Dynamic Systems and Applications 17 (2008) 283-302

ADDITIVE DECOMPOSITION OF MATRICES AND OPTIMIZATION PROBLEMS ON INFINITE TIME INTERVALS

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ABSTRACT. We discuss control systems over finite and countable state spaces defined on an infinite time horizon, where, typically, all the associated costs become unbounded as the time grows indefinitely. We consider the limit behavior, as $n \to \infty$, of the expression $\sum_{i=0}^{n-1} v(z_i, z_{i+1})$ for programs $\{z_i\}_{i=0}^{\infty}$ in a finite state space $X = \{x_i\}_{i=1}^{N}$, where $v(x_i, x_j)$ is the transition cost from state x_i to state x_j. To construct optimal programs we will establish and employ an additive decomposition formula which is of the form

$$V = \mu J + p\eta^T - \eta p^T + \Theta.$$

In this expression μ is a scalar, J is a matrix that satisfies $J_{ij} = 1$ for every $1 \leq i, j \leq N$, p and η are N-dimensional column vectors such that $\eta_i = 1$ for all $1 \leq i \leq N$, and Θ is a matrix satisfying $\min_{1\leq j\leq N} \Theta_{ij} = 0$ for every $1 \leq i \leq N$. We will show how to compute μ, p and Θ in time of order $O(N^5)$. Also, we will discuss infinite horizon optimization problems for certain non-autonomous control systems.

1. INTRODUCTION

In this research we will mainly deal with optimization problems on infinite horizon. We consider a state space $X = \{x_i\}_{i=1}^{N}$ of a plant that operates on unbounded time domain, where the controller has to decide at every instant of time whether or not to change the current state. We will consider a cost matrix V whose (i, j) entry, $v(x_i, x_j)$, represents the transition cost from state x_i to state x_j. In this manner, the controller creates a sequence of states $\mathbf{y} = \{y_i\}_{i=0}^{\infty}$ which we call a *program*. The cost of using program $\mathbf{y}$ till time N is

$$C_N(\mathbf{y}) = \sum_{i=0}^{N-1} v(y_i, y_{i+1}).$$

Our main goal is to find optimal programs for every initial value y_0.

The study of optimal control problems defined on infinite intervals has recently become a rapidly growing area of research. These problems arise in engineering [1], [2], in models of economic growth [7],[14],[18],[19],[20],[24], in infinite discrete models of solid-state physics related to dislocations in one-dimensional crystals [4],[23], and in the theory of thermodynamical equilibrium for materials [8],[11],[15],[16],[26],[27],[28].

Received October 19, 2006

It is not clear a-priori how optimality should be defined when the cost tends to infinity. We will employ the overtaking optimality notion which was first introduced by Gale [9], von Weizsäcker [21] and Astumi [3], and has been adopted by many other authors, see [10],[12],[17],[22],[25].

Definition 1. A program $\mathbf{z} = \{z_i\}_{i=0}^{\infty}$ is called an overtaking optimal program if for every program $\mathbf{s} = \{s_i\}_{i=0}^{\infty}$ that satisfies $s_0 = z_0$ the inequality

$$\limsup_{n\to\infty} \sum_{i=0}^{n-1} [v(z_i, z_{i+1}) - v(s_i, s_{i+1})] \leq 0 \tag{1.1}$$

holds.

The overtaking notion has several variants, of which we mention here the following one, which is intermediate between overtaking optimality and the widely used optimality of the long-run average cost. Another version of overtaking optimality will be introduced later below when required.

Definition 2. A program $\mathbf{z}=\{z_i\}_{i=0}^{\infty}$ is a bounding overtaking optimal program if there exists a constant C such that for every program $\mathbf{s}$ with $s_0 = z_0$ the following inequality

$$\sum_{i=0}^{n-1} v(z_i, z_{i+1}) \leq \sum_{i=0}^{n-1} v(s_i, s_{i+1}) + C \tag{1.2}$$

holds for every $n \geq 1$.

We comment that the notion of bounding overtaking optimality coincides with the notion of *good programs*, introduced by Gale and used in [10].

It was shown in [10] that every continuous function $v : K \times K \to R^1$, where $K \subset R^N$ is a compact set, can be written in the form

$$v(x, y) = \mu + \pi(x) - \pi(y) + \Theta(x, y)$$

where
(a) μ is a constant;
(b) $\pi : K \to R^1$ and $\Theta : K \times K \to R^1$ are continuous functions;
(c) Θ is nonnegative and

$$E(x) = \{y \in K : \Theta(x, y) = 0\}$$

is nonempty for every $x \in K$.

In the present work we display an analogous result for discrete spaces, namely, that every matrix V has an additive decomposition of the form

$$V = \mu J + p\eta^T - \eta p^T + \Theta. \tag{1.3}$$

In (1.3) μ is a constant, J is a matrix that satisfies

$$J_{ij} = 1, \quad \forall\, 1 \leq i, j \leq N,$$

p and η are N-dimensional column vectors where η satisfies

$$\eta_i = 1 \text{ for all } 1 \leq i \leq N,$$

and Θ is a matrix with the following property:

$$\min_{1 \leq j \leq N} \Theta_{ij} = 0 \quad \text{for every} \quad 1 \leq i \leq N.$$

Formula (1.3) is an adaptation to discrete spaces of formula (5.1) in [10]. Formula (1.3) is presented in section 2. This decomposition is a useful tool in the study of optimal programs. We will only describe the structure of the proof, since the details are similar to those in the proof presented in [10].

In section 3 we use the decomposition to show that in the countable space case, $X = \{x_i\}_{i=0}^{\infty}$, it is possible to find for every initial value z_0 a finite set $B = B(z_0) \subset X$ and a program $\mathbf{z}$ with the following properties:
(1) $\mathbf{z}$ is contained in B;
(2) $\mathbf{z}$ is a bounding overtaking optimal program for the initial value z_0.
All this, under the assumption that the cost matrix V satisfies the following condition:

$$\lim_{i+j \to \infty} V_{ij} = \infty.$$

Furthermore, it will be shown that if the matrix Θ in a decomposition of V has only one zero entry in every row, then the asserted program has certain additional optimality properties.

In Section 4, we will show that it is possible to compute the value of μ and the vector p in time which is of order $O(N^5)$. In fact, the proof is constructive and we will show how to compute these quantities.

The last section deals with certain Non Autonomous systems. We consider there a finite state space of size N and an infinite sequence $\{V^k\}_{k \geq 0}$, where each V^k is an $N \times N$ matrix whose (i, j) entry represents the cost transition from state x_i to state x_j in the k-th time epoch. We will assume that

$$V^k = V + \delta_k, \text{ where } \| \delta_k \|_\infty \leq \frac{C}{k^2} \text{ for every } k \geq 0.$$

for some constant $C > 0$. We will show that under these assumptions there exists a bounding overtaking optimal program for every initial value. Furthermore, as in section 3, an additional optimality property will be established when assuming that the matrix Θ has only one zero entry in each row.

2. ADDITIVE DECOMPOSITION OF A MATRIX

In this section we show that every matrix $V \in M_N(R)$ may be represented in the form

$$V = \mu J + p\eta^T - \eta p^T + \Theta \tag{2.1}$$

where μ is a constant, J is a matrix that satisfies

$$J_{i,j} = 1, \text{ for all } 1 \le i, j \le n,$$

p and η are N-dimensional column vectors where η satisfies

$$\eta_i = 1 \text{ for all } 1 \le i \le n,$$

and Θ is an $N \times N$ matrix that satisfies

$$\min_{1 \le j \le N} \Theta_{i,j} = 0 \text{ for every } 1 \le i \le N. \tag{2.2}$$

In terms of the entries $v_{i,j}$ of V we have the representation

$$v_{ij} = \mu + p_i - p_j + \Theta_{ij} \tag{2.3}$$

where $\mu,\ p\ \textit{and}\ \Theta$ have the above properties.

In [10] a decomposition formula for continuous functions on compact sets in R^n was established, and here we present an analogous result for $N \times N$ matrices. For details we refer the reader to [10].

In order to establish (2.1) we shall need Theorem 1 below.

Theorem 1. *There exist constants μ and $M > 0$ such that:*

1. *For every program $\{z_i\}_{i=0}^{\infty}$ and every integer $n \ge 0$ the inequality $\sum_{i=0}^{n}[v(z_i, z_{i+1}) - \mu] \ge -M$ holds.*
2. *For every program $\{z_i\}_{i=0}^{\infty}$ the sequence $\left\{ \sum_{i=0}^{n}[v(z_i, z_{i+1}) - \mu] \right\}_{n=0}^{\infty}$ is either bounded or diverges to infinity, and the set of all programs for which it is bounded is nonempty. Moreover, we can choose M such that for every initial value z_0 there exists a program $\{z_i^*\}_{i=0}^{\infty}$ with $z_0^* = z_0$ which satisfies*
$$\left| \sum_{i=0}^{n} \left[v\left(z_i^*, z_{i+1}^*\right) - \mu \right] \right| \le M\ , \quad \textit{for all } \ n \ge 0.$$

The assertion of the theorem will be a consequence of the following three lemmas. First, denote by $\lambda(n)$ the minimal average cost over all periodic programs of period n, namely

$$\lambda(n) = \min \left\{ \frac{1}{n} \sum_{i=0}^{n-1} v(z_i, z_{i+1}) \,\middle|\, z_0 = z_n \right\}. \tag{2.4}$$

Let μ_0 be defined as the infimal growth rate over all the programs, namely

$$\mu_0 = \inf_{\{z_i\}_{i=0}^{\infty}} \left[\liminf_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} v(z_i, z_{i+1}) \right]. \tag{2.5}$$

In order to compute μ_0, it is sufficient to look only at periodic programs of length $\leq N$. We will elaborate on this issue in section 4. The quantity μ_0 is a natural candidate to satisfy Theorem 1. In fact, if μ is the constant asserted in Theorem 1, then, according to the first claim in Theorem 1, every program $\mathbf{z} = \{z_i\}_{i=0}^{\infty}$ satisfies

$$\frac{1}{n} \sum_{i=0}^{n-1} v(z_i, z_{i+1}) \geq \frac{-M}{n} + \mu.$$

Hence, by letting n grow to infinity, we obtain that $\mu_0 \geq \mu$. In fact, by considering a periodic program $\{z_i\}$ that satisfies

$$\Theta(z_i, z_{i+1}) = 0 \text{ for every } i \geq 0,$$

it is easy to see that $\mu = \mu_0$.

The relation between the sequence $\{\lambda(N)\}_{N=0}^{\infty}$ and μ_0 is expressed in the following Lemmas.

Lemma 1. *The following relation holds:*

$$\mu_0 = \inf_{n\geq 1} \lambda(n).$$

Lemma 2. *The sequence $\{\lambda(n)\}_{n=0}^{\infty}$ converges to μ_0*

The next lemma gives an estimate to the convergence rate of $\lambda(n)$.

Lemma 3. *The following inequality holds:*

$$\limsup_{n\to\infty} n[\lambda(n) - \mu_0] < \infty.$$

For the proofs of these lemmas, as well as the proof of Theorem 1, see [10], section 3.

We will next establish the existence of an additive matrix decomposition as was asserted in the beginning of this section, using Theorem 1.

Theorem 2. *Let V be an $N \times N$ real matrix. Then V can be represented as in* (2.1)*.*

Proof. Let μ be as guaranteed by Theorem 1, and define $p : X \to R$ by

$$p(x) = \inf_{\mathbf{z}, z_0 = x} \{\liminf_{n\to\infty} m_n(\mathbf{z})\}. \tag{2.6}$$

Given any two states $x, y \in X$ we claim that

$$p(x) \leq [v(x, y) - \mu] + p(y). \tag{2.7}$$

This is because if we confine ourselves to programs $\mathbf{z} = \{z_i\}_{i=0}^{\infty}$ such that $z_0 = x, z_1 = y$ and compute the right hand side of (2.6) only over such programs, then we obtain $[v(x, y) - \mu] + p(y)$. Of course $p(x)$ is not greater than this value, which yields (2.7). We define $\Theta : X \times X \to R$ by

$$\Theta(x, y) = v(x, y) - p(x) + p(y) - \mu,$$

and obtain (2.1) with nonnegative Θ. It only remains to prove that there is a zero entry in every row of Θ. Suppose to the contrary that for some i

$$\min_{1 \le j \le n} \Theta_{i,j} = \delta > 0. \tag{2.8}$$

For convenience we denote state x_i by x. From (2.6) we conclude that there exists a program $\mathbf{z}$ such that

$$z_0 = x \quad \text{and} \quad \liminf_{n \to \infty} m_n(z) < p(x) + \frac{1}{2}\delta.$$

We compute:

$$\begin{aligned} p(x) + \frac{1}{2}\delta &> \liminf_{n \to \infty} m_n(z) \\ &= [\Theta(x, z_1) + p(x) - p(z_1)] + \liminf_{n \to \infty} \sum_{i=1}^{n} [v(z_i, z_{i+1}) - \mu] \\ &\ge [\delta + p(x) - p(z_1)] + p(z_1) \end{aligned}$$

implying that $p(x) + \frac{1}{2}\delta > p(x) + \delta$, which is a contradiction. It follows that (2.2) holds, and writing this in the matrix notation yields (2.1). □

3. EXTENSION TO COUNTABLE STATE SPACES

In this section we deal with an infinite countable state space $X^* = \{x_i\}_{i=1}^{\infty}$ and we will show that if the cost matrix satisfies certain assumptions, then for every initial value y in the state space, there exists a finite set $B = B(y) \subset X^*$ such that (in a sense that will be made precise below) we may restrict attention only to programs that start at y and are contained in B.

Theorem 3. *Let $V = V_{ij}, \quad i, j \ge 1$, be an infinite dimension cost matrix such that V_{ij} expresses the transition cost from x_i to x_j. Assume that*

$$\lim_{i+j \to \infty} V_{ij} = \infty. \tag{3.1}$$

Given a state $y \in X^*$ there exists a finite subset of the state space $B = B(y) \subset X^*$ which contains the state y, such that for every program $\{z_i\}_{i=0}^{\infty}$ which is not included in B and satisfies $z_0 = y$, there exists a sequence $\{s_i\}_{i=0}^{\infty} \in B$ such that $s_0 = y$ and

$$\sum_{i=0}^{n} v(s_i, s_{i+1}) < \sum_{i=0}^{n} v(z_i, z_{i+1}) \textit{ for all large enough } n. \tag{3.2}$$

In order to establish the theorem we need the next proposition.

Proposition 1. *Given a state* $y \in X^*$ *there exists a finite subset* $B_0 = B_0(y) \subset X^*$, $y \in B_0$, *and an integer* $N_0 = N_0(y)$ *such that for every sequence* $\{z_i\}_{i=0}^{\infty}$ *which contains* N_0 *consecutive members not belonging to* B_0, *there exists a sequence* $\{s_i\}_{i=0}^{\infty} \subset B_0$, $s_0 = y$, *so that* (3.2) *holds.*

Proof. Denote $\beta = v(y, y)$. Choose a subset $B_0 \subset X$ which is so large that it satisfies

$$(x_i, x_j) \notin B_0 \times B_0 \Rightarrow v(x_i, x_j) \geq \beta + 2. \tag{3.3}$$

Denote

$$a = \max_{(x_i, x_j) \in B_0 \times B_0} v(x_i, x_j),$$

and choose $N_0 = \lfloor 2(a - \beta) \rfloor + 1$ (where $\lfloor x \rfloor$ is the integer part of x). Now, if $\{z_i\}_{i=0}^{\infty}$ satisfies $z_p \in B_0$ and $z_{p+i} \notin B_0$ for all $1 \leq i \leq N_0$ for some p, then there are two possibilities: either $z_{p+i} \notin B_0$ for all $i \geq 1$, or there is a first integer so that $z_{p+l} \in B_0$. In the first possibility the cost for large N grows at least as $(\beta + 2)N$, hence any sequence $\{s_i\}_{i=0}^{\infty}$ which satisfies $s_i = y$ for every $i > p$, satisfies (3.2). In the second case we have:

$$z_p \in B_0, \quad z_{p+i} \notin B_0 \quad \mathit{for} \quad 1 \leq i \leq l-1, \quad z_{p+l} \in B_0 \quad \mathit{and} \quad l > N_0.$$

Therefore,

$$\sum_{i=p}^{p+l-1} v(z_i, z_{i+1}) \;\geq\; l(\beta + 2). \tag{3.4}$$

We define the sequence $\{s_k\}_{k=0}^{\infty}$ by

$$s_k = \begin{cases} y & p < k < p + l, \\ z_k & k \leq p \quad or \quad k \geq p + l, \end{cases}$$

and it follows that the program $\{s_k\}_{k=0}^{\infty}$ satisfies

$$\begin{aligned} \sum_{k=p}^{p+l-1} v(s_k, s_{k+1}) &= v(z_p, y) + (l-2)v(y, y) + v(y, z_l) \\ &\leq 2a + (l-2)\beta = 2(a - \beta) + l\beta < [2(a - \beta)] + 1 + l\beta \\ &= N_0 + l\beta < l + l\beta = l(\beta + 1). \end{aligned} \tag{3.5}$$

Then (3.2) holds in view of (3.4) and (3.5). ∎

Proof of Theorem 3. Let $B_0 = B_0(y) \quad \mathit{and} \quad N_0 = N_0(y)$ be as asserted in Proposition 1. By (3.1), for each $2 \leq k \leq N_0$, there exists a finite subset B_k such that if $\{z_0, \ldots, z_k\}$ is a subprogram that satisfies $z_0, z_k \in B_k$ and $z_l \notin B_k$ for $0 < l < k$ then $\sum_{i=0}^{k-1} v(z_i, z_{i+1}) > 2a + (k-2)\beta$. Any section of sequence of $k+1$ members $\{z_p, \ldots, z_{p+k}\}$

such that $k < N_0$ and satisfies $z_p, z_{p+k} \in B_0$ and $z_{p+i} \notin B_k$ for $0 < i < k$, can be replaced by $\{z_p, y, \ldots, y, z_{p+k}\}$ and thus diminishing the cost, since

$$v(z_0, y) + (k-2)v(y, y) + v(y, z_k) \leq 2a + (k-2)\beta.$$

Define $B = B_0 \cup \left(\bigcup_{k=2}^{N_0} B_k \right)$. Then any section of length $2 \leq k \leq N_0$ whose end points are in B_0 and the interior points are not in B is not contained in B_k either, and can be replaced by a constant section that consists of the state y (which belongs to B). The same holds for sections longer than N_0 which are not contained in B, as indicated by Proposition 1. This concludes the proof of the theorm. $\square$

3.1. **Improving the Result.** The previous result assures, for every initial value y, the existence of a finite subset $B = B(y) \subset X$ such that for every program $\mathbf{z}$ which is not included in B and satisfies $z_0 = y$, there is another program $\mathbf{s} = \mathbf{s}(\mathbf{z})$ which is contained in B and overtakes $\mathbf{z}$ for every large enough n. We will conclude from the property asserted in Theorem 3 the existence of certain optimal programs for each initial state.

Theorem 4. *Let $V = V_{ij}, \quad i, j \geq 1$, be an infinite dimension cost matrix such that V_{ij} expresses the transition cost from x_i to x_j. Assume that* (3.1) *holds. Given a state $y \in X^*$, there exists a finite subset of the state space $B^* = B^*(y) \subset X^*$ which contains the state y and a sequence $\{s_i\}_{i=0}^{\infty} \in B$ with $s_0 = y$ that is bounding overtaking optimal program* (*see Definition* 2). *Furthermore, $B^* = B$* (*see Theorem* 3).

Proof. We define a finite cost matrix V^* for the finite state space $B = \{b_i\}_{i=1}^{N}$ by

$$V^*_{ij} = v^*(b_i, b_j) = v(x_k, x_l) = V_{kl} \text{ if } b_i = x_k, b_j = x_l.$$

Since V^* is a finite dimension matrix, it has a decomposition as established in Theorem 2 in Section 2:

$$V^* = \mu^* J + p^* \eta^T - \eta {p^*}^T + \Theta^*$$

where $\mu^*,\ J,\ p^*,\ \Theta^*$ *and* η have the same properties as $\mu,\ J,\ p,\ \Theta$ and η in Theorem 2. Now, we construct a program $\mathbf{z} = \{z_i\}_{i=0}^{\infty}$ in the following recursive manner:

$$z_0 = y, \quad \text{and for every } k \geq 1,\ z_k \text{ is a state that satisfies}$$

$$\Theta^*(z_{k-1}, z_k) = 0.$$

Then in view of (2.3),

$$\begin{aligned} \sum_{i=0}^{n-1} v(z_i, z_{i+1}) &= \sum_{i=0}^{n-1} (\mu + p(z_i) - p(z_{i+1}) + \Theta(z_i, z_{i+1})) \\ &= n\mu + p(z_0) - p(z_n). \end{aligned}$$

For every other program $\mathbf{s}$ that satisfies $s_0 = y$ we have

$$\begin{aligned}\sum_{i=0}^{n-1} v(s_i, s_{i+1}) &= \sum_{i=0}^{n-1}(\mu + p(s_i) - p(s_{i+1}) + \Theta(s_i, s_{i+1})) \\ &= n\mu + p(s_0) - p(s_n) + \sum_{i=0}^{n-1} \Theta(s_i, s_{i+1}) \\ &= n\mu + p(z_0) - p(s_n) + \sum_{i=0}^{n-1} \Theta(s_i, s_{i+1}).\end{aligned}$$

When computing the difference between the costs of the two programs we obtain

$$\sum_{i=0}^{n-1}[v(z_i, z_{i+1}) - v(s_i, s_{i+1})] = p(s_n) - p(z_n) - \sum_{i=0}^{n-1} \Theta(s_i, s_{i+1})$$

$$\leq \quad p(s_n) - p(z_n) \leq \max_{1 \leq i,j \leq N}\{p(b_i) - p(b_j)\} = \max_{1 \leq i \leq N} p(b_i) - \min_{1 \leq j \leq N} p(b_j).$$

Hence, by choosing $C = \max_{1 \leq i,j \leq N}\{p(b_i) - p(b_j)\}$ we obtain the required result for programs that are contained in B. In order to show that $\mathbf{z}$ also boundedly overtakes any program $\mathbf{s} = \{s_i\}_{i=0}^{\infty}$ which is not contained in B, we compare $\mathbf{s}$ to a program $\mathbf{z^*}$ that is contained in B and overtakes $\mathbf{s}$, and this yields

$$\sum_{i=0}^{n-1} v(z_i, z_{i+1}) \leq \sum_{i=0}^{n-1} v(z_i^*, z_{i+1}^*) + C \leq \sum_{i=0}^{n-1} v(s_i, s_{i+1}) + C$$

for every large enough n. Hence, $\mathbf{z}$ is a bounding overtaking program. □

We will discuss now a special case of the decomposition of V^*. In case that Θ has only one zero entry in every row, we will establish a stronger optimality result. The property asserted in this result is known as *weakly-overtaking optimality*, which was introduced in [6], and which we define as follows.

Definition 3. A program $\mathbf{z} = \{z_i\}_{i=0}^{\infty}$ is called a weakly-overtaking program if for every $\mathbf{s} = \{s_i\}_{i=0}^{\infty}$ with $s_0 = x$, the inequality

$$\liminf_{n\to\infty} \sum_{k=0}^{n-1} \left[v^{k+1}(z_k, z_{k+1}) - v^{k+1}(s_k, s_{k+1})\right] \leq 0 \tag{3.6}$$

holds.

Theorem 5. *Suppose that the matrix V is such that in its decomposition* (2.1), Θ *has only one zero entry in every row. Then, for every initial value x there exists a weakly-overtaking optimal program. Namely, there exists a program $\boldsymbol{z} = \{z_i\}_{i=0}^{\infty}$ with $z_0 = x$ such that for every other program $\boldsymbol{s} = \{s_i\}_{i=0}^{\infty}$ that satisfies $s_0 = x$, inequality* (3.6) *holds.*

We need the following two definitions.

Definition 4. Let Θ be a matrix having only one zero entry in every row. A sequence $\Gamma = \{x_{i_1}, \ldots, x_{i_n}\}$ is called a cycle if

$$\Theta(x_{i_1}, x_{i_2}) = \ldots = \Theta(x_{i_{n-1}}, x_{i_n}) = \Theta(x_{i_n}, x_{i_1}) = 0.$$

For an initial value x we define

$$T_i(x) = \begin{cases} 0 & x \in \Gamma_i \\ \min\left\{\sum_{k=0}^{l-1} \Theta(z_k, z_{k+1}) : z_0 = x, z_l \in \Gamma_i\right\} & \text{otherwise} \end{cases} \tag{3.7}$$

We also define

$$\pi_i = \min_{y \in \Gamma_i} \{-p(y)\} = -\max_{y \in \Gamma_i} \{p(y)\}. \tag{3.8}$$

Proof of Theorem 5. Let $\{\Gamma_1, \ldots, \Gamma_k\}$ be the set of all cycles of Θ. We are interested in the index that attains $\min_{1 \le i \le k} \{\pi_i + T_i(x)\}$. Denote this index by j. Furthermore, denote by $\{z_0^{(j)}, \ldots, z_l^{(j)}\}$ the program that attains the minimum of $T_j(x)$. If $x \in \Gamma_j$ then this program is simply $\{x\}$. Denote by $\{x_{j_0}, \ldots, x_{j_{m-1}}\}$ the cycle Γ_j such that $z_l^{(j)} = x_{j_0}$ and let x_{j_r}, $0 \le r \le m-1$ be the state that achieves π_j. We construct a weakly overtaking program $\mathbf{z}$ for an initial value x as follows

$$z_p = \begin{cases} z_p^{(j)} & 0 \le p < l \\ z_l^{(j)} = x_{j_0} & p = l \\ x_{j_i} & p - l \equiv i \pmod{m}. \end{cases}$$

For every $n > 0$ we compute

$$\begin{aligned} \sum_{p=0}^{l+r+mn-1} v(z_p, z_{p+1}) &= (l+r+mn)\mu + p(z_0) - p(z_{l+r+mn}) + \sum_{p=0}^{l-1} \Theta(z_p, z_{p+1}) \\ &= (l+r+mn)\mu + p(x) + \pi_j + T_j(x) \\ &= (l+r+mn)\mu + p(x) + \min_{1 \le i \le k} \{\pi_i + T_i(x)\} \end{aligned}$$

for every large n. For an arbitrary program $\mathbf{s} = \{s_i\}_{i=0}^{\infty}$ with $s_0 = x$ we obtain

$$\begin{aligned} \sum_{i=0}^{l+r+mn-1} v(s_i, s_{i+1}) &= (l+r+mn)\mu + p(s_0) - p(s_{l+r+mn}) \\ &\quad + \sum_{i=0}^{l+r+nm-1} \Theta(s_i, s_{i+1}). \end{aligned} \tag{3.9}$$

We distinguish between two cases. Either there exists for $\mathbf{s}$ infinite number of integers i such that $\Theta(s_i, s_{i+1}) > 0$, or from a certain state on, $\mathbf{s}$ is contained in a cycle. In the first case, inequality (3.6) holds for n large enough since

$$\lim_{n\to\infty} \sum_{i=0}^{l+r+mn} \Theta(s_i, s_{i+1}) = \infty.$$

In the second case, (3.9) can be estimated by

$$\begin{aligned} &(l+r+mn)\mu + p(s_0) - p(s_{l+r+mn}) + \sum_{i=0}^{l+r+nm-1} \Theta(s_i, s_{i+1}) \\ \geq\ &(l+r+mn)\mu + p(x) + \min_{1\leq i\leq k} \{\pi_i + T_i(x)\} \end{aligned}$$

for all large enough n. Therefore, inequality (3.6) holds and $\mathbf{z}$ is a weakly overtaking program for the initial value x. □

4. COMPUTING μ AND p IN A POLYNOMIAL TIME FOR THE FINITE STATE SPACE CASE

In this section we establish that we can compute the decomposition (2.1) of any $N\times N$ matrix in a polynomial time. It will be convenient to use a definition introduced by Bapat and Raghavan [5], known as the ring of matrices with the *max algebra*. Actually we will employ a similar definition, the *min algebra*. According to our definition, this algebra consists of the set $\mathcal{L} = R \cup \{\infty\}$ and two binary operations, addition and multiplication, denoted by $\oplus$ and $\otimes$ respectively. The operations are defined as follows:

$$a \oplus b = \min(a, b)$$

$$a \otimes b = a + b.$$

Note that ∞ and 0 serve as identity elements for the operations $\oplus$ and $\otimes$ respectively. The operations on matrices will be defined as follows:

Definition 5. Let $A = (a_{ij}), B = (b_{ij})$ be two $n \times m$ matrices over $\mathcal{L}$. We define $A \oplus B$ to be the $n \times m$ matrix whose (i, j) entry is equal to

$$a_{ij} \oplus b_{ij} = \min(a_{ij}, b_{ij}).$$

If $A \in M_{m\times n}(\mathcal{L})$ and $B \in M_{n\times p}(\mathcal{L})$ we define $A \otimes B$ to be the $m \times p$ matrix whose (i, j) entry is

$$\oplus \sum_{k=1}^{n} a_{ik} \otimes b_{kj} = \min_{1\leq k\leq n} (a_{ik} + b_{kj}).$$

If $A \in M_{n\times n}(\mathcal{L})$ we define $A^{(k)}$ to be $\underbrace{A \otimes A \otimes \ldots \otimes A}_{k\ times}$.

The next Lemma will be useful in the sequel. Its proof follows from a straightforward computation, which we omit.

Lemma 4. *Consider square matrices in $M_{n\times n}(\mathcal{L})$. Then matrix multiplication over the min algebra is associative.*

As we saw in Section 2, the value of μ may be obtained by computing $\inf_{n\geq 1} \lambda(n)$ and we claim that for state space of size N, it is actually enough to compute $\min_{1\leq n\leq N} \lambda(n)$.

Theorem 6. *Let X be a finite state space, $|X| = N$. Then*

$$\mu = \min_{1\leq n\leq N} \lambda(n).$$

Proof. Denoting $\mu^* = \min_{1\leq n\leq N} \lambda(N)$, then $\mu \leq \mu^*$. We claim that for every $k > N$ we have

$$\lambda(k) \geq \mu^*, \tag{4.1}$$

and we prove (4.1) by induction on k. The proof below both establishes the claim for $k = N+1$ and the induction, step moving from value k to value $k+1$.

For some $k > N$ let $\mathbf{z} = \{z_i\}_{i=0}^{k}$, $z_0 = z_k$ be a k-length periodic program such that $\lambda(k)$ is achieved by $\mathbf{z}$. Since $|X| = N$, $\mathbf{z}$ contains a subinterval of size $1 \leq l_1 \leq N$, $\{z_i\}_{i=m}^{m+l_1}$ such that $z_m = z_{m+l_1}$. We compute

$$\begin{aligned} \lambda(k) &= \frac{1}{k}\sum_{i=1}^{k} v(z_i, z_{i+1}) \\ &= \frac{1}{k}\left[\sum_{i=1}^{m-1} v(z_i, z_{i+1}) + \sum_{i=m}^{m+l_1-1} v(z_i, z_{i+1}) + \sum_{i=m+l_1}^{k} v(z_i, z_{i+1})\right] \end{aligned} \tag{4.2}$$

where $z_m = z_{m+l_1}$, and therefore the first and the last term in the brackets above can be joined together to form a $(k-l_1)$-length program $\mathbf{w}$:

$$w_i = \begin{cases} z_i & 0 \leq i \leq m \\ z_{i+l_1} & m < i \leq k - l_1. \end{cases}$$

The fact that $\mathbf{z}$ is periodic implies that $\mathbf{w}$ is periodic too, since $w_0 = z_0 = z_k = w_{k-l_1}$ and by (4.2) we have

$$\lambda(k) = \frac{1}{k}\left[l_1 \cdot \frac{1}{l_1}\sum_{i=m}^{m+l_1-1} v(z_i, z_{i+1}) + (k-l_1)\cdot\frac{1}{k-l_1}\sum_{i=0}^{k-l_1-1} v(w_i, w_{i+1})\right] \tag{4.3}$$

$$\geq \frac{1}{k}\Big[l_1 \cdot \mu^* + (k-l_1)\lambda(k-l_1)\Big]. \tag{4.4}$$

For $k = N+1$ we have $k - l_1 < N$, and therefore $\lambda(k-l_1) \geq \mu^*$, and (4.3) implies that $\lambda(N+1) \geq \mu^*$. Thus the assertion is established for $k = N+1$.

To prove the induction step, assume that $\lambda(k') \geq \mu^*$ for every $1 \leq k' \leq k-1$, and it follows that

$$\lambda(k) \geq \frac{1}{k}\Big[l_1\cdot\mu^* + (k-l_1)\lambda(k-l_1)\Big] \geq \frac{1}{k}\Big[l_1\cdot\mu^* + (k-l_1)\mu^*\Big] = \mu^*,$$

establishing the claim for k too. This proves (4.1) for every $k \geq N+1$, which implies $\mu \geq \mu^*$, and concludes the proof of the theorem. $\square$

In the min algebra, $V_{i,j}^k$ represents the minimal transition cost from state x_i to state x_j in k steps, since by Lemma 4

$$(\underbrace{V \otimes V \otimes \ldots \otimes V}_{k \quad times})_{ij} = \min_{1 \leq i_j \leq N} \sum_{j=0}^{k-1} v_{i_j, i_{j+1}} \quad i_0 = i, i_k = j.$$

If we apply direct calculations, the number of operations required to compute the above minimum is exponential. Using the min algebra notation, it can be shown that the number of operations is polynomial. Consequently, μ can be computed in polynomial number of operations, as will be shown in the following result.

Theorem 7. *Let X be a finite state space, $|X| = N$. Then μ can be computed in number of operations that does not exceed N^5.*

In order to establish the theorem, we will need the two following lemmas.

Lemma 5. *For any two matrices $U, V \in M_{N \times N}(\mathcal{L})$, the number of operations needed to compute $U \otimes V$ is bounded by $2N^3$.*

Proof. We have to compute N^2 entries. We examine how many operations are needed to compute a certain $(U \otimes V)_{ij}$ for some $1 \leq i, j \leq N$. We need N operations to compute $U_{ik} + V_{kj}$, $1 \leq k \leq N$. Then, we need $N-1$ more operations to find the minimum, so we need $2N-1$ operations to compute $(U \otimes V)_{ij}$. It follows that we need less than $2N^3$ operations to compute all the N^2 entries. $\square$

Lemma 6. *For a matrix $V \in M_{N \times N}(\mathcal{L})$, the number of operations required to compute $V^{(k+1)}$ is bounded by $2kN^3$.*

Proof. We will prove this lemma by induction. For $k = 1$ the assertion follows directly from lemma 5. Assuming the claim holds for $V^{(k)}$, it follows from $V^{(k+1)} = V^{(k)} \otimes V$ and lemma 5, that the number of operations needed to compute $V^{(k+1)}$ is bounded by $2(k-1)N^3 + 2N^3 = 2kN^3$, and the Lemma is established. $\square$

Proof of Theorem 7. From Theorem 6, it is enough to find $\min_{1 \leq i,k \leq N} V_{ii}^{(k)}$. We count the number of operations required to compute the above minimum. The number of operations required to compute all the matrices $\{V^{(k)}\}_{k=1}^N$ is bounded by

$$\sum_{k=1}^{N-1} 2kN^3 = (N-1)N^4.$$

The array from which the minimum should be derived consists of N^2 terms (N diagonals of N matrices of dimensions $N \times N$). The number of operations required to find the minimum of this array is $N^2 - 1$. If we sum all the operations we see that the number of operations needed to compute μ is bounded by N^5. This concludes the proof of the theorem. $\square$

We next show how to compute the vector p in polynomial time. In this section, we will be particularly interested in the states which belong to a periodic program that realizes μ. We hence define the following notion:

Definition 6. For a finite state space X and a cost matrix V, we denote by X_V the set of all states which belong to a periodic program that realizes μ.

The next lemma follows immediately from Definition 6.

Lemma 7. *We have $x_i \in X_V$ if and only if there exists $1 \le k \le N$ such that*

$$\frac{1}{k} V_{ii}^{(k)} = \mu. \tag{4.5}$$

Using Lemma 7 we can easily construct the set X_V (recall that we already know the value of μ): if $\frac{1}{k} V_{ii}^{(k)} = \mu$ for some $1 \le i, k \le N$, then $x_i \in X_V$, and Lemma 7 assures that X_V consists of only these elements. We introduce now a new matrix

$$\bar{V} = V - \mu \cdot J$$

where J is the matrix in (2.1). Our method of computing p (in polynomial time) is based on the following result.

Theorem 8.

$$p(x_i) = \min \left\{ \bar{V}_{ij}^{(k)} \mid 1 \le k \le N,\ x_j \in X_V \right\}.$$

Proof. Denote

$$m = \min \left\{ \bar{V}_{ij}^{(k)} \mid 1 \le k \le N,\ x_j \in X_V \right\}. \tag{4.6}$$

According to the definition of p in (2.6), it suffices to show that

$$\inf_{\mathbf{z}, z_0 = x_i} \{ \liminf_{n \to \infty} m_n(\mathbf{z}) \} = m.$$

Since X is a finite set, the infimum is attained. Denote by $\mathbf{z}^{(i)} = \{z_j^{(i)}\}_{j=0}^{\infty}$ a program that attains this infimum, and by $\{z_{j_k}^{(i)}\}_{k=0}^{\infty}$ a subsequence along which $\mathbf{z}^{(i)}$ converges to this infimum. We claim that we may assume that $z_{j_k}^{(i)} = y \in X_V$ for every $k \ge 1$, and that $j_{k+1} - j_k \le N$ for every $k \ge 0$.

The assertion that there exists a subsequence $\{z_{j_k}\}$ such that $z_{j_k} = y$ for all k follows from the fact that X is finite. Considering the second assumption, if for some k we have that $j_{k+1} - j_k > N$, we can find a periodic subinterval $[l, m], j_k < l < m < j_{k+1}$, whose modified cost flow is equal to zero (otherwise the infimum would not be attained). But then we can reduce the program, obtaining a program that also realizes the infimum

$$(z_{j_k}^{(i)}, z_{j_k+1}^{(i)}, \ldots, z_{j_{k+1}}^{(i)}) \to (z_{j_k}^{(i)}, \ldots, z_l^{(i)}) = (z_m^{(i)}, \ldots, z_{j_{k+1}}^{(i)}).$$

In Addition, as was already mentioned above,

$$\sum_{l=j_k}^{j_{k+1}-1} [v(z_l^{(i)}, z_{l+1}^{(i)}) - \mu] = 0 \text{ for all } k \geq 1.$$

This reduction process may be continued as long as $j_{k+1} - j_k > N$, and therefore we reduced the problem to computing

$$\min \left\{ \bar{V}_{ij}^{(k)} \mid 1 \leq k \leq N, \ x_j \in X_V \right\}.$$

The proof of the theorem is complete. □

5. FINDING THE OPTIMAL PROGRAM IN THE NON AUTONOMOUS CASE

We consider now a state space $|X| = N$ and a sequence $\{V^k\}_{k\in N}$ of $N \times N$ matrices where the (i,j) entry of V^k represents the cost transition from state x_i to state x_j in the k-th time epoch. The main result of this section asserts the existence of certain optimal programs, and this by employing the matrix decomposition that we have established in Section 2.

Theorem 9. *Let $\{V^k\}_{k\in N}$ be a sequence of matrices such that there exist matrices V and δ_k, and a constant C satisfying*

$$V^k = V + \delta_k, \ \| \delta_k \|_\infty \leq \frac{C}{k^2} \text{ for every } k \in N. \tag{5.1}$$

Then, there exists a program $\boldsymbol{z} = \{z_i\}_{i=0}^\infty$ that is bounding overtaking optimal. Namely, there exist a constant $C^* > 0$ such that for all the programs $\boldsymbol{s} = \{s_i\}_{i=0}^\infty$ which satisfy $s_0 = z_0$ the inequality

$$\sum_{k=0}^{n-1} v^{k+1}(z_k, z_{k+1}) \leq \sum_{k=0}^{n-1} v^{k+1}(s_k, s_{k+1}) + C^* \tag{5.2}$$

holds for every $n \geq 1$.

In order to establish the theorem, we need the following lemma.

Lemma 8. *Let $\{V^k\}_{k\in N}$ be a sequence of matrices which satisfy* (5.1)*, and denote by*

$$V^k = \mu^k J + p^k \eta - \eta^T (p^k)^T + \Theta^k, \quad V = \mu J + p\eta - \eta^T p^T + \Theta \tag{5.3}$$

the decompositions of V^k and V which have been established in Section 2*. Then we can find decompositions such that*

1. $|\mu - \mu^k| \leq \dfrac{C_\mu}{k^2}$ *for some constant* $C_\mu > 0$,
2. $p^k = p + \delta_k^p$ *where* $\| \delta_k^p \|_\infty \leq \dfrac{C_p}{k^2}$ *for some constant* $C_p > 0$,
3. $\Theta^k = \Theta + \delta_k^\Theta$ *where* $\| \delta_k^\Theta \|_\infty \leq \dfrac{C_\Theta}{k^2}$ *for some constant* $C_\Theta > 0$.

Proof.

1. Let k be a given natural number. By Theorem 6 there exist two programs

$$\mathbf{z} = \{z_i\}_{i=0}^{l_1},\ z_0 = z_{l_1} \text{ and } \mathbf{s} = \{s_j\}_{j=0}^{l_2},\ s_0 = s_{l_2} \quad l_1, l_2 \le N$$

such that μ *and* μ^k are realized by $\mathbf{z}$ *and* $\mathbf{s}$ respectively. We estimate $|\mu - \mu^k|$ as follows. On the one hand

$$\begin{aligned}
\mu - \mu^k &= \frac{1}{l_1}\sum_{i=0}^{l_1-1} v(z_i, z_{i+1}) - \frac{1}{l_2}\sum_{i=0}^{l_2-1} v^k(s_i, s_{i+1}) \\
&\le \frac{1}{l_2}\sum_{i=0}^{l_2-1} v(s_i, s_{i+1}) - \frac{1}{l_2}\sum_{i=0}^{l_2-1} v^k(s_i, s_{i+1}) \\
&= \frac{1}{l_2}\sum_{i=0}^{l_2-1}[v(s_i, s_{i+1}) - v^k(s_i, s_{i+1})] \le \frac{1}{l_2}\sum_{i=0}^{l_2-1}\frac{C}{k^2} = \frac{C}{k^2}.
\end{aligned}$$

On the other hand

$$\begin{aligned}
\mu - \mu^k &= \frac{1}{l_1}\sum_{i=0}^{l_1-1} v(z_i, z_{i+1}) - \frac{1}{l_2}\sum_{i=0}^{l_2-1} v^k(s_i, s_{i+1}) \\
&\ge \frac{1}{l_1}\sum_{i=0}^{l_1-1} v(z_i, z_{i+1}) - \frac{1}{l_1}\sum_{i=0}^{l_1-1} v^k(z_i, z_{i+1}) \\
&= \frac{1}{l_1}\sum_{i=0}^{l_1-1}[v(z_i, z_{i+1}) - v^k(z_i, z_{i+1})] \ge \frac{1}{l_1}\sum_{i=0}^{l_1-1}\left(-\frac{C}{k^2}\right) = -\frac{C}{k^2}.
\end{aligned}$$

Therefore $|\mu - \mu^*| < C/k^2$, and the first assertion holds with $C_\mu = C$.

2. Since X is a finite state, $\sum_{i=0}^{k-1} v(z_i, z_{i+1})$ where $k \le N$ can have only a finite number of values. Therefore, for an arbitrary $x_i \in X$ there exists an integer k_0 such that the program which realizes $p(x_i)$ is the same program as the one that realizes $p^k(x_i)$ for $k > k_0$. Denote this program by

$$\{w_j\}_{j=0}^{l}, \quad w_0 = x_i,\ l \le N.$$

We compute

$$\begin{aligned}
\left|p(x_i) - p^k(x_i)\right| &= \left|\sum_{j=0}^{l-1}[v(w_i, w_{i+1}) - \mu] - \sum_{j=0}^{l-1}\left[v^k(w_i, w_{i+1}) - \mu^k\right]\right| \\
&\le \sum_{j=0}^{l-1}\left|v(w_i, w_{i+1}) - v^k(w_i, w_{i+1})\right| + l \cdot \left|\mu^k - \mu\right| \\
&\le l \cdot \left(\frac{C}{k^2} + \frac{C_\mu}{k^2}\right) \le \frac{2NC}{k^2}.
\end{aligned}$$

The second assertion of the Theorem is established with $C_p = 2NC$.

3. This part follows from the previous parts of the theorem. Using the decompositions of V and V^k presented in (2.3) we have

$$\begin{aligned}
\delta_k^{\Theta}(i,j) &= \left|\Theta_{ij}^k - \Theta_{ij}\right| \\
&= \left|V_{ij}^k - \mu^k - p^k(i) + p^k(j) - V_{ij} + \mu + p(i) - p(j)\right| \\
&\le \left|V_{ij}^k - V_{ij}\right| + \left|p(i) - p^k(i)\right| + \left|p^k(j) - p(j)\right| + \left|\mu - \mu^k\right| \\
&\le \frac{C}{k^2} + \frac{C_p}{k^2} + \frac{C_p}{k^2} + \frac{C_\mu}{k^2} \;=\; \frac{2C_p + 2C}{k^2}.
\end{aligned}$$

Thus, the last part of the theorem holds with $C_\Theta = 2C + 2C_p$.

This concludes the proof. □

From the third part of Lemma 8, it follows directly that if (i,j) is an entry which satisfies $\Theta_{ij} = 0$, then

$$\left|\Theta_{ij}^k\right| \le \frac{C_\Theta}{k^2} \text{ for every } k > 0. \tag{5.4}$$

This property is very useful for finding a bounding overtaking optimal program.

Proof of Theorem 9. For an initial value z_0 we construct in a recursive manner a program $\mathbf{z}$ which will be shown to satisfy (5.2). Assume that z_{k-1} has been already chosen, then we choose

$$z_k = x \text{ where } x \text{ is any state that satisfies } \Theta(z_{k-1}, x) = 0.$$

We set C^* to be

$$C^* = 2 \cdot \left(\max_{y \in X} (p(z_0) - p(y)) + (4C_p + C_\Theta)\frac{\pi^2}{6} \right). \tag{5.5}$$

Choose an arbitrary $n \ge 1$. We compute

$$\begin{aligned}
\sum_{k=0}^{n-1} v^{k+1}(z_k, z_{k+1}) &= \sum_{k=0}^{n-1} \left[\mu^{k+1} + p^{k+1}(z_k) - p^{k+1}(z_{k+1}) + \Theta^{k+1}(z_k, z_{k+1})\right] \\
&= \sum_{k=0}^{n-1} \mu^{k+1} + \sum_{k=0}^{n-1} p^{k+1}(z_k) - p^{k+1}(z_{k+1}) + \sum_{k=0}^{n-1} \Theta^{k+1}(z_k, z_{k+1}).
\end{aligned} \tag{5.6}$$

From the second part of Lemma 8 we obtain that

$$\begin{aligned}
\sum_{k=0}^{N-1} p^{k+1}(z_k) - p^{k+1}(z_{k+1}) &= \sum_{k=0}^{N-1} \left[p(z_k) + \delta_{k+1}^p(z_k) - \left(p(z_{k+1}) + \delta_{k+1}^p(z_{k+1})\right)\right] \\
&= \sum_{k=0}^{N-1} \left[p(z_k) - p(z_{k+1})\right] + \sum_{k=0}^{N-1} \left[\delta_{k+1}^p(z_k) - \delta_{k+1}^p(z_{k+1})\right] \\
&\le p(z_0) - p(z_N) + 2\sum_{k=0}^{N-1} \frac{C_p}{(k+1)^2}
\end{aligned}$$

$$\leq \quad p(z_0) - p(z_N) + 2\sum_{k=0}^{\infty} \frac{C_p}{(k+1)^2}$$

$$\leq \quad \max_{y \in X} (p(z_0) - p(y)) + 2C_p \frac{\pi^2}{6},$$

(recalling that $\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}$).

From the third part of Lemma 8, (5.4) and the way we chose $\mathbf{z}$ we obtain that

$$\sum_{k=0}^{N-1} \Theta^{k+1}(z_k, z_{k+1}) \leq \sum_{k=0}^{N-1} \frac{C_\Theta}{(k+1)^2} \leq \sum_{k=0}^{\infty} \frac{C_\Theta}{(k+1)^2} = C_\Theta \frac{\pi^2}{6}.$$

We have thus by (5.6) that

$$\sum_{k=0}^{N-1} v^{k+1}(z_k, z_{k+1}) \leq \sum_{k=0}^{N-1} \mu^{k+1} + \max_{y \in X} (p(z_0) - p(y)) + (2C_p + C_\Theta)\frac{\pi^2}{6}.$$

Let $\mathbf{s} = \{s_k\}_{k=0}^{\infty}$ be any program with $s_0 = z_0$. Computing as above we estimate as follows (recalling (5.5)):

$$\begin{aligned}
&\sum_{k=0}^{n-1} v^{k+1}(s_k, s_{k+1}) + C^* \\
&\quad = \sum_{k=0}^{n-1} \left[\mu^{k+1} + p^{k+1}(s_k) - p^{k+1}(s_{k+1}) + \Theta^{k+1}(s_k, s_{k+1})\right] + C^* \\
&\quad = \sum_{k=0}^{n-1} \mu^{k+1} + \sum_{k=0}^{N-1} \left[p^{k+1}(s_k) - p^{k+1}(s_{k+1})\right] + \sum_{k=0}^{N-1} \Theta^{k+1}(s_k, s_{k+1}) + C^* \\
&\quad \geq \sum_{k=0}^{n-1} \mu^{k+1} - \max_{y \in X} (p(z_0) - p(y)) - 2C_p \cdot \frac{\pi^2}{6} + C^* \\
&\quad = \sum_{k=0}^{n-1} \mu^{k+1} - \max_{y \in X} (p(z_0) - p(y)) - 2C_p \cdot \frac{\pi^2}{6} \\
&\qquad\quad + 2(\max_{y \in X} (p(z_0) - p(y)) + (4C_p + C_\Theta)\frac{\pi^2}{6}) \\
&\quad = \sum_{k=0}^{n-1} \mu^{k+1} + \max_{y \in X} (p(z_0) - p(y)) + (2C_p + C_\Theta)\frac{\pi^2}{6} \\
&\quad \geq \sum_{k=0}^{n-1} v^{k+1}(z_k, z_{k+1}).
\end{aligned}$$

Hence, (5.2) holds for every $n \geq 1$. $\square$

Actually, employing the method used in section 3 to establish Theorem 5, a stronger result can be established when assuming that in the decomposition of V in (2.1), Θ has only one zero entry in every row (see [13]).

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