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MULTIPLE NONTRIVIAL SOLUTIONS FOR SEMILINEAR ELLIPTIC NEUMANN PROBLEMS WITH INDEFINITE LINEAR PART

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ABSTRACT. We consider a semilinear Neumann problem with indefinite linear part and a nonsmooth potential (hemivariational inequality).Using a nonsmooth variant of the reduction technique, we prove a multiplicity theorem for problems with subquadratic potential.

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1. INTRODUCTION

Let $Z \subseteq \mathbb{R}^{\mathbb{N}}$ be a bounded domain with a C^2-boundary ∂Z. In this paper, we study the existence of nontrivial solutions for the following semilinear elliptic Neumann problem with a nonsmooth potential (hemivariational inequality):

$$\left\{ \begin{array}{l} -\triangle x(z) - a(z)x(z) \in \partial j(z, x(z)) \text{ a.e. on } Z, \\ \frac{\partial x}{\partial n} = 0 \text{ on } \partial Z. \end{array} \right\} \tag{1.1}$$

Here $a \in L^{\frac{N}{2}}(Z)$ $(N \geq 3)$ and $j(z, x)$ is jointly measurable and locally Lipschitz and in general nonsmooth in $x \in \mathbb{R}$. By $\partial j(z, x)$ we denote the generalized subdifferential of $x \to j(z, x)$ (see Section 2). We do not assume that $a(\cdot)$ is positive and so the linear part of the problem (1.1) is indefinite. Rabinowitz [11] examined problem (1.1) with $a(\cdot)$ positive, Dirichlet boundary conditions and a continuous right hand side nonlinearity (hence the potential function is C^1). He also had a parameter $\lambda \in \mathbb{R}$ (i.e. $\lambda a(z)x(z)$) and proved existence of nontrivial solutions for every value of the parameter, using the Ambrosetti-Rabinowitz condition on the nonlinearity, which implies that the potential exhibits a strictly superquadratic growth in $x \in \mathbb{R}$. The result of Rabinowitz was extended to unbounded domains with indefinite linear part by Li-Chen [9], to problems with indefinite nonlinearity by Badiale-Nabana [2] and to problems with nonsmooth potential (hemivariational inequalities) by Barletta-Marano [3]. Hemivariational inequalities seem to be a suitable model for various problems in mechanics and engineering (see Naniewicz-Panagiotopoulos [10]).

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2. MATHEMATICAL BACKGROUND

Let X be a Banach space, X^* its topological dual and by $\langle \cdot, \cdot \rangle$ we denote the duality brackets for the pair (X, X^*). Given a locally Lipschitz function $\varphi : X \to \mathbb{R}$, the generalized directional derivative of φ at $x \in X$ in the direction $h \in X$, is defined by

$$\varphi^0(x; h) = \limsup_{\substack{x' \to x \\ \lambda \downarrow 0}} \frac{\varphi(x' + \lambda h) - \varphi(x')}{\lambda}.$$

It is easy to see that $\varphi^0(x, \cdot)$ is sublinear, continuous. So by the Hahn-Banach theorem it is the support function of a nonempty, w^*-compact and convex set $\partial\varphi(x) \subseteq X^*$, defined by

$$\partial\varphi(x) = \{x^* \in X^* : \langle x^*, h \rangle \le \varphi^0(x; h) \text{ for all } h \in X\}.$$

The multifunction $x \to \partial\varphi(x)$ is the generalized subdifferential of φ and $x \in X$ is a critical point of φ, if $0 \in \partial\varphi(x)$. Note that if φ is continuous, convex, then it is locally Lipschitz and the generalized subdifferential coincides with the subdifferential in the sense of convex analysis. Also, if $\varphi \in C^1(X)$, then it is locally Lipschitz and $\partial\varphi(x) = \{\varphi'(x)\}$. For details we refer to Clarke [6] and Gasinski-Papageorgiou [7].

Consider the following linear eigenvalue problem:

$$\left\{ \begin{array}{l} -\triangle x(z) - \alpha(z)x(z) = \lambda x(z) \text{ a.e. on } Z, \\ \frac{\partial x}{\partial n} = 0 \text{ on } \partial Z. \end{array} \right\} \tag{2.1}$$

By virtue of Corollary 7.D, p.78 of Showalter [12], we know that problem (2.1) has a sequence of eigenvalues $\{\lambda_n\}_{n \ge 1}$ (counting multiplicities), such that $-\infty < \lambda_1 < \lambda_2 \le \cdots \le \lambda_n \to \infty$ as $n \to \infty$ and a corresponding sequence of eigenfunctions which form an orthonormal basis of $L^2(Z)$ and an orthogonal basis of $L^2(Z)$. Using these eigenvalues, we consider the following orthogonal direct sum decomposition of $H^1(Z)$, $H^1(Z) = H_- \oplus H_0 \oplus H_+$, where

$$H_- = \text{span}\{x \in H^1(Z) : -\triangle x - ax = \lambda x, \ \lambda < 0\}, \ H_0 = \text{Ker}(-\triangle - aI)$$
$$\text{and } H_+ = \overline{span}\{x \in H^1(Z) : -\triangle x - ax = \lambda x, \lambda > 0\}$$

We know that both H_- and H_0 are finite dimensional subspaces of $H^1(Z)$. In what follows by $\lambda_k < 0$ we denote the largest negative eigenvalue and by $\lambda_m > 0$ the smallest positive eigenvalue.

The hypotheses on the nonsmooth potential $j(z, x)$ are the following:

$\underline{H(j)}$: $j : Z \times \mathbb{R} \to \mathbb{R}$ is a function such that

(i): for all $x \in \mathbb{R}$, $z \to j(z, x)$ is measurable;

(ii): for almost all $z \in Z$, $x \to j(z, x)$ is locally Lipschitz;

(iii): for almost all $z \in Z$, all $x \in \mathbb{R}$ and all $u \in \partial j(z,x)$, we have

$$|u| \leq a(z) + c|x|^{r-1} \text{ with } a \in L^\infty(Z)_+, c > 0,\ 1 \leq r < 2^* = \frac{N}{N-2};$$

(iv): there exist a measurable set $C \subseteq Z$ with $|C|_N > 0$ ($|\cdot|_N$ being the Lebesgue measure on $\mathbb{R}^{\mathbb{N}}$) and $h_0 \in L^1(Z)$ such that

$$j(z,x) \to +\infty \text{ for a.a. } z \in C \text{ as } |x| \to \infty$$
$$\text{and } h_0(z) \leq j(z,x) \text{ for a.a. } z \in Z \text{ and all } x \in \mathbb{R};$$

(v): there exists $\beta \in L^\infty(Z)_+$ such that $\beta(z) \leq \lambda_m$ a.e. on Z with strict inequality on a set of positive measure and for almost all $z \in Z$, all $x, y \in \mathbb{R}$ and all $u_1 \in \partial j(z,x), u_2 \in \partial j(z,y)$, we have

$$(u_1 - u_2)(x - y) \leq \beta(z)|x - y|^2;$$

(vi): there exist $\delta > 0$ and $\gamma \in L^\infty(Z)$ with $\gamma(z) \leq 0$ a.e. on Z and the inequality is strict on a set of positive measure, such that

$$\frac{\lambda_k}{2}x^2 \leq j(z,x) \leq \frac{\gamma(z)}{2}x^2 \text{ for a.a. } z \in Z \text{ and all } |x| \leq \delta.$$

Remark 2.1. Hypothesis $H(j)(vi)$ implies that $j(z,0) = 0$ for a.a. $z \in Z$. Also hypothesis $H(j)(v)$ implies that $x \to j(z,x)$ is subquadratic at infinity, while hypothesis $H(j)(iv)$ makes the Euler functional of the problem indefinite and so the PS-condition can not be verified. This creates serious difficulties in the implementation of minimax techniques and in order to overcome them, we develop a nonsmooth variant of the so-called reduction technique, originally due to Amann [1], Castro-Lazer [4] and Thews [14].

The following two potential functions satisfy hypotheses $H(j)$ (for simplicity we drop the z-dependence)

$$j_1(x) = \frac{c_0}{2}\min\{x^2, |x|\} - c_1\frac{x^2}{1+x^2}$$
$$\text{with } 0 < \frac{c_0}{2} < c_1 < \min\{\frac{c_0 - \lambda_k}{2}, \frac{16(\lambda_m - c_0)}{3(6+\sqrt{3})}\}$$
$$\text{and } j_2(x) = \frac{c_3}{2}x^2 - c_4\ln(x^2+1) \text{ with } c_3 < 2c_4.$$

The Euler functional for problem (1.1), is the function $\varphi : H^1(Z) \to \mathbb{R}$ defined by

$$\varphi(x) = \frac{1}{2}\|Dx\|_2^2 - \frac{1}{2}\int_Z ax^2dz - \int_Z j(z,x(z))dz \text{ for all } x \in H^1(Z).$$

We know that φ is Lipschitz continuous on bounded sets, hence locally Lipschitz.

3. AUXILIARY RESULTS

In what follows $Y = H_- \oplus H_0$ (hence $dim Y < \infty$) and if $x \in H^1(Z)$, then $x = \overline{x} + x^0 + \widehat{x}$, $\overline{x} \in H_-$, $x^0 \in H_0$, $\widehat{x} \in H_+$.

Lemma 3.1.

- *(a) If $\beta \in L^\infty(Z)_+$, $\beta(z) \le \lambda_m$ a.e. on Z with strict inequality on a set of positive measure, then there exists $\xi > 0$ such that*
$$\psi_1(\widehat{x}) = \|D\widehat{x}\|_2^2 - \int_Z a\widehat{x}^2 dz - \int_Z \beta\widehat{x}^2 dz \ge \xi\|\widehat{x}\|^2 \text{ for all } \widehat{x} \in H_+.$$
- *(b) If $\gamma \in L^\infty(Z)$, $\gamma(z) \le 0$ a.e. on Z with strict inequality on a set of positive measure, then there exists $\widehat{\xi} > 0$ such that*
$$\psi_2(v) = \|Dv\|_2^2 - \int_Z a v^2 dz - \int_Z \gamma v^2 dz \ge \widehat{\xi}\|v\|^2 \text{ for all } v \in H_0 \oplus H_+.$$

Proof. From the variational characterization of $\lambda_m > 0$, we have $\psi_1 \ge 0$. Suppose that the Lemma is not true. Since ψ_1 is 2-homogeneous, we can find $\{\widehat{x}\}_{n\ge1} \subseteq H_+$ with $\|\widehat{x}_n\| = 1$ such that $\psi_1(\widehat{x}_n) \downarrow 0$. We may assume that
$$\widehat{x}_n \xrightarrow{w} \widehat{x} \text{ in } H^1(Z),\ \widehat{x}_n \xrightarrow{w} \widehat{x} \text{ in } L^{2^*}(Z) \text{ and } \widehat{x}_n(z) \to \widehat{x}(z) \text{ a.e. on } Z.$$

Note that $\int_Z (\widehat{x}_n^2)^{\frac{N}{N-2}} dz = \int_Z |\widehat{x}_n|^{2^*} dz \le M_1 < +\infty$ for all $n \ge 1$. So we infer that
$$\widehat{x}_n^2 \xrightarrow{w} \widehat{x}^2 \text{ in } L^{\frac{N}{N-2}}(Z), \text{ hence } \int_Z a\widehat{x}_n^2 dz \to \int_Z a\widehat{x}^2 dz \text{ (since } \tfrac{2}{N} + \tfrac{N-2}{N} = 1).$$

Moreover, we have
$$\|D\widehat{x}\|_2^2 \le \liminf_{n\to\infty} \|D\widehat{x}_n\|_2^2 \text{ and } \int_Z \beta\widehat{x}_n^2 dz \to \int_Z \beta\widehat{x}^2 dz.$$

So in the limit as $n \to \infty$, we obtain
$$\psi_1(\widehat{x}) = \|D\widehat{x}\|_2^2 - \int_Z a\widehat{x}^2 dz - \int_Z \beta\widehat{x}^2 dz \le 0,\ \widehat{x} \in H_+, \tag{3.1}$$
$$\Rightarrow \|D\widehat{x}\|_2^2 - \int_Z a\widehat{x}^2 dz \le \lambda_m \|\widehat{x}\|_2^2.$$

Since $\widehat{x} \in H_+$, from the variational characterization of $\lambda_m > 0$, we have
$$\|D\widehat{x}\|_2^2 - \int_Z a\widehat{x}^2 dz = \lambda_m \|\widehat{x}\|_2^2 \text{ and } \widehat{x} \in E(\lambda_m) = \text{the eigenspace for } \lambda_m > 0.$$

Note that if $\widehat{x} = 0$, then $\widehat{x}_n \to 0$ in $H^1(Z)$, a contradiction to the fact that $\|\widehat{x}_n\| = 1$ for all $n \ge 1$. So from the unique continuation property, it follows that $\widehat{x}(z) \neq 0$ a.e. on Z. Hence (3.1) implies
$$\|D\widehat{x}\|_2^2 - \int_Z a\widehat{x}^2 dz < \lambda_m \|\widehat{x}\|_2^2,$$
a contradiction to the variational characterization of $\lambda_m > 0$.

(b) The proof is similar, using the orthogonality of the component spaces and since $\|Dx^0\|_2^2 = \int_Z a(x^0)^2 dz$ for all $x^0 \in H_0$. $\square$

Proposition 3.2. If hypotheses $H(j)$ hold, then there exists a continuous map $\eta : Y \to H_+$ such that for every $y \in Y$

$$\min[\varphi(y+\widehat{x}) : \widehat{x} \in H_+] = \varphi(y+\eta(y))$$

and $\eta(y) \in H_+$ is the unique solution of the operator inclusion

$$0 \in p_{H_+^*}\partial\varphi(y+\widehat{x})$$

with $y \in Y$ fixed and $p_{H_+^*}$ the orthogonal projection of $H^1(Z)^*$ onto $H_+^* = (Y^*)^\perp$.

Proof. We fix $y \in Y$ and consider the function $\varphi_y : H^1(Z) \to \mathbb{R}$ defined by $\varphi_y(w) = \varphi(y+w)$ for all $w \in H^1(Z)$. Evidently $\varphi_y(\cdot)$ is locally Lipschitz and for every $w, h \in H^1(Z)$, we have

$$\varphi_y^0(w;h) = \limsup_{\substack{w' \to w \\ \lambda \downarrow 0}} \frac{\varphi(y+w'+\lambda h) - \varphi(y+w')}{\lambda} = \varphi^0(y+w;h),$$

$$\Rightarrow \partial\varphi_y(w) = \partial\varphi(y+w) \text{ for all } w \in H^1(Z). \tag{3.2}$$

Let $i : H_+ \to H^1(Z)$ be the inclusion map and let $\widehat{\varphi}_y : H_+ \to \mathbb{R}$ be defined by $\widehat{\varphi}_y = \varphi_y \circ i$. From the nonsmooth chain rule (see Clarke [], pp.45-46) and since $i^* = p_{H_+^*}$, we have

$$\partial\widehat{\varphi}_y(\widehat{x}) \subseteq p_{H_+^*}\partial\varphi_y(i(x)) = p_{H_+^*}\partial\varphi(y+x^*) \text{ for all} \widehat{x} \in H_+ \text{ (see (3.2))}. \tag{3.3}$$

Let $\langle\cdot,\cdot\rangle$ be the duality brackets for the pair $(H^1(Z), H^1(Z)^*)$ and let $A \in \mathcal{L}(H^1(Z), H^1(Z)^*)$ be defined by $\langle A(x), y\rangle = \int_Z (Dx, Dy)_{\mathbb{R}^N} dz$ for all $x, y \in H^1(Z)$. Clearly A is monotone, coercive and for all $x^* \in \partial\varphi(x)$, we have

$$x^* = A(x) - ax - u, \text{ with } u \in L^{r'}(Z) \ (\frac{1}{r} + \frac{1}{r'} = 1),\ u(z) \in \partial j(z, x(z)) \text{ a.e. on } Z \tag{3.4}$$

(since Clarke [6], p. 83). Let $\langle\cdot,\cdot\rangle_{H_+}$ denote the duality brackets for the pair (H_+, H_+^*). Then for any $\widehat{x}_1, \widehat{x}_2 \in H_+$ and any $x_1^* \in \partial\widehat{\varphi}_y(\widehat{x}_1)$, $x_2^* \in \partial\widehat{\varphi}_y(\widehat{x}_2)$, using (3.4), hypothesis $H(j)(v)$ and Lemma 3.1(a), we have

$$\langle x_1^* - x_2^*, \widehat{x}_1 - \widehat{x}_2\rangle_{H_+} \geq \|D(\widehat{x}_1 - \widehat{x}_2)\|_2^2 - \int_Z a(\widehat{x}_1 - \widehat{x}_2)^2 dz$$

$$- \int_Z \beta(\widehat{x}_1 - \widehat{x}_2)^2 dz \geq c_5\|\widehat{x}_1 - \widehat{x}_2\|^2,\ c_5 > 0,$$

$$\Rightarrow \widehat{x} \to \widehat{\varphi}_y(\widehat{x}) - \frac{c_5}{2}\|\widehat{x}\|^2, \text{ is convex on } H_+. \tag{3.5}$$

In addition, if $x^* \in \partial\varphi_y(\widehat{x})$ and $y^* \in \partial\varphi_y(0)$, we have

$$\langle x^*, \widehat{x}\rangle_{H_+} = \langle x^* - y^*, \widehat{x}\rangle_{H_+} + \langle y^*, \widehat{x}\rangle_{H_+} \geq c_5\|\widehat{x}\|^2 - c_6\|\widehat{x}\|,\ c_6 > 0,$$
$$\Rightarrow \widehat{x} \to \partial\varphi_y(\widehat{x}) \ \text{ is coercive on } H_+.$$

Also because of (3.5), $\partial\varphi(\cdot)$ is maximal monotone on H_+. A maximal monotone, coercive map is surjective (see Gasinski-Papageorgiou [7], p.320). Thus we can find $\widehat{x}_0 \in H_+$ such that $0 \in \partial\widehat{\varphi}_y(\widehat{x}_0)$ and so $\widehat{x}_0$ is a minimizer of $\widehat{\varphi}_y$ (since $\widehat{\varphi}_y$ is convex) and in fact a unique minimizer, due to the strong convexity of $\widehat{\varphi}_y(\cdot)$ (see (3.5)). So we can define the map $\eta : Y \to H_+$ which to a given $y \in Y$ assigns the unique solution $\widehat{x}_0 = \widehat{x}_0(y) \in H_+$ of the problem $\min[\varphi(y+\widehat{x}) : \widehat{x} \in H_+]$. Using (3.3) we have

$$0 \in \partial\widehat{\varphi}_y(\widehat{x}_0) \subseteq p_{H_+^*}\partial\varphi(y+\eta(y))$$
$$\text{and } \ \varphi(y+\eta(y)) = \min[\varphi(y+\widehat{x}) : \widehat{x} \in H_+].$$

We need to show that η is continuous. To this end, suppose $y_n \to y$ in Y. First we show that $\{\eta(y_n)\}_{n\geq 1} \subseteq H_+$ is bounded. For some $M_2 > 0$ and all $n \geq 1$, we have

$$\begin{aligned} &\frac{1}{2}\|D(y_n+\eta(y_n))\|_2^2 - \frac{1}{2}\int_Z a(y_n+\eta(y_n))^2dz - \int_Z j(z, y_n+\eta(y_n))dz \\ &\qquad = \varphi(y_n+\eta(y_n)) \leq \varphi(y_n) \leq M_2. \end{aligned} \tag{3.6}$$

From Rademacher's theorem, we know that for a.a. $z \in Z$ and a.a. $\tau \in \mathbb{R}$, $\frac{d}{d\tau}j(z,\tau)$ exists and belongs in $\partial j(z,\tau)$. So using the nonsmooth chain rule and hypotheses $H(j)(v)$ and (iii), we have

$$\begin{aligned} j(z,(y_n+\eta(y_n))(z)) &= j(z,y_n(z)) + \int_0^1 \frac{d}{d\tau}j(z,(y_n+\tau\eta(y_n))(z))dz \\ &\leq j(z,y_n(z)) + \frac{\beta(z)}{2}\eta(y_n)(z)^2 + c_7|\eta(y_n)(z)| \\ &\qquad \text{for a.a. } z \in Z, \ \text{ with } \ c_7 > 0. \end{aligned} \tag{3.7}$$

Using (3.7) in (3.6) and exploiting the orthogonality of the component spaces, we obtain

$$\begin{aligned} M_2 &\geq \frac{1}{2}\|D\eta(y_n)\|_2^2 - \frac{1}{2}\int_Z a\eta(y_n)^2dz - \frac{1}{2}\int_Z \beta\eta(y_n)^2dz \\ &\qquad - c_7\int_Z |\eta(y_n)|dz - c_8,\ c_8 > 0, \\ &\geq \frac{\xi}{2}\|\eta(y_n)\|^2 - c_9\|\eta(y_n)\| - c_8,\ c_9 > 0 \ \text{ (see Lemma 3.1(a))}, \\ &\Rightarrow \{\eta(y_n)\}_{n\geq 1} \subseteq H_+ \ \text{ is bounded.} \end{aligned}$$

Therefore we may assume that $\eta(y_n) \xrightarrow{w} w$ in $H^1(Z)$. Because of the weak lower semicontinuity of φ, we have

$$\varphi(y+w) \leq \liminf_{n\to\infty} \varphi(y_n+\eta(y_n)). \tag{3.8}$$

From the definition of η, we also have

$$\limsup_{n\to\infty} \varphi(y_n + \eta(y_n)) \leq \varphi(y + \widehat{x}) \ \text{ for all } \ \widehat{x} \in H_+. \tag{3.9}$$

Combining (3.8) (3.9), we see that $w = \eta(y)$. So for the original sequence, we have

$$\eta(y_n) \xrightarrow{w} \eta(y) \ \text{ in } \ H^1(Z) \ \text{ and } \ \eta(y_n) \to \eta(y) \ \text{ in } \ L^2(Z). \tag{3.10}$$

Recall that $0 \in p_{H_+^*}\partial\varphi(y_n + \eta(y_n))$ and so $p_{H_+^*}A(y_n + \eta(y_n)) - p_{H_+^*}a(y_n + \eta(y_n)) = p_{H_+^*}u_n$ with $u_n \in L^{r'}(Z)$, $u_n(z) \in \partial j(z, (y_n + \eta(y_n))(z))$ a.e. on Z, $n \geq 1$ (see (3.3)). Hence

$$\begin{aligned} &\langle A(y_n + \eta(y_n)), \eta(y_n) - \eta(y)\rangle \\ &= \int_Z u_n(\eta(y_n) - \eta(y))dz + \int_Z a(y_n + \eta(y_n))(\eta(y_n) - \eta(y))dz \to 0 \end{aligned}$$

and $A(y_n + \eta(y_n)) \xrightarrow{w} A(y + \eta(y))$ in $H^1(Z)^*$ (see (3.10)). Therefore

$$\begin{aligned} &\langle A(y_n + \eta(y_n)), \eta(y_n)\rangle \to \langle A(y + \eta(y)), \eta(y)\rangle, \\ &\Rightarrow \|D(y_n + \eta(y_n))\|_2 \to \|D(y + \eta(y))\|_2. \end{aligned}$$

Since $D(y_n + \eta(y_n)) \xrightarrow{w} D(y + \eta(y))$ in $L^2(Z, \mathbb{R}^{\mathbb{N}})$, from the Kadec-Klee property, we have

$$\begin{aligned} &D(y_n + \eta(y_n)) \to D(y + \eta(y)) \ \text{ in } \ L^2(Z, \mathbb{R}^{\mathbb{N}}), \\ &\qquad \Rightarrow \eta(y_n) \to \eta(y) \ \text{ in } \ H^1(Z) \text{ (see (3.10)), i.e. } \eta \text{ is continuous.} \end{aligned}$$

□

Let $\overline{\varphi} : Y \to \mathbb{R}$ be defined by $\overline{\varphi}(y) = \varphi(y + \eta(y))$. It is easy to see that $\overline{\varphi}$ is locally Lispchitz.

Proposition 3.3. If hypotheses $H(j)$ hold, then $\partial\overline{\varphi}(y) \subseteq p_{Y^*}\partial\varphi(y + \eta(y))$ for all $y \in Y$, with p_{Y^*} the orthogonal projection of $H^1(Z)^*$ onto $Y^* = (H_+^*)^\perp$.

Proof. For all $y, h \in Y$, we have

$$\overline{\varphi}^0(y; h) = \limsup_{\substack{y' \to y \\ \lambda \downarrow 0}} \frac{\varphi(y' + \lambda h + \eta(y' + \lambda h))) - \varphi(y' + \eta(y'))}{\lambda}.$$

$$\leq \limsup_{\substack{y' \to y \\ \lambda \downarrow 0}} \frac{\varphi(y' + \lambda h + \eta(y'))) - \varphi(y' + \eta(y'))}{\lambda} = \varphi^0(y + \eta(y); h) \tag{3.11}$$

(since η is continuous by Proposition 3.2). Let $i_0 : Y \to H^1(Z)$ be the inclusion map and let $\langle \cdot, \cdot \rangle_Y$ denote the duality brackets for the pair (Y, Y^*). We know $i_0^* = p_{Y^*}$. So if $x_0^* \in \partial\overline{\varphi}(y)$, using (3.11), we have

$$\begin{aligned} \langle x_0^*, h\rangle_Y &\leq \overline{\varphi}^0(y;h) \leq \varphi^0(y+\eta(y);h) \\ &= \sup[\langle p_{Y^*}(x^*), h\rangle_Y : x^* \in \partial\varphi(y+\eta(y))] \text{ for all } h \in Y, \\ \Rightarrow & \partial\overline{\varphi}(y) \subseteq p_{Y^*}\partial(y+\eta(y)) \text{ for all } y \in Y. \end{aligned}$$

□

We set $\psi = -\overline{\varphi} : Y \to \mathbb{R}$ and show that ψ satisfies the local linking geometry (see Gasinski-Papageorgiou [7], p. 661).

Proposition 3.4. If hypotheses $H(j)$ hold, then we can find $p > 0$ such that

$$\begin{cases} \psi(x^0) \leq 0 & \text{if } x^0 \in H_0, \|x^0\| \leq p \\ \psi(\overline{x}) \geq 0 & \text{if } \overline{x} \in H_-, \|\overline{x}\| \leq p \end{cases}.$$

Proof. From hypotheses $H(j)(iii)$ and (iv), we obtain

(3.12)
$$j(z,x) \leq \frac{\gamma(z)}{2}x^2 + c_{10}|x|^\theta \text{ for a.a. } z \in Z, \text{ all } x \in \mathbb{R} \text{ and with } c_{10} > 0,\ 2 < \theta < 2^*.$$

Then using (3.12) and Lemma 3.1(b), for all $x^0 \in H_0$ we have

(3.13)
$$\psi(x^0) = -\overline{\varphi}(x^0) = -\varphi(x^0+\eta(x^0)) \leq -\frac{\widehat{\xi}}{2}\|x^0+\eta(x^0)\|^2 + c_{11}\|x^0+\eta(x^0)\|^\theta,\ c_{11} > 0.$$

Hypothesis $H(j)(vi)$ implies that $x = 0$ is a local maximizer of $j(z, \cdot)$ for a.a. $z \in Z$, hence $0 \in \partial j(z, 0)$ a.e. on Z. So from hypothesis $H(j)(v)$ we have $ux \leq \beta(z)x^2$ for a.a. $z \in Z$, all $x \in \mathbb{R}$ and all $u \in \partial j(z,x)$. From this it follows that $j(z,x) \leq \frac{\beta(z)}{2}x^2$ for a.a. $z \in Z$, all $x \in \mathbb{R}$. Then because of Lemma 3.1(a), we have

$$\varphi(\widehat{x}) \geq \frac{\xi}{2}\|\widehat{x}\|^2 \text{ for all } \widehat{x} \in H_+, \text{ i.e. } \inf_{H_+}\varphi = 0 \text{ and so } \eta(0) = 0.$$

Therefore from (3.13), the continuity of η and since $\theta > 2$, we can find $p_1 > 0$ small such that

$$\psi(x^0) \leq 0 \text{ for all } x^0 \in H_0,\ \|x^0\| \leq p_1.$$

Also, if $\dim H_- \neq 0$, then due to the finite dimensionality of $H_- \subseteq C(\overline{Z})$ all norms are equivalent. Hence $\|\overline{x}\|_\infty \leq c_{12}\|\overline{x}\|$ for all $\overline{x} \in H_-$, $c_{12} > 0$. Therefore if $p_2 = \frac{\delta}{c_{12}}$ and $\|\overline{x}\| \leq p_2$, $\overline{x} \in H_-$, then $|\overline{x}(z)| \leq \delta$ for all $z \in \overline{Z}$ and so using hypothesis $H(j)(vi)$ and the variational characterization of $\lambda_k < 0$, we have

$$\psi(\overline{x}) \geq 0 \text{ for all } \overline{x} \in H_-, \|\overline{x}\| \leq p_2.$$

Finally take $p = \min\{p_1, p_2\}$. □

Lemma 3.5. *Given $\varepsilon > 0$, we can find $\mu_\varepsilon > 0$ such that $|\{z \in Z : |x^0(z)| \leq \mu_\varepsilon \|x^0\|\}|_N < \varepsilon$ for all $x^0 \in H_0$.*

Proof. We may assume that $\dim H_0 \neq 0$. Suppose the lemma is not true. Then we can find $\varepsilon > 0$ and a sequence $\{x_n^0\}_{n\geq 1} \subseteq H_0$ such that $|D_n|_N = |\{z \in Z : |x_n^0(z)| < \frac{1}{n}\|x_n^0\|\}|_N \geq \varepsilon$ for all $n \geq 1$. We set $y_n^0 = \frac{x_n^0}{\|x_n^0\|} \in H_0$, $n \geq 1$. Since $\dim H_0 < +\infty$, we may assume that $y_n^0 \to y^0$ in $H^1(Z)$, hence $\|y^0\| = 1$. Let $\widehat{D}_0 = \{z \in Z : y^0(z) = 0\}$. We have $\limsup\limits_{n\to\infty} D_n \subseteq \widehat{D}_0$ and so $0 < \varepsilon \leq \limsup\limits_{n\to\infty} |D_n|_N \leq |\limsup\limits_{n\to\infty} D_n|_N \leq |\widehat{D}_0|_N$. But since $y_0 \in H_0$, $y_0 \neq 0$, from the unique continuation property, we have $y^0(z) \neq 0$ a.e. on Z and so $|\widehat{D}_0|_N = 0$, a contradiction. $\square$

Using this lemma, we can show the coercivity of ψ.

Proposition 3.6. If hypotheses $H(j)$ hold, then $\psi : Y \to \mathbb{R}$ is coercive

Proof. Since $\psi \geq -\varphi|_Y$, it suffices to show that $-\varphi|_Y$ is coercive. Suppose not. We can find $\{y_n\}_{n\geq 1} \subseteq Y$ and $M_3 > 0$ such that $-M_3 \leq \varphi(y_n)$ for all $n \geq 1$ and $\|y_n\| \to \infty$. Because of hypothesis $H(j)(iv)$ and Tang-Wu [13], given $\varepsilon > 0$, we can find $D_\varepsilon \subseteq C$ measurable with $|C \backslash D_\varepsilon|_N < \varepsilon$ such that

$$j(z, x) \to +\infty \text{ uniformly for all } z \in D_\varepsilon \text{ as } |x| \to \infty.$$

Hence from Tang-Wu [13], we have

$$j(z, x) \geq G(x) - h(z) \text{ for a.a. } z \in D_\varepsilon \text{ and all } x \in \mathbb{R}, \tag{3.14}$$

where $G \in C(\mathbb{R})$, $G \geq 0$, G is subadditive, coercive, $G(x) \leq 4 + |x|$ for all $x \in \mathbb{R}$ and $h \in L^1(Z)_+$.

We write $y_n = y_n^0 + \overline{y}_n$, $y_n^0 \in H_0, \overline{y}_n \in H_-$, $n \geq 1$. Then using the orthogonality of the component spaces, hypothesis $H(j)(iv)$ and (3.14), we have

$$\begin{aligned} -M_3 \leq \varphi(y_n) &\leq -c_{13}\|\overline{y}_n\|^2 - \int_{D_\varepsilon} G(y_n(z))dz + \|h\|_1 + \|h_0\|_1 \\ &\leq -c_{13}\|\overline{y}_n\|^2 + c_{14}, \quad c_{13}, c_{14} > 0 \\ &\Rightarrow \{\overline{y}_n\}_{n\geq 1} \subseteq H_- \text{ is bounded.} \end{aligned} \tag{3.15}$$

Since $\|y_n\| \leq \|y_n^0\| + \|\overline{y}_n\|$ and $\|y_n\| \to \infty$, it follows that $\|y_n^0\| \to \infty$. Fix $\delta > 0$. From Lemma 3.5, we can find $\mu_{\delta_0} > 0$ such that $|\{z \in Z : |x^0(z)| < \mu_{\delta_0}\|x^0\|\}|_N < \delta_0$ for all $x^0 \in H_0$. For every $n \geq 1$, we set $S_n = \{z \in Z : |y_n^0(z)| \geq \mu_{\delta_0}\|y_n^0\|\}$. We know that $|Z \backslash S_n|_N < \delta_0$ for all $n \geq 1$. Since $\dim H_- < \infty$ and $\{\overline{y}_n\}_{n\geq 1} \subseteq H_- \subseteq C(\overline{Z})$, we can find $c_{14} > 0$ such that $\|\overline{y}_n\|_\infty \leq c_{14}$ for all $n \geq 1$. Recall that G is coercive. So given $\omega > 0$, we can find $M_0 = M_0(\omega) > 0$ such that $G(x) \geq \omega$ for all $|x| \geq M_0$. Let $E_n = \{z \in Z : |y_n(z)| \geq M_0\}$, $n \geq 1$. Note that $|y_n(z)| \geq |y_n^0(z)| - |\overline{y}_n(z)| \geq$

$\mu_{\delta_0}\|y_n^0\| - c_{14}$ for all $z \in S_n$ and all $n \geq 1$. Since $\|y_n^0\| \to \infty$, it follows that $|y_n(z)| \geq M_0$ for all $z \in S_n$ and all $n \geq n_0$, hence $S_n \subseteq E_n$ for all $n \geq n_0$. Then

$$\begin{aligned}\int_{D_\varepsilon} G(y_n(z))dz &= \int_{D_\varepsilon \cap E_n} G(y_n(z))dz + \int_{D_\varepsilon \setminus E_n} G(y_n(z))dz \\ &\geq \int_{D_\varepsilon \cap E_n} G(y_n(z))dz \ \ (\text{since } G \geq 0) \\ &\geq \omega|D_\varepsilon \cap E_n|_N \geq \omega|D_\varepsilon \cap S_n|_N \ \text{ for all } \ n \geq n_0. \end{aligned} \tag{3.16}$$

But $|D_\varepsilon \cap S_n|_N = |D_\varepsilon|_N - |D_\varepsilon \setminus S_n|_N \geq |C|_N - \varepsilon - \delta_0 > 0$ for $\varepsilon, \delta_0 > 0$ small. Hence from (3.16) and since $\omega > 0$ was arbitrary, we conclude that $\lim\limits_{x\to\infty} G(y_n(z))dz = +\infty$. Returning to (3.15) and passing to the limit as $n \to \infty$, we reach a contradiction. □

4. MULTIPLICITY THEOREM

Using the auxiliary results of Section 3 we can state and prove the multiplicity theorem.

Theorem 4.1. *If hypotheses $H(j)$ hold and $dim H_0 \neq 0$, then problem (1.1) has at least two nontrivial solutions $x_1, x_2 \in C^1(\overline{Z})$.*

Proof. We have $\psi(0) = -\overline{\varphi}(0) = -\varphi(0 + \eta(0)) = -\varphi(0) = 0$ and so $\inf\limits_Y \psi \leq 0$.

If $\inf\limits_Y \psi = 0$, then by virtue of Proposition 3.4, all $x^0 \in H_0$ with $\|x^0\| \leq p$ are minimizers of ψ and so ψ has a continuum of nontrivial critical points.

If $\inf\limits_Y \psi < 0$, then because of Proposition 3.6 and since $Y = H_0 \oplus H_-$ is finite dimensional, we infer that ψ is bounded below and satisfies the nonsmooth PS-condition (see Chang [5]). So we can apply the nonsmooth local linking theorem of Kandilakis-Kourogenis-Papageorgiou [8] and deduce that ψ has at least two nontrivial critical points.

So we see that in any case, ψ has at least two nontrivial critical points. Let $y \in Y$, $y \neq 0$, be a critical point of ψ. From Propositions 3.2 and 3.3 we have

$$\begin{aligned} &0 \in \partial\widehat{\varphi}_y(\eta(y)) \subseteq p_{H_+^*}\partial\varphi(y + \eta(y)) \\ \text{and } \ &0 \in \partial\psi(y) = -\partial\overline{\varphi}(y) \subseteq p_{Y^*}\partial\varphi(y + \eta(y)), \\ \Rightarrow &0 \in \partial\varphi(y + \eta(y)) \ \ (\text{since } H^1(Z)^* = Y^* \oplus H_+^*), \\ &\text{i.e. } \ y + \eta(y) \text{ is a nontrivial critical point of } \varphi. \end{aligned}$$

Therefore, if $y_1, y_2 \in Y$ are the nontrivial critical points of ψ, then $x_1 = y_1 + \eta(y_1)$, $x_2 = y_2 + \eta(y_2)$ are two nontrivial critical points of φ, hence two nontrivial solutions of (1.1). Standard regularity theory (see for example Gasinski-Papageorgiou [7]), implies $x_1, x_2 \in C^1(\overline{Z})$. □

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