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ANTIPODAL FIXED POINT THEORY FOR VOLTERRA MAPS

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ABSTRACT. New antipodal fixed point theorems for compact Kakutani maps between Fréchet spaces are presented.

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1. INTRODUCTION

This paper presents new antipodal fixed point theorem for compact Kakutani maps between Fréchet spaces. The proofs rely on an antipodal fixed point theorem for compact Kakutani maps due to O'Regan and Peran [4] and viewing a Fréchet space as the projective limit of a sequence of Banach spaces. In the literature [1, 2] one usually assumes the map F is defined on a subset X of a Fréchet space E and its restriction (again called F) is well defined on $\overline{X_n}$ (see Section 2). In general of course for Volterra operators the restriction is always defined on X_n and in most applications it is in fact defined on $\overline{X_n}$ and usually even on E_n (see Section 2). In this paper we make use of the fact that the restriction is well defined on X_n and we only assume it admits an extension (satisfying certain properties) on $\overline{X_n}$. We also show in Section 2 how easily one can extend antipodal fixed point theory in Banach spaces to fixed point theory in Fréchet spaces.

Existence in Section 2 will be based on an antipodal fixed point theorem for compact Kakutani maps due to O'Regan and Peran [4]. We state a particular case of it here for the convenience of the reader. Let X and Y be topological vector spaces. We say $F : X \to CK(Y)$ is a Kakutani map if F is upper semicontinuous; here $CK(Y)$ denotes the family of nonempty compact convex subsets of Y. We write $F \in K(X, Y)$ if F is a compact Kakutani map.

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Theorem 1.1. *Let E be a Banach space and M a closed, bounded, symmetric subset of E with $0 \in M$ and let $F \in K(M, E)$ with $F(x) \cap (-F(-x)) \neq \emptyset$ for all $x \in \partial M$. Then F has a fixed point in M.*

Now let I be a directed set with order $\leq$ and let $\{E_\alpha\}_{\alpha \in I}$ be a family of locally convex spaces. For each $\alpha \in I$, $\beta \in I$ for which $\alpha \leq \beta$ let $\pi_{\alpha,\beta} : E_\beta \to E_\alpha$ be a continuous map. Then the set

$$\left\{ x = (x_\alpha) \in \prod_{\alpha \in I} E_\alpha : \; x_\alpha = \pi_{\alpha,\beta}(x_\beta) \; \forall\, \alpha,\, \beta \in I,\, \alpha \leq \beta \right\}$$

is a closed subset of $\prod_{\alpha \in I} E_\alpha$ and is called the projective limit of $\{E_\alpha\}_{\alpha \in I}$ and is denoted by $\lim_{\leftarrow} E_\alpha$ (or $\lim_{\leftarrow} \{E_\alpha, \pi_{\alpha,\beta}\}$ or the generalized intersection [3, pp. 439] $\bigcap_{\alpha \in I} E_\alpha$.)

2. FIXED POINT THEORY IN FRÉCHET SPACES

Let $E = (E, \{|\cdot|_n\}_{n \in N})$ be a Fréchet space with the topology generated by a family of seminorms $\{|\cdot|_n : n \in N\}$; here $N = \{1, 2, \ldots\}$. We assume that the family of seminorms satisfies

$$|x|_1 \leq |x|_2 \leq |x|_3 \leq \cdots\cdots \quad \text{for every } \; x \in E. \tag{2.1}$$

A subset X of E is bounded if for every $n \in N$ there exists $r_n > 0$ such that $|x|_n \leq r_n$ for all $x \in X$. For $r > 0$ and $x \in E$ we denote $B(x, r) = \{y \in E : |x - y|_n \leq r \, \forall n \in N\}$. To E we associate a sequence of Banach spaces $\{(\mathbf{E}_n, |\cdot|_n)\}$ described as follows. For every $n \in N$ we consider the equivalence relation $\sim_n$ defined by

$$x \sim_n y \quad \text{iff} \quad |x - y|_n = 0. \tag{2.2}$$

We denote by $\mathbf{E}^n = (E / \sim_n, |\cdot|_n)$ the quotient space, and by $(\mathbf{E}_n, |\cdot|_n)$ the completion of $\mathbf{E}^n$ with respect to $|\cdot|_n$ (the norm on $\mathbf{E}^n$ induced by $|\cdot|_n$ and its extension to $\mathbf{E}_n$ are still denoted by $|\cdot|_n$). This construction defines a continuous map $\mu_n : E \to \mathbf{E}_n$. Now since (2.1) is satisfied the seminorm $|\cdot|_n$ induces a seminorm on $\mathbf{E}_m$ for every $m \geq n$ (again this seminorm is denoted by $|\cdot|_n$). Also (2.2) defines an equivalence relation on $\mathbf{E}_m$ from which we obtain a continuous map $\mu_{n,m} : \mathbf{E}_m \to \mathbf{E}_n$ since $\mathbf{E}_m / \sim_n$ can be regarded as a subset of $\mathbf{E}_n$. Now $\mu_{n,m}\, \mu_{m,k} = \mu_{n,k}$ if $n \leq m \leq k$ and $\mu_n = \mu_{n,m}\, \mu_m$ if $n \leq m$. We now assume the following condition holds:

$$\begin{cases} \text{for each } \; n \in N, \quad \text{there exists a Banach space } \; (E_n, |\cdot|_n) \\ \text{and an isomorphism (between normed spaces) } \; j_n : \mathbf{E}_n \to E_n. \end{cases} \tag{2.3}$$

Remark 2.1. (i). For convenience the norm on E_n is denoted by $|\cdot|_n$.

(ii). In our applications $\mathbf{E}_n = \mathbf{E}^n$ for each $n \in N$.

(iii). Note if $x \in \mathbf{E}_n$ (or $\mathbf{E}^n$) then $x \in E$. However if $x \in E_n$ then x is not necessaily in E and in fact E_n is easier to use in applications (even though E_n is isomorphic to $\mathbf{E}_n$). For example if $E = C[0,\infty)$, then $\mathbf{E}^n$ consists of the class of functions in E which coincide on the interval $[0,n]$ and $E_n = C[0,n]$.

Finally we assume

$$\text{(2.4)} \qquad \begin{cases} E_1 \supseteq E_2 \supseteq \cdots\cdots \quad \text{and for each} \quad n \in N, \\ |j_n\, \mu_{n,n+1}\, j_{n+1}^{-1}\, x|_n \leq |x|_{n+1}\, \forall\, x \in E_{n+1} \end{cases}$$

(here we use the notation from [3] i.e. decreasing in the generalized sense). Let $\lim_{\leftarrow} E_n$ (or $\cap_1^\infty E_n$ where $\cap_1^\infty$ is the generalized intersection [3]) denote the projective limit of $\{E_n\}_{n\in N}$ (note $\pi_{n,m} = j_n\, \mu_{n,m}\, j_m^{-1} : E_m \to E_n$ for $m \geq n$) and note $\lim_{\leftarrow} E_n \cong E$, so for convenience we write $E = \lim_{\leftarrow} E_n$.

For each $X \subseteq E$ and each $n \in N$ we set $X_n = j_n\, \mu_n(X)$, and we let $\overline{X_n}$, $int\, X_n$ and ∂X_n denote respectively the closure, the interior and the boundary of X_n with respect to $|\cdot|_n$ in E_n. Also the pseudo-interior of X is defined by

$$pseudo - int\,(X) = \{x \in X :\ j_n\, \mu_n(x) \in \overline{X_n} \setminus \partial X_n \ \text{ for every } \ n \in N\}.$$

The set X is pseudo-open if $X = pseudo - int\,(X)$. For $r > 0$ and $x \in E_n$ we denote $B_n(x,r) = \{y \in E_n :\ |x-y|_n \leq r\}$.

Let $M \subseteq E$ and consider the map $F : M \to 2^E$. Assume for each $n \in N$ and $x \in M$ that $j_n\, \mu_n\, F\,(x)$ is closed. Let $n \in N$ and $M_n = j_n\, \mu_n(M)$. Since we only consider Volterra type operators we assume

$$\text{(2.5)} \qquad \text{if } \ x,\, y \in E \ \text{ with } \ |x-y|_n = 0 \ \text{ then } \ H_n(F\,(x), F\,(y)) = 0;$$

here H_n denotes the appropriate generalized Hausdorff distance (alternatively we could assume $\forall n \in N, \forall x, y \in M$ if $j_n\, \mu_n\,(x) = j_n\, \mu_n\,(y)$ then $j_n\, \mu_n\, F\,(x) = j_n\, \mu_n\, F\,(y)$ and of course here we do not need to assume that $j_n\, \mu_n\, F\,(x)$ is closed for each $n \in N$ and $x \in M$). Now (2.5) guarantees that we can define (a well defined) F_n on M_n as follows:

For $y \in M_n$ there exists a $x \in M$ with $y = j_n\, \mu_n(x)$ and we let

$$F_n\,(y) = j_n\, \mu_n\, F\,(x)$$

(we could of course call it $F\,y$ since it is clear in the situation we use it); note $F_n : M_n \to C(E_n)$ and note if there exists a $z \in M$ with $y = j_n\, \mu_n(z)$ then $j_n\, \mu_n\, F\,(x) = j_n\, \mu_n\, F\,(z)$ from (2.5) (here $C(E_n)$ denotes the family of nonempty closed subsets of E_n). In this paper we assume F_n will be defined on $\overline{M_n}$ i.e. we assume the F_n described above admits an extension (again we call it F_n) $F_n : \overline{M_n} \to 2^{E_n}$ (we will assume certain properties on the extension).

We now show how easily one can extend fixed point theory in Banach spaces to applicable fixed point theory in Fréchet spaces.

Theorem 2.1. *Let E and E_n be as described in the beginning of Section 2, M a bounded symmetric subset of E, $F : Y \to 2^E$ with $Y \subseteq E$ and with $j_n\,\mu_n(0) \in \overline{M_n}$ and $\overline{M_n} \subseteq Y_n$ for each $n \in N$. Also assume for each $n \in N$ and $x \in Y$ that $j_n\,\mu_n\,F(x)$ is closed and in addition for each $n \in N$ that $F_n : \overline{M_n} \to 2^{E_n}$ is as described above. Suppose the following conditions are satisfied:*

$$\text{for each } n \in N,\ F_n \in K(\overline{M_n}, E_n) \tag{2.6}$$

$$\text{for each } n \in N, \text{ we have } F_n(x) \cap (-F_n(-x)) \neq \emptyset \text{ for } x \in \partial\,\overline{M_n} \tag{2.7}$$

and

$$\begin{cases} \text{for each } n \in \{2, 3, \dots\} \text{ if } y \in \overline{M_n} \text{ solves } y \in F_n\,y \text{ in } E_n \\ \text{then } j_k\,\mu_{k,n}\,j_n^{-1}(y) \in \overline{M_k} \text{ for } k \in \{1, \dots, n-1\}. \end{cases} \tag{2.8}$$

Then F has a fixed point in E.

Remark 2.2. Note in Theorem 2.1 if $x \in \overline{M_n}$ then $x \in Y_n$ so there exists a $y \in Y$ with $x = j_n\,\mu_n\,(y)$ and so $F_n\,(x) = j_n\,\mu_n\,F(y)$.

PROOF: Fix $n \in N$. We would like to apply Theorem 1.1. To do so we need to show

$$\overline{M_n} \text{ is a bounded symmetric subset of } E_n. \tag{2.9}$$

To see that M_n is symmetric let $\hat{x} \in \mu_n(M)$. Then for every $x \in \mu_n^{-1}(\hat{x})$ we have $-x \in M$ since M is symmetric and of course $-\hat{x} = -\mu_n(x)$. Now its easy to see $-\mu_n(x) = \mu_n(-x)$ so

$$-\hat{x} = -\mu_n(x) = \mu_n(-x) \in \mu_n(M).$$

As a result $\mu_n(M)$ is symmetric and since j_n is linear (in particular $j_n(-y) = -j_n(y)$ for $y \in \mathbf{E}_n$) we have that $M_n = j_n\,\mu_n(M)$ is symmetric and so $\overline{M_n}$ is symmetric. Also M_n is bounded (so $\overline{M_n}$ is bounded) since M is bounded (note if $y \in M_n$ then there exists $x \in M$ with $y = j_n \mu_n(x)$). Thus (2.9) holds.

For each $n \in N$ (see Theorem 1.1) there exists $y_n \in \overline{M_n}$ with $y_n \in F_n\,(y_n)$. Lets look at $\{y_n\}_{n \in N}$. Notice $y_1 \in \overline{M_1}$ and $j_1\,\mu_{1,k}\,j_k^{-1}\,(y_k) \in \overline{M_1}$ for $k \in N\backslash\{1\}$ from (2.8). Note $j_1\,\mu_{1,n}\,j_n^{-1}(y_n) \in F_1\,(j_1\,\mu_{1,n}\,j_n^{-1}(y_n))$ in E_1; to see note for $n \in N$ fixed there exists a $x \in E$ with $y_n = j_n\,\mu_n\,(x)$ so $j_n\,\mu_n\,(x) \in F_n\,(y_n) = j_n\,\mu_n\,F(x)$ on E_n so on E_1 we have

$$\begin{aligned} j_1\,\mu_{1,n}\,j_n^{-1}(y_n) &= j_1\,\mu_{1,n}\,j_n^{-1}\,j_n\,\mu_n\,(x) \in j_1\,\mu_{1,n}\,j_n^{-1}\,j_n\,\mu_n\,F(x) \\ &= j_1\,\mu_{1,n}\,\mu_n\,F(x) = j_1\,\mu_1\,F(x) = F_1(j_1\,\mu_1\,(x)) \\ &= F_1(j_1\,\mu_{1,n}\,j_n^{-1}\,j_n\,\mu_n\,(x)) = F_1\,(j_1\,\mu_{1,n}\,j_n^{-1}(y_n)). \end{aligned}$$

As a result $j_1\,\mu_{1,n}\,j_n^{-1}(y_n) \in F_1\,(j_1\,\mu_{1,n}\,j_n^{-1}(y_n))$ in E_1, $j_1\,\mu_{1,n}\,j_n^{-1}\,(y_n) \in \overline{M_1}$ for $n \in N$, together with (2.6) implies there is a subsequence $N_1^\star$ of N and a $z_1 \in \overline{M_1}$ with $j_1\,\mu_{1,n}\,j_n^{-1}\,(y_n) \to z_1$ in E_1 as $n \to \infty$ in $N_1^\star$ and $z_1 \in F_1\,(z_1)$ since F_1 is upper semicontinuous. Let $N_1 = N_1^\star \setminus \{1\}$. Now $j_2\,\mu_{2,n}\,j_n^{-1}(y_n) \in \overline{M_2}$ for $n \in N_1$ together with (2.6) guarantees that there exists a subsequence $N_2^\star$ of N_1 and a $z_2 \in \overline{M_2}$ with $j_2\,\mu_{2,n}\,j_n^{-1}(y_n) \to z_2$ in E_2 as $n \to \infty$ in $N_2^\star$ and $z_2 \in F_2(z_2)$. Note from (2.4) and the uniqueness of limits that $j_1\,\mu_{1,2}\,j_2^{-1}\,z_2 = z_1$ in E_1 since $N_2^\star \subseteq N_1$ (note $j_1\,\mu_{1,n}\,j_n^{-1}(y_n) = j_1\,\mu_{1,2}\,j_2^{-1}\,j_2\,\mu_{2,n}\,j_n^{-1}(y_n)$ for $n \in N_2^\star$). Let $N_2 = N_2^\star \setminus \{2\}$. Proceed inductively to obtain subsequences of integers

$$N_1^\star \supseteq N_2^\star \supseteq \cdots\cdots, \quad N_k^\star \subseteq \{k, k+1, \dots\}$$

and $z_k \in \overline{M_k}$ with $j_k\,\mu_{k,n}\,j_n^{-1}(y_n) \to z_k$ in E_k as $n \to \infty$ in $N_k^\star$ and $z_k \in F_k(z_k)$. Note $j_k\,\mu_{k,k+1}\,j_{k+1}^{-1}\,z_{k+1} = z_k$ in E_k for $k \in \{1, 2, \dots\}$. Also let $N_k = N_k^\star \setminus \{k\}$.

Fix $k \in N$. Now $z_k \in F_k(z_k)$ in E_k. Note as well that

$$\begin{aligned} z_k &= j_k\,\mu_{k,k+1}\,j_{k+1}^{-1}\,z_{k+1} = j_k\,\mu_{k,k+1}\,j_{k+1}^{-1}\,j_{k+1}\,\mu_{k+1,k+2}\,j_{k+2}^{-1}\,z_{k+2} \\ &= j_k\,\mu_{k,k+2}\,j_{k+2}^{-1}\,z_{k+2} = \cdots = j_k\,\mu_{k,m}\,j_m^{-1}\,z_m = \pi_{k,m}\,z_m \end{aligned}$$

for every $m \geq k$. We can do this for each $k \in N$. As a result $y = (z_k) \in \lim_{\leftarrow} E_n = E$ and also note $z_k \in \overline{M_k} \subseteq Y_k$ for each $k \in N$. Thus for each $k \in N$ we have

$$j_k\,\mu_k(y) = z_k \in F_k(z_k) = j_k\,\mu_k\,F(y) \quad \text{in} \quad E_k$$

so $y \in F(y)$ in E. $\square$

In Theorem 2.1 it is possible to replace $\overline{M_n} \subseteq Y_n$ with $\overline{M_n}$ a subset of the closure of Y_n in E_n provided Y is a closed subset of E so in this case we could have $Y = M$ if M was closed. To see this note from $y = (z_k) \in \lim_{\leftarrow} E_n = E$ and $\pi_{k,m}(y_m) \to z_k$ in E_k as $m \to \infty$ we can conclude that $y \in \overline{Y} = Y$ (note $q \in \overline{Y}$ iff for every $k \in N$ there exists $(x_{k,m}) \in Y$, $x_{k,m} = \pi_{k,n}(x_{n,m})$ for $n \geq k$ with $x_{k,m} \to j_k\,\mu_k(q)$ in E_k as $m \to \infty$). Thus $z_k = j_k\,\mu_k(y) \in Y_k$ and so $j_k\,\mu_k(y) \in j_k\,\mu_k\,F(y)$ in E_k as before. For completeness we state these results.

Theorem 2.2. *Let E and E_n be as described in the beginning of Section 2, M a bounded symmetric subset of E, $F : Y \to 2^E$ with Y a closed subset of E and with $j_n\,\mu_n(0) \in \overline{M_n}$ and $\overline{M_n}$ a subset of the closure of Y_n in E_n for each $n \in N$. Also assume for each $n \in N$ and $x \in Y$ that $j_n\,\mu_n\,F(x)$ is closed and in addition for each $n \in N$ that $F_n : \overline{M_n} \to 2^{E_n}$ is as described above. Suppose* (2.6), (2.7) *and* (2.8) *hold. Then F has a fixed point in E.*

Corollary 2.1. *Let E and E_n be as described in the beginning of Section 2, M a closed bounded symmetric subset of E, $F : M \to 2^E$ with $j_n\,\mu_n(0) \in \overline{M_n}$ for each $n \in N$. Also assume for each $n \in N$ and $x \in M$ that $j_n\,\mu_n\,F(x)$ is closed and in addition for each $n \in N$ that $F_n : \overline{M_n} \to 2^{E_n}$ is as described above. Suppose* (2.6), (2.7) *and* (2.8) *hold. Then F has a fixed point in E.*

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