subject to diverse boundary conditions, see Agarwal and O'Regan [1-2], Kelley and Peterson [3], Rodriguez [8], Thompson and Tisdell [9] and the references therein. However none of these consider the problem under the 'asymptotic nonuniform nonresonance' condition, and no uniqueness results were proved.

2. EXISTENCE OF EXTREMAL SOLUTIONS

Let

$$D := \{\big(0, u(a+1), \ldots, u(b+1), 0\big) \mid u(t) \in \mathbb{R},\ t \in \mathbb{T}\}. \tag{2.1}$$

Then D is a Hilbert space under the inner product

$$\langle u, v\rangle = \sum_{t=a+1}^{b+1} u(t)v(t),$$

and the corresponding norm is

$$||u|| := \sqrt{\langle u, u\rangle} = \Big(\sum_{t=a+1}^{b+1} u(t)u(t)\Big)^{1/2}.$$

Notice that D is also a Hilbert space under the inner product

$$\langle u, v\rangle_1 = \sum_{t=a}^{b+1} \Delta u(t)\Delta v(t),$$

and the corresponding norm is

$$||u||_1 := \sqrt{\langle u, u\rangle_1} = \Big(\sum_{t=a}^{b+1} \Delta u(t)\Delta u(t)\Big)^{1/2}.$$

For $u \in D$, let us write

$$u(t) = \bar{u}(t) + \tilde{u}(t) \tag{2.2}$$

where

$$\bar{u}(t) = \sum_{j=1}^{m} \langle u, \psi_j \rangle \psi_j(t), \qquad \tilde{u}(t) = \sum_{j=m+1}^{N} \langle u, \psi_j \rangle \psi_j(t) \tag{2.3}$$

Obviously, $D = \bar{D} \oplus \tilde{D}$ with

$$\bar{D} = \operatorname{span}\{\psi_1, \dots, \psi_m\}, \qquad \tilde{D} = \operatorname{span}\{\psi_{m+1}, \dots, \psi_N\}. \tag{2.4}$$

Lemma 2.1 ([4, Lemma 2.3]) Let $u, w \in D$. Then

$$\sum_{k=a+1}^{b+1} w(k) \Delta^2 u(k-1) = -\sum_{k=a}^{b+1} \Delta u(k) \Delta w(k).$$

Lemma 2.2 Let $\alpha : \mathbb{T} \to \mathbb{R}$ be a function such that

$$\mu_m \le \alpha(t), \qquad t \in \mathbb{T}, \tag{2.5}$$

with

$$\mu_m < \alpha(\tau_\alpha), \quad \text{for some } \ \tau_\alpha \in \mathbb{T} \setminus \{t \in \mathbb{T} \,|\, \psi_m(t) = 0\}. \tag{2.6}$$

Then there is a positive number $\delta_1 > 0$ such that for any $p : \mathbb{T} \to \mathbb{R}$ satisfying

$$\alpha(t) \le p(t), \quad t \in \mathbb{T} \tag{2.7}$$

and $\bar{u} \in \bar{D}$, we have

$$\sum_{t=a+1}^{b+1} \big(- \Delta^2 \bar{u}(t-1) - p(t)\bar{u}(t) \big) \bar{u}(t) \le -\delta_1 \sum_{a+1}^{b+1} [\bar{u}(t)]^2. \tag{2.8}$$

Proof. We divided the proof into two cases.

Case 1. $m \ge 2$.

By assumption (2.7), it is sufficient to prove that

$$\sum_{t=a+1}^{b+1} \big(- \Delta^2 \bar{u}(t-1) - \alpha(t)\bar{u}(t) \big) \bar{u}(t) \le -\delta_1 \sum_{t=a+1}^{b+1} [\bar{u}(t)]^2, \qquad u \in \bar{D}. \tag{2.9}$$

Suppose on the contrary that there is no such $\delta_1 > 0$. Then there is a sequence $\{\bar{u}_k\} \in \bar{D}$ with $||\bar{u}_k|| = 1$ and

$$-k^{-1} \le \sum_{t=a+1}^{b+1} \big(- \Delta^2 \bar{u}_k(t-1) - \alpha(t)\bar{u}_k(t) \big) \bar{u}_k(t), \qquad k = 1, 2, \dots. \tag{2.10}$$

For each $k \in \mathbb{N}$, we will write

$$\bar{u}_k = v_k + w_k \tag{2.11}$$

where

$$v_k \in \operatorname{span}\{\psi_1, \dots, \psi_{m-1}\}, \quad w_k \in \operatorname{span}\{\psi_m\}. \tag{2.12}$$

Now since $\alpha(t) \geq \mu_m$ on $\mathbb{T}$, we have from (2.10) that

$$-k^{-1} \leq \sum_{t=a+1}^{b+1} \big(-\Delta^2 \bar{u}_k(t-1) - \mu_m \bar{u}_k(t)\big)\bar{u}_k(t), \qquad k = 1, 2, \ldots.$$

which reduces to

$$\begin{aligned} -k^{-1} &\leq \sum_{t=a+1}^{b+1} \big(-\Delta^2 v_k(t-1) - \mu_m v_k(t)\big) v_k(t) \\ &\leq (\mu_{m-1} - \mu_m)||v_k||^2. \end{aligned} \tag{2.13}$$

so that $||v_k|| \to 0$ as $k \to \infty$.

Now since $1 = ||\bar{u}_k||^2 = ||v_k||^2 + ||w_k||^2$, we have that a subsequence of $\{\bar{u}_k\}$, which we may relabel as $\{\bar{u}_k\}$, converges strongly to some $w^* := \nu\psi_m$ with $\nu \in \{+, -\}$. Consequently,

$$\begin{aligned} -k^{-1} &\leq \sum_{t=a+1}^{b+1} \big(-\Delta^2 \bar{u}_k(t-1) - \alpha(t)\bar{u}_k(t)\big)\bar{u}_k(t) \\ &= \sum_{t=a+1}^{b+1} \big(-\Delta^2 v_k(t-1) - \alpha(t) v_k(t)\big) v_k(t) - 2\sum_{t=a+1}^{b+1} \alpha(t) v_k(t) w_k(t) \\ &\quad + \sum_{t=a+1}^{b+1} \big(-\Delta^2 w_k(t-1) - \alpha(t) w_k(t)\big) w_k(t) \\ &\leq -(\mu_m - \mu_{m-1})||v_k||^2 - 2\sum_{t=a+1}^{b+1} \alpha(t) v_k(t) w_k(t) \\ &\quad + \sum_{t=a+1}^{b+1} \big(\mu_m - \alpha(t)\big)|w_k(t)|^2. \end{aligned} \tag{2.14}$$

Using $v_k \to 0$ in $\bar{D}$ and $w_k \to w^*$ as $k \to \infty$, we obtain

$$-k^{-1} \leq \sum_{t=a+1}^{b+1} \big(\mu_m - \alpha(t)\big)|w^*(t)|^2 = \sum_{t=a+1}^{b+1} \big(\mu_m - \alpha(t)\big)|\psi_m(t)|^2, \tag{2.15}$$

and since $\mu_m \leq \alpha(t)$ on $\mathbb{T}$, we have

$$\sum_{t=a+1}^{b+1} \big(\mu_m - \alpha(t)\big)|\psi_m(t)|^2 = 0. \tag{2.16}$$

However, (2.16) contradicts (2.6). This contradiction proves (2.9) and hence (2.8) hold.

Case 2. $m = 1$.

In this case, (2.11) reduces to

$$\bar{u}_k = v_k + w_k = 0 + w_k$$

with $v_k \in \{0\}$ and $w_k \in \operatorname{span}\{\psi_1\}$. It is easy to see that the argument used in Case 1 still works in this case. $\square$

Using a similar method with obvious changes, we get the following

Lemma 2.3 Let $\beta : \mathbb{T} \to \mathbb{R}$ be a function such that

$$\beta(t) \leq \mu_{m+1}, \qquad t \in \mathbb{T},$$

with

$$\beta(\tau_\alpha) < \mu_{m+1}, \quad \text{for some } \ \tau_\alpha \in \mathbb{T} \setminus \{t \in \mathbb{T} \,|\, \psi_{m+1}(t) = 0\}.$$

Then there is a positive number $\delta_2 > 0$ such that for any $p : \mathbb{T} \to \mathbb{R}$ satisfying

$$p(t) \leq \beta(t), \quad t \in \mathbb{T}$$

and $\tilde{u} \in \tilde{D}$, we have

$$\sum_{t=a+1}^{b+1} (-\Delta^2 \tilde{u}(t-1) - p(t)\tilde{u}(t))\tilde{u}(t) \geq \delta_2 \sum_{t=a+1}^{b+1} [\tilde{u}(t)]^2. \qquad \square$$

Lemma 2.4 Let α, $\beta : \mathbb{T} \to \mathbb{R}$ be functions such that

$$\mu_m \leq \alpha(t) \leq \beta(t) \leq \mu_{m+1}, \qquad t \in \mathbb{T}, \tag{2.17}$$

$$\mu_m < \alpha(\tau_\alpha), \qquad \text{for some } \ \tau_\alpha \in \mathbb{T} \setminus \{t \in \mathbb{T} \,|\, \psi_m(t) = 0\}, \tag{2.18}$$

$$\beta(\tau_\beta) < \mu_{m+1}, \qquad \text{for some } \ \tau_\beta \in \mathbb{T} \setminus \{t \in \mathbb{T} \,|\, \psi_{m+1}(t) = 0\}. \tag{2.19}$$

Then there are $\delta > 0$ and $\epsilon > 0$ such that for any $p : \mathbb{T} \to \mathbb{R}$ satisfying

$$\alpha(t) - \epsilon \leq p(t) \leq \beta(t) + \epsilon, \quad t \in \mathbb{T}, \tag{2.20}$$

we have

$$||(-\Delta^2 \rho - p)u(\cdot)|| \geq \delta ||u||, \quad u \in D, \tag{2.21}$$

where

$$\rho u(t) := u(t-1), \qquad \Delta^2 \rho u(a) := 0, \qquad \Delta^2 \rho u(b+2) := 0,$$

(so that $(-\Delta^2 \rho - p)u \in D$ for each $u \in D$).

Proof. Suppose the conclusion of the lemma is false. Then there exists a sequence $\{u_k\}$ in D with $||u_k|| = 1$ and a sequence $\{p_k : \mathbb{T} \to \mathbb{R}\}$ with

$$\alpha(t) - \frac{1}{k} \leq p_k(t) \leq \beta(t) + \frac{1}{k}, \quad t \in \mathbb{T}, \tag{2.22}$$

for $k \in \mathbb{N}$, and

$$||(-\Delta^2 \rho - p)u_k|| \leq \frac{1}{k}, \qquad k = 1, 2, \ldots. \tag{2.23}$$

That is

$$-\Delta^2 u_k(t-1) - pu_k(t) = f_k(t), \quad t \in \mathbb{T} \tag{2.24}$$

with $f_k \in D$, $||f_k|| \le k^{-1}$ and $||u_k|| = 1$ for $k \in \mathbb{N}$.

Write $u_k = \bar{u}_k + \tilde{u}_k$, where $\bar{u}_k \in \bar{D}$ and $\tilde{u}_k \in \tilde{D}$ for $k \in \mathbb{N}$. By (2.24), we have that

$$\sum_{t=a+1}^{b+1} \big(-\Delta^2 u_k(t-1) - p_k(t)u_k(t)\big)(\tilde{u}_k(t) - \bar{u}_k(t)) = \sum_{t=a+1}^{b+1} f_k(t)(\tilde{u}_k(t) - \bar{u}_k(t))$$

which reduces upon expansion to

$$\begin{aligned} &\sum_{t=a+1}^{b+1} \big(-\Delta^2 \tilde{u}_k(t-1) - p_k(t)\tilde{u}_k(t)\big)\tilde{u}_k(t) \\ &\quad - \sum_{t=a+1}^{b+1} \big(-\Delta^2 \bar{u}_k(t-1) - p_k(t)\bar{u}_k(t)\big)\bar{u}_k(t) \\ &= \sum_{t=a+1}^{b+1} f_k(t)(\tilde{u}_k(t) - \bar{u}_k(t)). \end{aligned} \tag{2.25}$$

Now by (2.22),

$$\alpha(t) \le p_k(t) + \frac{1}{k}, \quad p_k(t) - \frac{1}{k} \le \beta(t), \quad t \in \mathbb{T}, \tag{2.26}$$

for $k \in \mathbb{N}$. So, using Lemma 2.2 and 2.3, we obtain the existence of $\delta_1 > 0$ and $\delta_2 > 0$ such that for all $k \in \mathbb{N}$, we have

$$\sum_{t=a+1}^{b+1} \big(-\Delta^2 \bar{u}_k(t-1) - \big(p_k(t) + \frac{1}{k}\big)\bar{u}_k(t)\big)\bar{u}_k(t) \le -\delta_1 ||\bar{u}_k||^2,$$

and

$$\sum_{t=a+1}^{b+1} \big(-\Delta^2 \tilde{u}_k(t-1) - \big(p_k(t) - \frac{1}{k}\big)\tilde{u}_k(t)\big)\tilde{u}_k(t) \ge \delta_2 ||\tilde{u}_k||^2.$$

Combining with (2.25) and using Schwarz inequality, this gives

$$\delta_2 ||\tilde{u}_k||^2 - \frac{1}{k}||\tilde{u}_k||^2 + \delta_1 ||\bar{u}_k||^2 - \frac{1}{k}||\bar{u}_k||^2 \le ||f_k||\, ||\tilde{u}_k - \bar{u}_k||,$$

and hence

$$\delta_2 ||\tilde{u}_k||^2 + \delta_1 ||\bar{u}_k||^2 \le \frac{4}{k}$$

which implies that $u_k = \bar{u}_k + \tilde{u}_k$ converges strongly to zero. This contradicts $||u_k|| = 1$ and thus proves the lemma. ☐

3. PROOF OF THE MAIN RESULTS

We now proceed to the proofs of Theorem 1.1 and 1.2 stated in the introduction.

Proof of Theorem 1.1. Let $\delta > 0$ and $\epsilon > 0$ be given by Lemma 2.4. By (1.7) we can find $r > 0$ such that for $t \in \mathbb{T}$ and all u with $|u| \ge r$, we have

$$\alpha(t) - \epsilon \le u^{-1} g(t, u) \le \beta(t) + \epsilon. \tag{3.1}$$

This implies that

$$|g(t,u)| \le \max\{|\mu_{m+1}| + \epsilon, |\mu_m| + \epsilon\}|u| + h_r(t), \quad t \in \mathbb{T},\ u \in \mathbb{R},$$

where $h_r : \mathbb{T} \to \mathbb{R}$ is a fixed function. Consequently, the mapping G defined on D by

$$(Gu)(t) = g(t,u(t)), \quad t \in \mathbb{T};\ (Gu)(a) = (Gu)(b+2) = 0,$$

will map D continuously into itself and take bounded sets into bounded sets.

Using that fact that D is finite dimensional and according to the Leray-Schauder continuation theorem [6], we have that (1.6) have solutions if the set of possible solutions of the family of equations

$$\begin{cases} \Delta^2 u(t-1) + (1-\lambda)\dfrac{\alpha(t)+\beta(t)}{2}u(t) + \lambda g(t,u(t)) = \lambda h(t), \\ u(a) = u(b+2) = 0 \end{cases} \tag{3.2}$$

$(t \in \mathbb{T},\ \lambda \in [0,1])$, is a priori bounded independently of λ. Define Γ on $\mathbb{T} \times \mathbb{R}$ by

$$\Gamma(t,u) = \begin{cases} u^{-1}g(t,u), & |u| \ge r, \\ r^{-1}g(t,r)(u/r) + (1-u/r)\dfrac{\alpha(t)+\beta(t)}{2}, & 0 \le u \le r, \\ r^{-1}g(t,-r)(u/r) + (1+u/r)\dfrac{\alpha(t)+\beta(t)}{2}, & -r \le u \le 0, \end{cases}$$

and σ on $\mathbb{T} \times \mathbb{R}$ by

$$\sigma(t,u) = g(t,u) - \Gamma(t,u)u.$$

It is easy to check that

$$\alpha(t) - \epsilon \le \Gamma(t) \le \beta(t) + \epsilon, \qquad t \in \mathbb{T},\ u \in \mathbb{R}, \tag{3.3}$$

and there exists a constant $M_1 \ge 0$, such that

$$|\sigma(t,u)| \le M_1. \tag{3.4}$$

If $u : D \to \mathbb{R}$ is a solution of (3.2) for some $\lambda \in [0,1]$, then

$$\begin{cases} \Delta^2 u(t-1) + [(1-\lambda)\dfrac{(\alpha(t)+\beta(t))}{2} + \lambda\Gamma(t,u(t))]u(t) \\ \qquad\qquad\qquad\qquad = \lambda[h(t) - \sigma(t,u(t))], \\ u(a) = u(b+2) = 0. \end{cases} \tag{3.5}$$

$(t \in \mathbb{T},\ \lambda \in [0,1])$. Using (3.3) and the fact that

$$\alpha(t) \le \frac{(\alpha(t)+\beta(t))}{2} \le \beta(t), \quad t \in \mathbb{T},$$

it follows that

$$\alpha(t) - \epsilon \le [(1-\lambda)\frac{(\alpha(t)+\beta(t))}{2} + \lambda\Gamma(t,u(t))] \le \beta(t) + \epsilon,$$

and hence using Lemma 2.4, (3.5) and (3.4), we obtain

$$\begin{aligned} M_1 + ||h|| &\geq ||\lambda[h(\cdot) - \sigma(\cdot, u(\cdot))]|| \\ &= ||\Delta^2 u(\cdot - 1) + [(1-\lambda)\frac{\alpha(\cdot) + \beta(\cdot)}{2} + \lambda\Gamma(\cdot, u(\cdot))]u(\cdot)|| \\ &\geq \delta||u||, \end{aligned}$$

i.e.

$$||u|| \leq \delta^{-1}(M_1 + ||h||).$$

This completes the proof of Theorem 1.1. □

Proof of Theorem 1.2. Obviously, existence follows as in Theorem 1.1.

Now let u and v are solutions of (1.6), then, letting $w = u - v$ will be a solution of

$$\begin{cases} \Delta^2 w(t-1) + [g(t, u(t)) - g(t, v(t))] = 0, & t \in \mathbb{T}, \\ w(a) = w(b+2) = 0. \end{cases} \tag{3.6}$$

Setting

$$\gamma(t, w(t)) = \begin{cases} w^{-1}[g(t, u(t)) - g(t, v(t))], & w \neq 0, \\ \dfrac{\alpha(t) + \beta(t)}{2}, & w = 0, \end{cases}$$

we see that (3.6) can be written

$$\begin{cases} \Delta^2 w(t-1) + \gamma(t, w(t))w(t) = 0, & t \in \mathbb{T}, \\ u(a) = u(b+2) = 0, \end{cases} \tag{3.7}$$

with

$$\alpha(t) \leq \gamma(t, w(t)) \leq \beta(t), \qquad t \in \mathbb{T}. \tag{3.8}$$

Therefore, by Lemma 2.4, there exist $\delta > 0$ and $\bar{\epsilon} > 0$ such that for any $p : \mathbb{T} \to \mathbb{R}$ satisfying

$$\bar{\alpha}(t) - \bar{\epsilon} \leq p(t) \leq \beta(t) + \bar{\epsilon}, \quad t \in \mathbb{T}, \tag{3.9}$$

we have

$$||(-\Delta^2\rho - p)u(\cdot)|| \geq \delta||u||, \quad u \in D, \tag{3.10}$$

where $\rho u(t) := u(t-1)$. This together with (3.8) and implies that

$$||(-\Delta^2\rho - \gamma(t, w(t)))w(\cdot)|| \geq \delta||w||. \tag{3.11}$$

Combining (3.11) with (3.7), it follows that

$$0 \geq \delta||w||,$$

and subsequently $w = 0$, i.e. $u = v$, and the proof is complete. □

4. EXAMPLE AND REMARK

From [3, Example 4.1], we know that the linear eigenvalues and the eigenfunctions of the problem

$$\begin{cases} \Delta^2 y(t-1) + \mu y(t) = 0, \quad t \in \mathbb{T}_0, \\ u(0) = u(4) = 0 \end{cases} \tag{4.1}$$

are as follows:

$$\begin{aligned} &\mu_1 = 2 - \sqrt{2}, \quad && \psi_1(t) = \sin(\frac{\pi}{4}t), \quad t \in \mathbb{T}_1 := \{1,2,3\}; \\ &\mu_2 = 2, && \psi_2(t) = \sin(\frac{\pi}{2}t), \quad t \in \mathbb{T}_1; \\ &\mu_3 = 1 - \sqrt{2}, && \psi_3(t) = \sin(\frac{3\pi}{4}t), \quad t \in \mathbb{T}_1. \end{aligned}$$

Obviously

$$\{t \in \mathbb{T}_1 \mid \psi_1(t) = 0\} = \emptyset; \quad \{t \in \mathbb{T}_1 \mid \psi_2(t) = 0\} = \{2\}; \quad \{t \in \mathbb{T}_1 \mid \psi_3(t) = 0\} = \emptyset.$$

Example 4.1. Let us consider the discrete boundary value problem

$$\begin{cases} \Delta^2 y(t-1) + g_0(t, y(t)) = h(t), \quad t \in \mathbb{T}_1, \\ u(0) = u(4) = 0 \end{cases} \tag{4.2}$$

where $h : \mathbb{T}_1 \to \mathbb{R}$, and

$$g_0(t,s) = \frac{\mu_1 + \mu_2}{2} + \frac{\mu_2 - \mu_1}{2} \sin(\pi t)(s + \frac{s}{1+s^2}). \tag{4.3}$$

It is easy to verify that g_0 satisfies all conditions of Theorem 1.1 with

$$\alpha(t) = \frac{\mu_1 + \mu_2}{2} + \frac{\mu_2 - \mu_1}{2} \sin(\pi t) = \beta(t).$$

Therefore (4.2) has at least one solution for every $h : \mathbb{T}_1 \to \mathbb{R}$. □

Remark 4.1 Condition (1.9) and (1.10) are necessary to guarantee the existence of solutions of (1.6). In fact, the function $g(t,s) = \mu_1 s$ satisfies all conditions in Theorem 1.1 except the condition (1.9). Since $\sum_{t=1}^{3}[\psi_1(t)]^2 \neq 0$, we see from Lemma 1.1 (ii) that the problem

$$\begin{cases} \Delta^2 y(t-1) + \mu_1 y(t) = \psi_1(t), \quad t \in \mathbb{T}_1, \\ u(0) = u(4) = 0 \end{cases} \tag{4.4}$$

has no solutions. □

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