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# NONOCCURRENCE OF THE LAVRENTIEV PHENOMENON FOR MANY INFINITE DIMENSIONAL LINEAR CONTROL PROBLEMS WITH NONCONVEX INTEGRANDS

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**ABSTRACT.** In this paper we establish nonoccurrence of gap for two large classes of infinite-dimensional linear control systems in a Hilbert space with nonconvex integrands. These classes are identified with the corresponding complete metric spaces of integrands which satisfy a growth condition common in the literature. For most elements of the first space of integrands (in the sense of Baire category) we establish the existence of a minimizing sequence of trajectory-control pairs with bounded controls. We also establish that for most elements of the second space (in the sense of Baire category) the infimum on the full admissible class of trajectory-control pairs is equal to the infimum on a subclass of trajectory-control pairs whose controls are bounded by a certain constant.

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## 1. INTRODUCTION

In this paper we consider two large classes of optimal linear control systems in infinite-dimensional Hilbert spaces with nonconvex integrands. These classes are identified with the corresponding complete metric spaces of integrands which satisfy a growth condition common in the literature. For most elements of the first space of integrands (in the sense of Baire category) we establish the existence of a minimizing sequence of trajectory-control pairs with bounded controls. We also establish that for most elements of the second space (in the sense of Baire category) the infimum on the full admissible class of trajectory-control pairs is equal to the infimum on a subclass of trajectory-control pairs whose controls are bounded by a certain constant. The results of the paper show that for these classes of integrands the Lavrentiev phenomenon does not occur for most elements.

The Lavrentiev phenomenon in the calculus of variations was discovered in 1926 by M. Lavrentiev in [11]. There it was shown that it is possible for the variational integral of a two-point Lagrange problem, which is sequentially weakly lower semi-continuous on the admissible class of absolutely continuous functions, to possess an infimum on the dense subclass of $C^1$ admissible functions that is strictly greater than

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its minimum value on the admissible class. Since this seminal work the Lavrentiev phenomenon is of great interest. See, for instance, [1, 2, 4, 5, 7-10, 12-14, 16-19, 21, 22] and the references mentioned there. Mania [13] simplified the original example of Lavrentiev. Ball and Mizel [4, 5] demonstrated that the Lavrentiev phenomenon can occur with fully regular integrands. Sarychev [16] constructed a broad class of integrands that exhibit the Lavrentiev phenomenon. Nonoccurrence of the Lavrentiev phenomenon in the calculus of variations was studied in [1, 2, 8-10, 12, 17, 18, 19, 22]. Clarke and Vinter [8] showed that the Lavrentiev phenomenon cannot occur when a variational integrand $f(t,x,u)$ is independent of $t$. Sychev and Mizel [18] considered a class of integrands $f(t,x,u)$ which are convex with respect to the last variable. For this class of integrands they established that the Lavrentiev phenomenon does not occur. Sarychev and Torres [17] studied a class of optimal control problems with control-affine dynamics and with continuously differentiable integrands $f(t,x,u)$. For this class of problems they established Lipschitzian regularity of minimizers which implies nonoccurrence of the Lavrentiev phenomenon. Ferriero [10] showed that the Lavrentiev phenomenon cannot occur for a class of higher-order variational problems with integrands which are convex with respect to the last variable. In [22] we studied nonoccurrence of the Lavrentiev phenomenon for a large class of nonconvex optimal control problems with integrands which belong to a complete metric space of functions. We established that for most problems (integrands) in the sense of Baire category the Lavrentiev phenomenon does not occur [22]. It should be mentioned that in [22] we consider optimal control problems with the unconstrained state variable $x$ which belongs to a Banach space $E$, with the constrained control variable $u$ which belongs to a Banach space $F$ and with the right-hand side of differential equations determined by a continuous mapping $G : [T_1, T_2] \times E \times F \to E$ which satisfies Lipschitzian conditions with respect to $x$ and with respect to $u$. In this paper we extend the main results of [22] to infinite-dimensional constrained optimal linear control systems with nonconvex nonautonomous integrands. More precisely, here we consider optimal control problems with the constrained state variable $x$ which belongs to a Hilbert space $X$, with the constrained control variable $u$ which belongs to a normed space $Y$ and with the differential equation $x' = Ax + Bu$, where $A$ is a given possible unbounded closed and densely defined operator in $X$ which is a generator of a strongly continuous semigroup $\{S(t) : \ t \in [0, \infty)\}$ on $X$ and $B : Y \to X$ is a bounded linear operator.

In the sequel we say that a property of elements of a complete metric space $Z$ is generic (typical) in $Z$ if the set of all elements of $Z$ which possess this property contains an everywhere dense $G_\delta$ subset of $Z$. In this case we also say that the property holds for a generic (typical) element of $Z$ or that a generic (typical) element of $Z$ possesses the property [3, 15, 20]. Our results show that the Lavrentiev phenomenon does not occur for a generic element of large classes of constrained optimal linear control systems.

Assume that $(Z, ||\cdot||)$ is a Banach space. Let $-\infty < \tau_1 < \tau_2 < \infty$. Denote by $W^{1,1}(\tau_1, \tau_2; Z)$ the set of all functions $x : [\tau_1, \tau_2] \to Z$ for which there exists a Bochner integrable function $u : [\tau_1, \tau_2] \to Z$ such that

$$x(t) = x(\tau_1) + \int_{\tau_1}^{t} u(s)ds,\ t \in (\tau_1, \tau_2]$$

(see, e.g., [6]). It is known that if $x \in W^{1,1}(\tau_1, \tau_2; Z)$, then this equation defines a unique Bochner integrable function $u$ which is called the derivative of $x$ and is denoted by $x'$.

We denote by $\mathrm{mes}(\Omega)$ the Lebesgue measure of a Lebesgue measurable set $\Omega \subset R^1$ and denote by $\mathrm{int}(E)$ the interior of a set $E \subset Z$ in the norm topology.

For each $x \in Z$ and each $r > 0$ set

$$B_Z(x, r) = \{y \in Z :\ ||y - x|| \le r\} \text{ and } B_Z(r) = B_Z(0, r).$$

Let $a, b \in R^1$ satisfy $a < b$, $X$ be a Hilbert space equipped with a scalar (inner) product $< x, y >$, $x, y \in X$ and with the norm induced by the scalar (inner) product and let $(Y, ||\cdot||)$ be a norm space.

Let $A$ be a given possible unbounded closed and densely defined operator in $X$ which is a generator of a strongly continuous semigroup $\{S(t) :\ \ t \in [0, \infty)\}$ on $X$ and let $B : Y \to X$ be a bounded linear operator.

In the sequel we assume that $H$ is a convex subset of $X$ with the nonempty interior $\mathrm{int}(H)$ and that for each $t \in [a, b]$ a set $U(t)$ is a nonempty convex subset of $Y$.

We consider the following optimal linear control problem

$$(P) \qquad \int_a^b f(t, x(t), u(t))dt \to \min,$$

$$x'(t) = Ax(t) + Bu(t),\ t \in [a, b] \text{ almost everywhere (a.e.)},$$

$$x(0) = z_0 \text{ and } x(t) \in H,\ t \in [a, b],$$

$$u(t) \in U(t),\ t \in [a, b] \text{ almost everywhere},$$

where $z_0 \in X$, $x \in W^{1,1}(a, b; X)$, $u : [a, b] \to Y$ is a Bochner integrable function and an integrand $f : [a, b] \times X \times Y \to R^1$ satisfies the conditions stated below. Here $x(\cdot)$ is the mild solution of the equation

$$x'(t) = Ax(t) + Bu(t),\ t \in [a, b] \text{ almost everywhere}.$$

Namely

$$(1.1) \qquad x(t) = S(t)z_0 + \int_a^t S(t - s)Bu(s)ds \text{ for all } t \in [a, b].$$

A function $x \in W^{1,1}(a,b;X)$ is called a trajectory if there exists a Bochner integrable function $u : [a,b] \to Y$ (referred to as a control) such that the pair $(x,u)$ satisfies (1.1) with $z_0 = x(a)$ and

$$x(t) \in H \text{ for all } t \in [a,b] \text{ and } u(t) \in U(t),\ t \in [a,b] \text{ (a.e.)}. \tag{1.2}$$

For each $z \in H$ denote by $\mathcal{A}(z)$ the set of all trajectory-control pairs $(x,u)$ satisfying $x(a) = z$ and denote by $\mathcal{A}_L(z)$ the set of all trajectory-control pairs $(x,u) \in \mathcal{A}(z)$ for which there is $M_u > 0$ such that

$$||u(t)|| \le M_u \text{ for almost every } t \in [a,b].$$

Let $\phi : [0,\infty) \to [0,\infty)$ be an increasing function such that

$$\lim_{t\to\infty} \phi(t)t^{-1} = \infty. \tag{1.3}$$

In [21] we studied the problem (P) with $H = X$ and $U(t) = Y$, $t \in [a,b]$ for two classes of integrands. The first class of integrands considered in [21] contains the set of all functions $f : [a,b] \times X \times Y \to R^1$ which satisfy the following assumptions.

(A1) $f$ is measurable with respect to the $\sigma$-algebra generated by products of Lebesgue measurable subsets of $[a,b]$ and Borel subsets of $X \times Y$.

(A2) $f(t,x,u) \ge \phi(||u||)$ for all $(t,x,u) \in [a,b] \times X \times Y$.

(A3) The set $\{f(t,0,0) :\ t \in [a,b]\}$ is bounded from above.

(A4) For each $M, \epsilon > 0$ there exists $\delta > 0$ such that for every $t \in [a,b]$

$$|f(t,x_1,u_1) - f(t,x_2,u_2)| \le \epsilon$$

for each $x_1, x_2 \in B_X(M)$ and each $u_1, u_2 \in B_Y(M)$ satisfying $||x_1 - x_2||,\ ||u_1 - u_2|| \le \delta$.

(A5) For each $M, \epsilon > 0$ there exist $\Gamma, \delta > 0$ such that for every $t \in [a,b]$

$$|f(t,x_1,u) - f(t,x_2,u)| \le \epsilon \max\{|f(t,x_1,u)|, |f(t,x_2,u)|\}$$

for each $u \in Y$ satisfying $||u|| \ge \Gamma$ and each $x_1, x_2 \in B_X(M)$ satisfying $||x_1 - x_2|| \le \delta$.

In [21] we showed that if an integrand $f$ belongs to this class of functions, $z_0 \in X$, $H = X$ and $U(t) = Y$ for all $t \in [a,b]$, then for the problem (P) there exist a minimizing sequence of trajectory-control pairs $\{(x_i,u_i)\}_{i=1}^{\infty}$ and a sequence of positive numbers $\{M_i\}_{i=1}^{\infty}$ such that for each integer $i \ge 1$

$$||u_i(t)|| \le M_i,\ t \in [a,b] \text{ (a.e.)}$$

The second class of integrands studied in [21] contains all integrands $f : [a,b] \times X \times Y \to [0,\infty)$ which satisfy the following assumptions.

(B1) $f$ is measurable with respect to the $\sigma$-algebra generated by products of Lebesgue measurable subsets of $[a,b]$ and Borel subsets of $X \times Y$.

(B2) $f(t,x,u) \ge \phi(||u||)$ for all $(t,x,u) \in [a,b] \times X \times Y$.

(B3) The set $\{f(t,0,0): t \in [a,b]\}$ is bounded from above.

(B4) For each $M > 0$ there exists $L > 0$ such that for each $t \in [a,b]$, each $x_1, x_2 \in B_X(M)$ and each $u_1, u_2 \in B_Y(M)$ the following inequality holds:

$$|f(t,x_1,u_1) - f(t,x_2,u_2)| \le L(||x_1 - x_2|| + ||u_1 - u_2||).$$

(B5) For each $M > 0$ there exist $\delta, L > 0$ and an integrable scalar function $\psi_M(t) \ge 0$, $t \in [a,b]$ such that for each $t \in [a,b]$, each $u \in Y$ and each $x_1, x_2 \in B_X(M)$ satisfying $||x_1 - x_2|| \le \delta$ the inequality $|f(t,x_1,u) - f(t,x_2,u)| \le ||x_1 - x_2||L(f(t,x_1,u) + \psi_M(t))$ holds.

In [21] we showed that if an integrand $f$ belongs to this class of functions, $z_0 \in X$, $H = X$ and $U(t) = Y$, $t \in [a,b]$, then for the optimal linear control problem (P) there exist a minimizing sequence of trajectory-control pairs $\{(x_i, u_i)\}_{i=1}^{\infty}$ and a positive number $M$ such that for each integer $i \ge 1$

$$||u_i(t)|| \le M,\ t \in [a,b] \text{ (a,e.)}.$$

One of our goals in this paper is to extend the results of [21] obtained for unconstrained optimal linear control problems (with $H = X$ and $U(t) = Y$, $t \in [a,b]$) to the class of constrained linear control problems (P).

Our second goal is to answer the question if the extensions of the results of [21] hold for constrained linear control problems (P) with many integrands. In order to meet this goal we introduce the following spaces of integrands.

Denote by $\mathcal{M}_A$ the set of all functions $f : [a,b] \times X \times Y \to R^1$ which satisfy assumptions (A1)-(A4). We equip the set $\mathcal{M}_A$ with the uniformity determined by the base

$$\begin{gathered}\mathcal{E}_{As}(N,\epsilon) = \{(f,g) \in \mathcal{M}_A \times \mathcal{M}_A : \\ |g(t,x,u) - f(t,x,u)| \le \epsilon \text{ for all } (t,x,u) \in [a,b] \times B_X(N) \times B_Y(N)\} \\ \cap\{(f,g) \in \mathcal{M}_A \times \mathcal{M}_A :\ |(f-g)(t,x_1,u_1) - (f-g)(t,x_2,u_2)| \\ \le \epsilon(||x_1 - x_2|| + ||u_1 - u_2||) \\ \text{for each } t \in [a,b],\ \text{each } x_1, x_2 \in B_X(N) \text{ and each } u_1, u_2 \in B_Y(N)\}, \end{gathered} \tag{1.4}$$

where $N, \epsilon > 0$. Clearly, the space $\mathcal{M}_A$ with this uniformity is metrizable and complete. We equip the space $\mathcal{M}_A$ with the topology induced by this uniformity. This topology will be called the strong topology of $\mathcal{M}_A$.

We also equip the space $\mathcal{M}_A$ with the uniformity determined by the following base:

$$\begin{gathered}\mathcal{E}_{Aw}(N,\epsilon) = \{(f,g) \in \mathcal{M}_A \times \mathcal{M}_A :\ |g(t,x,u) - f(t,x,u)| \le \epsilon \\ \text{for all } (t,x,u) \in [a,b] \times B_X(N) \times B_Y(N)\}, \end{gathered} \tag{1.5}$$

where $N, \epsilon > 0$. Clearly, the space $\mathcal{M}_A$ with this uniformity is metrizable and complete. We equip the space $\mathcal{M}_A$ with the topology induced by this uniformity. This topology will be called the weak topology of $\mathcal{M}_A$. Denote by $\mathcal{L}_A$ the set of all functions $f \in \mathcal{M}_A$ which satisfy (A5). Clearly, for each $f \in \mathcal{M}_A$ and each trajectory-control pair $(x, u)$ the function $f(t, x(t), u(t))$, $t \in [a, b]$ is Lebesgue measurable.

Now we define the second space of integrands. Denote by $\mathcal{M}_B$ the set of all functions $f : [a, b] \times X \times Y \to R^1$ which satisfy assumptions (B1)-(B4). Clearly, $\mathcal{M}_B \subset \mathcal{M}_A$.

We equip the set $\mathcal{M}_B$ with the uniformity determined by the base

$$\mathcal{E}_B(N, \epsilon) = \{(f, g) \in \mathcal{M}_B \times \mathcal{M}_B : \ |g(t, x, u) - f(t, x, u)| \le \epsilon$$

$$\text{for all } (t, x, u) \in [a, b] \times B_X(N) \times B_Y(N)\}$$

$$\cap\{(f, g) \in \mathcal{M}_B \times \mathcal{M}_B : \ |(f - g)(t, x_1, u_1) - (f - g)(t, x_2, u_2)|$$

$$\le \epsilon(||x_1 - x_2|| + ||u_1 - u_2||)$$

$$\text{for each } t \in [a, b], \ \text{each } x_1, x_2 \in B_X(N) \text{ and each } u_1, u_2 \in B_Y(N)\}, \tag{1.6}$$

where $N, \epsilon > 0$. Clearly, the space $\mathcal{M}_B$ with this uniformity is metrizable and complete. We equip the space $\mathcal{M}_B$ with the topology induced by this uniformity. Denote by $\mathcal{L}_B$ the set of all functions $f \in \mathcal{M}_B$ which satisfy (B5).

Note that assumptions (A1)-(A4) and (B1)-(B4) are not very restrictive. They are common in the literature and the spaces $\mathcal{M}_A$ and $\mathcal{M}_B$ contain many integrands. Therefore it is natural to ask a question if the Lavrentiev phenomenon does not occur for many integrands in these spaces. This goal is achieved by Theorems 1.1, 1.3, 1.4 and 1.5.

For each $f \in \mathcal{M}_A$ and each trajectory-control pair $(x, u)$ set

$$I^f(x, u) = \int_a^b f(t, x(t), u(t))dt.$$

For each $f \in \mathcal{M}_A$ and each $z \in H$ we study a problem

$$I^f(x, u) \to \min, \ (x, u) \in \mathcal{A}(z)$$

and put

$$U^f(z) = \inf\{I^f(x, u) : \ (x, u) \in \mathcal{A}(z)\}. \tag{1.7}$$

In this paper we assume that there exists a strongly measurable function $\xi_* : [a, b] \to Y$ such that

$$\xi_*(t) \in U(t), \ t \in [a, b] \text{ (a.e.)} \ \text{ and } \sup\{||\xi_*(t)|| : \ t \in [a, b]\} < \infty. \tag{1.8}$$

Denote by $\mathcal{M}_{Ac}$ the set of all continuous functions $f \in \mathcal{M}_A$ and by $\mathcal{M}_{Bc}$ the set of all continuous functions $f \in \mathcal{M}_B$. Set

$$\mathcal{L}_{Bc} = \mathcal{L}_B \cap \mathcal{M}_{Bc},\ \mathcal{L}_{Ac} = \mathcal{L}_A \cap \mathcal{M}_{Ac}.$$

Clearly, $\mathcal{M}_{Ac}$ is closed subset of $\mathcal{M}_A$ with the weak topology and $\mathcal{M}_{Bc}$ is a closed subset of $\mathcal{M}_B$. We consider the topological subspace $\mathcal{M}_{Bc} \subset \mathcal{M}_B$ with the relative topology and the topological subspace $\mathcal{M}_{Ac} \subset \mathcal{M}_A$ with the relative weak and strong topologies.

For each $\rho > 0$ put

$$H_\rho = \{x \in H :\ B_X(x,\rho) \subset H\}. \tag{1.9}$$

Let $\rho, M$ be positive numbers. Denote by $\tilde{H}_{\rho,M}$ the set of all $z \in H$ for which there exists a trajectory-control pair $(x,u)$ such that

$$x(a) = z,\ x(t) \in H_\rho \text{ for all } t \in [a,b], \tag{1.10}$$

$$||u(t)|| \le M,\ t \in [a,b] \text{ (a.e.)}.$$

It is not difficult to see that if $f \in \mathcal{M}_A$, $\rho, M > 0$ and $z \in \tilde{H}_{\rho,M}$, then $U^f(z) < \infty$.

The next theorem establishes that if an integrand $f$ belongs to $\mathcal{L}_B$, then the Lavrentiev phenomenon does not occur for any integrand which is contained in a certain neighborhood of $f$ in $\mathcal{M}_B$. It should be mentioned that this neighborhood is not necessarily small.

**Theorem 1.1.** *Let $f \in \mathcal{L}_B$ and let $\rho, M, q$ be positive numbers. Then there exists $K > 0$ such that for each $g \in \mathcal{M}_B$ satisfying $(f,g) \in \mathcal{E}_B(K,q)$ each $z \in \tilde{H}_{\rho,M} \cap B_X(M)$ and each $(x,u) \in \mathcal{A}(z)$ the following assertion holds:*

*If mes$(\{t \in [a,b] :\ ||u(t)|| \ge K\}) > 0$, then there exists $(y,v) \in \mathcal{A}(z)$ such that $I^g(y,v) < I^g(x,u)$ and $||v(t)|| \le K$ for almost every $t \in [a,b]$.*

Theorem 1.1 will be proved in Section 3. In Section 5 we will prove the following useful result.

**Lemma 1.2.** *The set $\mathcal{L}_B$ ($\mathcal{L}_{Bc}$ respectively) is an everywhere dense subset of $\mathcal{M}_B$ ($\mathcal{M}_{Bc}$ respectively) and the set $\mathcal{L}_A$ ($\mathcal{L}_{Ac}$ respectively) is an everywhere dense subset of $\mathcal{M}_A$ ($\mathcal{M}_{Ac}$ respectively) with the strong topology.*

The next two theorems which will be proved in Section 6 show nonoccurrence of the Lavrentiev phenomenon for most elements of $\mathcal{M}_B$.

**Theorem 1.3.** *Let $\rho, M, q$ be positive numbers and let $\mathcal{M}$ be ether $\mathcal{M}_B$ or $\mathcal{M}_{Bc}$. Then there exists an open everywhere dense subset $\mathcal{F} \subset \mathcal{M}$ such that for each $f \in \mathcal{F}$ the following assertion holds:*

*There is a number $K > 0$ such that for each $g \in \mathcal{M}$ satisfying $(f, g) \in \mathcal{E}_B(K, q)$, each $z \in \tilde{H}_{\rho,M} \cap B_X(M)$ and each $(x, u) \in \mathcal{A}(z)$ satisfying $mes(\{t \in [a, b] : \ ||u(t)|| \geq K\}) > 0$ there is $(y, v) \in \mathcal{A}(z)$ such that $I^g(y, v) < I^g(x, u)$ and $||v(t)|| \leq K$ for almost every $t \in [a, b]$.*

**Theorem 1.4.** *Let $\mathcal{M}$ be either $\mathcal{M}_B$ or $\mathcal{M}_{Bc}$ . Then there exists a set $\mathcal{F} \subset \mathcal{M}$ which is a countable intersection of open everywhere dense subsets of $\mathcal{M}$ such that for each $f \in \mathcal{F}$ and each triplet of positive numbers $M, q, \rho$ the following assertion holds:*

*There is a number $K > 0$ such that for each $g \in \mathcal{M}$ satisfying $(f, g) \in \mathcal{E}_B(K, q)$, each $z \in \tilde{H}_{\rho,M} \cap B_X(M)$ and each $(x, u) \in \mathcal{A}(z)$ satisfying $mes(\{t \in [a, b] : \ ||u(t)|| \geq K\}) > 0$ there is $(y, v) \in \mathcal{A}(z)$ such that $I^g(y, v) < I^g(x, u)$ and $||v(t)|| \leq K$ for almost every $t \in [a, b]$.*

The next theorem which will be also proved in Section 6 shows nonoccurrence of the Lavrentiev phenomenon for most elements of $\mathcal{M}_A$.

**Theorem 1.5.** *Let $\mathcal{M}$ be either $\mathcal{M}_A$ or $\mathcal{M}_{Ac}$. Then there exists a set $\mathcal{F} \subset \mathcal{M}$ which is a countable intersection of open (in the weak topology) everywhere dense (in the strong topology) subsets of $\mathcal{M}$ such that for each $f \in \mathcal{F}$ and each $z \in \cup\{\tilde{H}_{\rho,M} : \ \rho, M > 0\}$,*

$$\inf\{I^f(x, u) : \ (x, u) \in \mathcal{A}(z)\} = \inf\{I^f(x, u) : \ (x, u) \in \mathcal{A}_L(z)\}.$$

## 2. AUXILIARY RESULTS

Put

$$D_0 = \sup\{||S(t)|| : \ t \in [a, b]\}. \tag{2.1}$$

It is well-known that $D_0$ is a finite number.

**Lemma 2.1.** *Let $f \in \mathcal{M}_A$ and let $M, \rho, q$ be positive numbers. Then there exists $M_1 > 0$ such that for each $g \in \mathcal{M}_A$ satisfying*

$$(f, g) \in \mathcal{E}_{Aw}(M(D_0 + 1)(1 + (b - a)||B||), q) \tag{2.2}$$

*and each $z \in \tilde{H}_{\rho,M} \cap B_X(M)$ the inequality $U^g(z) \leq M_1$ holds.*

*Proof.* Set

$$M_1 = (b - a)[q + \sup\{f(s, y, v) :$$

$$(s, y, v) \in [a, b] \times B_X(D_0 M(1 + (b - a)||B||)) \times B_Y(M)\}].$$

Let

$$z \in \tilde{H}_{\rho,M} \cap B_X(M) \tag{2.3}$$

and $g \in \mathcal{M}_A$ satisfy (2.2). By (2.3) and the definition of $\tilde{H}_{\rho,M}$ (see (1.9) and (1.10)) there exists a trajectory-control pair $(x, u)$ such that

$$x(a) = z,\ x(t) \in H_\rho,\ t \in [a, b] \text{ and } ||u(t)|| \le M,\ t \in [a, b] \text{ (a.e.)}. \tag{2.4}$$

Since $(x, u)$ is a trajectory-control pair it follows from (1.1), (2.1) and (2.3) that for all $\tau \in [a, b]$

$$||x(\tau)|| = ||S(\tau)z + \int_a^\tau S(\tau - t)Bu(t)dt|| \le D_0 M + (b - a)D_0||B||M. \tag{2.5}$$

In view of (2.2), (2.4), (2.5) and the choice of $M_1$ for $t \in [a, b]$ a.e. we have

$$g(t, x(t), u(t)) \le f(t, x(t), u(t)) + q \le$$

$$\le \sup\{f(s, y, v) : (s, y, v) \in [a, b] \times B_X(D_0 M(1 + (b - a)||B||)) \times B_Y(M)\} + q$$

and

$$U^g(z) \le I^g(x, u) \le (b - a)[q + \sup\{f(s, y, v) :$$

$$(s, y, v) \in [a, b] \times B_X(D_0 M(1 + (b - a)||B||)) \times B_Y(M)\}] = M_1.$$

Lemma 2.1 is proved. □

**Lemma 2.2.** *Let $f \in \mathcal{M}_A$ and let $M, \rho, q$ be positive numbers. Then there exists $M_0 > 0$ such that for each $g \in \mathcal{M}_A$ satisfying (2.2), each*

$$z \in \tilde{H}_{\rho,M} \cap B_X(M) \tag{2.6}$$

*and each $(x, u) \in \mathcal{A}(z)$ satisfying $I^g(x, u) \le U^g(z) + 1$, the inequality $||x(t)|| \le M_0$ holds for all $t \in [a, b]$.*

*Proof.* Let $M_1 > 0$ be as guaranteed by Lemma 2.1. In view of (1.3) there is $c_0 \ge 1$ such that

$$\phi(t) \ge t \text{ for all } t \ge c_0. \tag{2.7}$$

Set

$$M_0 = M + MD_0 + D_0||B||c_0(b - a) + D_0||B||(M_1 + 1). \tag{2.8}$$

Assume that $z \in X$ satisfies (2.6), $g \in \mathcal{M}_A$ satisfies (2.2) and $(x, u) \in \mathcal{A}(z)$ satisfies

$$I^g(x, u) \le U^g(z) + 1. \tag{2.9}$$

By (2.6), (2.2), the choice of $M_1$ and Lemma 2.1, $U^g(z) \le M_1$. Together with (2.9) this inequality implies that

$$I^g(x, u) \le M_1 + 1. \tag{2.10}$$

Let $\tau \in (a, b]$ and set

$$E_1 = \{t \in [a, \tau] : ||u(t)|| \ge c_0\},\ E_2 = [a, \tau] \setminus E_1. \tag{2.11}$$

It follows from (1.1), (2.1), (2.11), (2.6) and (2.7) that

$$||x(\tau)|| = ||S(\tau)z + \int_a^\tau S(\tau - s)Bu(s)ds|| \le ||z||D_0 + D_0||B|| \int_a^\tau ||u(s)||ds$$

$$\le ||z||D_0 + D_0||B||(\int_{E_1} ||u(s)||ds + \int_{E_2} ||u(s)||ds)$$

$$\le MD_0 + D_0||B||c_0(b-a) + D_0||B|| \int_{E_1} ||u(s)||ds$$

$$\le MD_0 + D_0||B||c_0(b-a) + D_0||B|| \int_{E_1} \phi(||u(s)||)ds.$$

Combined with (A2), (2.10) and (2.8) this inequality implies that

$$||x(\tau)|| \le MD_0 + D_0||B||c_0(b-a) + D_0||B|| \int_a^b \phi(||u(s)||)ds$$

$$\le MD_0 + D_0||B|||c_0(b-a) + D_0||B||I^g(x,u)$$

$$\le MD_0 + D_0||B||c_0(b-a) + D_0||B||(M_1 + 1) \le M_0.$$

Lemma 2.2 is proved. □

The following result is proved analogously to Lemma 2.2 of [21].

**Lemma 2.3.** *Let $f \in \mathcal{L}_A$ and let $\epsilon, M > 0$. Then there exist $\Gamma, \delta > 0$ such that*

$$|f(t,x_1,u) - f(t,x_2,u)| \le \epsilon \min\{f(t,x_1,u),\ f(t,x_2,u)\}$$

*for each $t \in [a,b]$, each $u \in Y$ satisfying $||u|| \ge \Gamma$ and each $x_1, x_2 \in B_X(M)$ satisfying $||x_1 - x_2|| \le \delta$.*

## 3. PROOF OF THEOREM 1.1

In this section we establish the following result which easily implies Theorem 1.1.

**Theorem 3.1.** *Let $f \in \mathcal{L}_B$ and let $\rho, M, q$ be positive numbers. Then there exist $K, \Delta_1 > 0$ such that for each $g \in \mathcal{M}_B$ satisfying $(f,g) \in \mathcal{E}_B(K,q)$, each $z \in \tilde{H}_{\rho,M} \cap B_X(M)$ and each $(x,u) \in \mathcal{A}(z)$ satisfying $I^g(x,u) \le U^g(z) + 1$ the following assertion holds:*

*If the set $E := \{t \in [a,b] :\ ||u(t)|| \ge K\}$ has a positive Lebesgue measure, then there exists $(y,v) \in \mathcal{A}(z)$ such that $||v(t)|| \le K$ for almost every $t \in [a,b]$ and the following inequalities hold:*

$$I^g(y,v) < I^g(x,u) - M \int_E ||u(t)||dt,$$

$$||x(t) - y(t)|| \le \Delta_1 \int_E ||u(t)||dt \text{ for all } t \in [a,b],$$

$$||u(t) - v(t)|| \le \Delta_1 \int_E ||u(t)||dt,\ t \in [a,b] \setminus E \text{ (a.e.)}$$

*Proof.* Recall that $\xi_* : [a,b] \to Y$ is a strongly measurable function which satisfies (1.8). In view of (1.8) there is a number $N_0 > 0$ such that

$$||\xi_*(t)|| \le N_0 \text{ for all } t \in [a,b]. \tag{3.1}$$

By Lemma 2.1 there exists $M_1 > 0$ such that

$$U^g(z) \le M_1 \tag{3.2}$$

for each $z \in \tilde{H}_{\rho,M} \cap B_X(M)$ and each $g \in \mathcal{M}_B$ satisfying

$$(f,g) \in \mathcal{E}_B(M(D_0+1)(1+(b-a)||B||), q). \tag{3.3}$$

By Lemma 2.2 there exists $M_0 > 0$ such that for each $g \in \mathcal{M}_B$ satisfying (3.3), each $z \in \tilde{H}_{\rho,M} \cap B_X(M)$ and each $(x,u) \in \mathcal{A}(z)$ satisfying $I^g(x,u) \le U^g(z)+1$ the following inequality holds:

$$||x(t)|| \le M_0 \text{ for all } t \in [a,b]. \tag{3.4}$$

We may assume without loss of generality that

$$M_0 > (M+1)(N_0+1)(D_0+1)(1+(b-a)||B||). \tag{3.5}$$

By (B5) there are $\delta_0 \in (0,1)$, $L_0 > 0$ and an integrable scalar function $\psi_0(t) \ge 0$, $t \in [a,b]$ such that for each $t \in [a,b]$, each $u \in Y$ and each $x_1, x_2 \in X$ satisfying

$$||x_1||, ||x_2|| \le M_0 + 4, \; ||x_1 - x_2|| \le \delta_0$$

the following inequality holds:

$$|f(t,x_1,u) - f(t,x_2,u)| \le ||x_1 - x_2|| L_0(f(t,x_1,u) + \psi_0(t)). \tag{3.6}$$

Set

$$\Delta_0 = \sup\{f(t,x,u) : \; t \in [a,b], \; x \in B_X(M_0+1), \; y \in B_Y(M_0+1)\}. \tag{3.7}$$

Clearly, $\Delta_0 < \infty$. Choose a positive number $\gamma_0 < 1$ and a number $K_0 > 1$ such that

$$(D_0+1)(||B||+1)(M+1)(M_1+1)\gamma_0 < \rho/4, \tag{3.8}$$

$$K_0 > M + 1 + M_0 + \rho, \tag{3.9}$$

$$\phi(t)/t \ge \gamma_0^{-1} \text{ for all } t \ge K_0. \tag{3.10}$$

By (B4) there exists $L_1 > 1$ such that for each $t \in [a,b]$, each $x_1, x_2 \in B_X(2K_0)$ and each $u_1, u_2 \in B_Y(2K_0)$,

$$|f(t,x_1,u_1) - f(t,x_2,u_2)| \le L_1(||x_1 - x_2|| + ||u_1 - u_2||). \tag{3.11}$$

Set

$$\Delta_1 = \max\{D_0||B||(3 + 8M_0\rho^{-1}), \; 8D_0||B||\rho^{-1}K_0\}. \tag{3.12}$$

Choose a number $\gamma_1 \in (0,1)$ such that

$$(M_0+1)8(D_0+1)(||B||+1)(\min\{1,\rho\})^{-1}\gamma_1(M_1+2) < \delta_0, \tag{3.13}$$

$$\gamma_1^{-1} > 4[24D_0||B||L_1\rho^{-1}K_0(1+q)(b-a)$$

$$+8D_0||B||(1+M_0\rho^{-1})(L_0(M_1+1)+q(L_0+1)(b-a)+L_0\int_a^b \psi_0(t)dt)+2M]. \tag{3.14}$$

Choose a number $K > 0$ such that

$$K > 4+(M+1)(D_0+1)(1+(b-a)||B||)+2M_0+2(M+N_0)+2K_0+8(q+\Delta_0), \tag{3.15}$$

$$\phi(t)/t \geq \gamma_1^{-1} \text{ for a ll } t \geq K. \tag{3.16}$$

Assume that

$$g \in \mathcal{M}_B,\ (f,g) \in \mathcal{E}_B(K,q),\ z \in \tilde{H}_{\rho,M} \cap B_X(M),\ (x,u) \in \mathcal{A}(z), \tag{3.17}$$

$$\mathrm{mes}(\{t \in [a,b]:\ ||u(t)|| \geq K\}) > 0, \tag{3.18}$$

$$I^g(x,u) \leq U^g(z)+1. \tag{3.19}$$

In view of (3.17), (3.15) and the choice of $M_1$, the inequality (3.2) is true. Together with (3.19) the inequality (3.2) implies that

$$I^g(x,u) \leq M_1+1. \tag{3.20}$$

By (3.17), (3.15), (3.19) and the choice of $M_0$ the inequality (3.4) holds. Set

$$E_1 = \{t \in [a,b]:\ ||u(t)|| \geq K\},\ E_2 = \{t \in [a,b]:\ ||u(t)|| \leq K_0\},$$

$$E_3 = [a,b] \setminus (E_1 \cup E_2), \tag{3.21}$$

$$d = \int_{E_1} ||u(t)||dt. \tag{3.22}$$

Relations (3.21), (3.22) and (3.18) imply that

$$d > 0. \tag{3.23}$$

It follows from (3.17) and the definition of $\tilde{H}_{\rho,M}$ (see (1.10)) that there exists a trajectory-control pair $(\tilde{x},\tilde{u}) \in \mathcal{A}(z)$ such that

$$\tilde{x}(a) = z,\ \tilde{x}(t) \in H_\rho \text{ for all } t \in [a,b], \tag{3.24}$$

$$||\tilde{u}(t)|| \leq M,\ t \in [a,b] \text{ a.e..} \tag{3.25}$$

Since $(\tilde{x},\tilde{u}) \in \mathcal{A}(z)$ it follows from (1.1), (2.1), (3.17) and (3.25) that for each $\tau \in [a,b]$

$$||\tilde{x}(\tau)|| = ||S(\tau)z + \int_a^\tau S(\tau-t)B\tilde{u}(t)dt|| \leq D_0M+(b-a)D_0||B||M. \tag{3.26}$$

Combined with (3.5) this relation implies that

$$||\tilde{x}(t)|| \le M_0 \text{ for all } t \in [a, b]. \tag{3.27}$$

We estimate the number $d$. By (3.22), (3.21), (3.15), (B2) and (3.20),

$$d = \int_{E_1} ||u(t)||dt \le \int_{E_1} \gamma_1 \phi(||u(t)||)dt$$

$$\le \gamma_1 \int_a^b \phi(||u(t)||)dt \le \gamma_1 \int_a^b g(t, x(t), u(t))dt \le \gamma_1(M_1 + 1). \tag{3.28}$$

In view of (3.21), (3.22) and (3.28)

$$\mathrm{mes}(E_1) \le K^{-1} \int_{E_1} ||u(t)||dt \le K^{-1}d \le K^{-1}\gamma_1(M_1 + 1). \tag{3.29}$$

Put

$$\alpha = 4D_0||B||d\rho^{-1}. \tag{3.30}$$

By (3.30), (3.28) and (3.13)

$$\alpha < 4D_0||B||\rho^{-1}\gamma_1(M_1 + 1) < 1. \tag{3.31}$$

It follows from (3.21), (3.15), (3.10), (B2) and (3.20) that

$$\int_{E_1 \cup E_3} ||u(t)||dt \le \int_{E_1 \cup E_3} \gamma_0 \phi(||u(t)||)dt \le \gamma_0 \int_a^b \phi(||u(t)||)dt$$

$$\le \gamma_0 I^g(x, u) \le \gamma_0(M_1 + 1). \tag{3.32}$$

By (3.21), (3.15) and (3.32)

$$\mathrm{mes}(E_1 \cup E_3) \le K_0^{-1} \int_{E_1 \cup E_3} ||u(t)||dt \le K_0^{-1}\gamma_0(M_1 + 1). \tag{3.33}$$

Since the sets $U(t)$, $t \in [a, b]$ and $H$ are convex it follows from (1.1) that

$$(\alpha\tilde{x} + (1 - \alpha)x, \alpha\tilde{u} + (1 - \alpha)u)$$

is a trajectory-control pair. The inclusions $(\tilde{x}, \tilde{u})$, $(x, u) \in \mathcal{A}(z)$ imply that

$$(\alpha\tilde{x} + (1 - \alpha)x, \alpha\tilde{u} + (1 - \alpha)u) \in \mathcal{A}(z). \tag{3.34}$$

Let $t \in [a, b]$. Relation (3.24) and the inclusion $x(t) \in H$ imply that

$$\alpha\tilde{x}(t) + (1 - \alpha)x(t) \in H_{\alpha\rho}. \tag{3.35}$$

Thus (3.35) is true for all $t \in [a, b]$. Define

$$v(t) = \xi_*(t),\ t \in E_1,\ v(t) = \alpha\tilde{u}(t) + (1 - \alpha)u(t),\ t \in E_2, \tag{3.36}$$

$$v(t) = u(t),\ t \in E_3,$$

$$y(t) = S(t)z + \int_a^t S(t - s)Bv(s)ds,\ t \in [a, b]. \tag{3.37}$$

Since $\xi_* : [a,b] \to Y$ is a strongly measurable function it follows from (3.34) and (3.36) that $v : [a,b] \to Y$ is also a strongly measurable function. Relations (1.8), (3.34) and (3.36) imply that

$$v(t) \in U(t),\ t \in [a,b] \text{ (a.e.)}. \tag{3.38}$$

Let $t \in [a,b]$. By (3.37), (3.34), (1.1) and (3.36)

$$\begin{aligned}
&||y(t) - (\alpha\tilde{x}(t) + (1-\alpha)x(t))|| \\
&= ||S(t)z + \int_a^t S(t-s)Bv(s)ds - S(t)z - \int_a^t S(t-s)B(\alpha\tilde{u}(s) + (1-\alpha)u(s))ds|| \\
&= ||\sum_{i=1}^{3}\int_{[a,t]\cap E_i} S(t-s)Bv(s)ds - \sum_{i=1}^{3}\int_{[a,t]\cap E_i} S(t-s)B(\alpha\tilde{u}(s) + (1-\alpha)u(s))ds|| \\
&\le ||\int_{[a,t]\cap E_1} S(t-s)Bv(s)ds - \int_{[a,t]\cap E_1} S(t-s)B(\alpha\tilde{u}(s) + (1-\alpha)u(s))ds|| \\
&\quad + ||\int_{[a,t]\cap E_3} S(t-s)Bv(s)ds - \int_{[a,t]\cap E_3} S(t-s)B(\alpha\tilde{u}(s) + (1-\alpha)u(s))ds|| \\
&\le ||\int_{[a,t]\cap E_1} S(t-s)Bv(s)ds|| + ||\int_{[a,t]\cap E_1} S(t-s)B(\alpha\tilde{u}(s) + (1-\alpha)u(s))ds|| \\
&\quad + ||\int_{[a,t]\cap E_3} S(t-s)B(\alpha(u(s) - \tilde{u}(s))ds|| \\
&\le D_0||B||\int_{E_1} ||\xi_*(s)||ds + D_0||B||\int_{E_1} ||\alpha\tilde{u}(s) + (1-\alpha)u(s)||ds \\
&\quad + D_0||B||\alpha\int_{E_3} ||u(s) - \tilde{u}(s)||ds.
\end{aligned}$$

It follows from this relation, (3.1), (3.25), (3.29), (3.22), (3.33), (3.32), (3.15), (3.8) and (3.30) that

$$\begin{aligned}
||y(t) - (\alpha\tilde{x}(t) + (1-\alpha)x(t))|| &\le D_0||B||N_0\mathrm{mes}(E_1) + D_0||B||\alpha\int_{E_1} ||\tilde{u}(s)||ds \\
&\quad + D_0||B||\int_{E_1} ||u(s)||ds + D_0||B||\alpha(\mathrm{mes}(E_3)M + \int_{E_3} ||u(s)||ds) \\
&\le D_0||B|||N_0K^{-1}d + D_0||B||\alpha MK^{-1}d + D_0||B||d \\
&\quad + D_0||B||\alpha(M\gamma_0(M_1+1) + \gamma_0(M_1+1)) \\
&= dD_0||B||[N_0K^{-1} + \alpha MK^{-1} + 1] + D_0||B||\alpha(M+1)(M_1+1)\gamma_0 \\
&\le 2dD_0||B|| + \alpha\rho/4 = 3D_0||B||d.
\end{aligned}$$

Thus

$$||y(t) - (\alpha\tilde{x}(t) + (1-\alpha)x(t))|| \le 3D_0||B||d \text{ for all } t \in [a,b]. \tag{3.39}$$

In view of (3.39) and (3.30),

$$||y(t) - (\alpha\tilde{x}(t) + (1-\alpha)x(t))|| \le (3/4)\alpha\rho \text{ for all } t \in [a,b].$$

Together with (3.35) this inequality implies that

$$y(t) \in H \text{ for all } t \in [a,b]. \tag{3.40}$$

Relations (3.40), (3.38) and (3.37) imply that

$$(y,v) \in \mathcal{A}(z). \tag{3.41}$$

Now we estimate $I^g(y,v) - I^g(x,u)$. By (3.4), (3.27) and (3.30) for each $t \in [a,b]$

$$||x(t) - (\alpha\tilde{x}(t) + (1-\alpha)x(t))|| = \alpha||x(t) - \tilde{x}(t)|| \le 2\alpha M_0 = 8M_0 D_0||B||d\rho^{-1}.$$

Combined with (3.39) this implies that

$$||y(t) - x(t)|| \le dD_0||B||(3 + 8M_0\rho^{-1}) \text{ for all } t \in [a,b]. \tag{3.42}$$

Relations (3.42), (3.28) and (3.13) imply that for all $t \in [a,b]$

$$||y(t) - x(t)|| \le 8D_0||B||(1 + M_0\rho^{-1})\gamma_1(M_1+1) < \delta_0 < 1. \tag{3.43}$$

It follows from (3.36), (3.25), (3.21), (3.8) and (3.30) that for all $t \in E_2$,

$$||v(t) - u(t)|| = ||\alpha\tilde{u}(t) + (1-\alpha)u(t) - u(t)|| = \alpha||\tilde{u}(t) - u(t)||$$

$$\le \alpha(K_0 + M) \le 2\alpha K_0 = 8dD_0||B||\rho^{-1}K_0. \tag{3.44}$$

In view of (3.43) and (3.4)

$$||y(t)|| \le ||x(t)|| + 1 \le M_0 + 1,\ t \in [a,b]. \tag{3.45}$$

By (3.44), (3.21), (3.28) and (3.13) for all $t \in E_2$

$$\begin{aligned} ||v(t)|| &\le ||u(t)|| + 8dD_0||B||\rho^{-1}K_0 \le K_0(1 + 8dD_0||B||\rho^{-1}) \\ &\le K_0(1 + 8\gamma_1(M_1+1)D_0||B||\rho^{-1}) \le 2K_0. \end{aligned} \tag{3.46}$$

Together with (3.21), (3.36), (3.1) and (3.15) this implies that

$$||v(t)|| \le K \text{ for all } t \in [a,b]. \tag{3.37}$$

Clearly,

$$I^g(x,u) - I^g(y,v) = \sum_{i=1}^{3}[\int_{E_i} g(t,x(t),u(t))dt - \int_{E_i} g(t,y(t),v(t))dt].$$

It follows from (B2), (3.21) and (3.16) that

$$\int_{E_1} g(t,x(t),u(t))dt \ge \int_{E_1} \phi(||u(t)||)dt \ge \gamma_1^{-1}\int_{E_1} ||u(t)||dt, \tag{3.48}$$

$$\int_{E_1} g(t,x(t),u(t))dt \ge \gamma_1^{-1}K\mathrm{mes}(E_1). \tag{3.49}$$

In view of (3.45), (3.36), (3.7), (3.1) and (3.5),

$$f(t,y(t),v(t)) \le \Delta_0,\ t \in E_1. \tag{3.50}$$

By (3.45), (3.36), (3.1), (3.5), (3.9), (3.30) and (3.50) for $t \in E_1$,

$$g(t, y(t), v(t)) \le q + f(t, y(t), v(t)) \le q + \Delta_0. \tag{3.51}$$

It follows from (3.51), (3.15) and (3.49) that

$$\int_{E_1} g(t, y(t), v(t))dt \le (q + \Delta_0)\mathrm{mes}(E_1)$$

$$\le 8^{-1}K\mathrm{mes}(E_1) \le 8^{-1}\int_{E_1} g(t, x(t), u(t))dt. \tag{3.52}$$

The inequality (3.52) implies that

$$\int_{E_1} g(t, x(t), u(t))dt - \int_{E_1} g(t, y(t), v(t))dt \ge (7/8)\int_{E_1} g(t, x(t), u(t))dt. \tag{3.53}$$

Let $t \in E_2$. It follows from (3.45), (3.4), (3.46), (3.35), (3.15), (3.17) and (1.6) that

$$|g(t, x(t), u(t)) - g(t, y(t), v(t))| \le$$

$$|f(t, x(t), u(t)) - f(t, y(t), v(t))| + |(g - f)(t, x(t), u(t)) - (g - f)(t, y(t), v(t))| \tag{3.54}$$

$$\le |f(t, x(t), u(t)) - f(t, y(t), v(t))| + q||x(t) - y(t)|| + q||u(t) - v(t)||.$$

By the choice of $L_1$ (see (3.11)), (3.45), (3.4), (3.22), (3.46), (3.9), (3.42) and (3.44),

$$|f(t, y(t), v(t)) - f(t, x(t), u(t))| \le L_1(||y(t) - x(t)|| + ||v(t) - u(t)||)$$

$$\le L_1(\alpha D_0||B||8(1 + M_0\rho^{-1}) + 8dD_0||B||\rho^{-1}K_0) \le 8dD_0||B||L_1(3\rho^{-1}K_0). \tag{3.55}$$

Combined with the inequality $L_1 > 1$, (3.54), (3.42), (3.44) and (3.9) the inequality (3.55) implies that

$$|g(t, x(t), u(t)) - g(t, y(t), v(t))|$$

$$\le 8dD_0||B||L_1 3\rho^{-1}K_0 + q||x(t) - y(t)|| + q||u(t) - v(t)||$$

$$\le 8dD_0||B||L_1 3\rho^{-1}K_0 + qdD_0||B||8(1 + M_0\rho^{-1}) + 8qdD_0||B||\rho^{-1}K_0$$

$$\le 24dD_0||B||L_1\rho^{-1}K_0(1 + q). \tag{3.56}$$

This inequality implies that

$$|\int_{E_2} g(t, x(t), u(t))dt - \int_{E_2} g(t, y(t), v(t))dt|$$

$$\le 24dD_0||B||L_1\rho^{-1}K_0(1 + q)(b - a). \tag{3.57}$$

Let $t \in E_3$. It follows from (3.36), the choice of $L_0$ (see (3.6)), (3.45), (3.4) and (3.43) that

$$|f(t, x(t), u(t)) - f(t, y(t), v(t))| = |f(t, x(t), u(t)) - f(t, y(t), u(t))|$$

$$\le ||x(t) - y(t)||L_0(f(t, x(t), u(t)) + \psi_0(t)). \tag{3.58}$$

By (3.21), (3.4), (3.45), (3.15), (3.36), (3.17) and (1.6)

$$|g(t,x(t),u(t)) - g(t,y(t),v(t))|$$

$$\leq |f(t,x(t),u(t)) - f(t,y(t),v(t))| + |(g-f)(t,x(t),u(t)) - (g-f)(t,y(t),v(t))|$$

$$\leq |f(t,x(t),u(t)) - f(t,y(t),v(t))| + q||x(t)-y(t)||. \tag{3.59}$$

Combined with (3.58) and (3.42) this inequality implies that

$$|g(t,x(t),u(t)) - g(t,y(t),v(t))| \leq ||x(t)-y(t)||[L_0(f(t,x(t),u(t)) + \psi_0(t)) + q]$$

$$\leq 8dD_0||B||(1+M_0\rho^{-1})[L_0(f(t,x(t),u(t)) + \psi_0(t)) + q]. \tag{3.60}$$

Together with (3.4), (3.21), (3.15), (3.17) and (1.6) this inequality implies that

$$|g(t,x(t),u(t)) - g(t,y(t),v(t))|$$

$$\leq 8dD_0||B||(1+M_0\rho^{-1})[L_0(g(t,x(t),u(t)) + q + \psi_0(t)) + q] \tag{3.61}$$

for all $t \in E_3$. It follows from (3.61) that

$$|\int_{E_3} g(t,x(t),u(t))dt - \int_{E_3} g(t,y(t),v(t))dt|$$

$$\leq 8dD_0||B||(1+M_0\rho^{-1})[L_0\int_a^b g(t,x(t),u(t))dt + q(L_0+1)(b-a) + L_0\int_a^b \psi_0(t)dt]. \tag{3.62}$$

By (3.21), (3.53), (3.57), (3.62) and (3.20),

$$I^g(x,u) - I^g(y,v) \geq (7/8)\int_{E_1} g(t,x(t),u(t))dt - 24dD_0||B||L_1\rho^{-1}K_0(1+q)(b-a)$$

$$-8dD_0||B||(1+M_0\rho^{-1})[L_0(M_1+1) + q(L_0+1)(b-a) + L_0\int_a^b \psi_0(t)dt]. \tag{3.63}$$

In view of (B2), (3.21), (3.16) and (3.22)

$$\int_{E_1} g(t,x(t),u(t))dt \geq \int_{E_1} \phi(||u(t)||)dt \geq \gamma_1^{-1}\int_{E_1} ||u(t)||t = \gamma_1^{-1}d. \tag{3.64}$$

Relations (3.63), (3.64), (3.14) and (3.22) imply that

$$I^g(x,u) - I^g(y,v) \geq (2\gamma_1)^{-1}d - d[24D_0||B||L_1\rho^{-1}K_0(1+q)(b-a)$$

$$+8D_0||B||(1+M_0\rho^{-1})(L_0(M_1+1) + q(L_0+1)(b-a) + L_0\int_a^b \psi_0(t)dt)]$$

$$\geq (4\gamma_1)^{-1}d > Md = M\int_{E_1} ||u(t)||dt. \tag{3.65}$$

By (3.42), (3.12) and (3.22)

$$||x(t)-y(t)|| \leq \Delta_1 d = \Delta_1 \int_{E_1} ||u(t)||dt \text{ for all } t \in [a,b].$$

By (3.36), (3.21), (3.44) and (3.12)

$$||u(t) - v(t)|| \le \Delta_1 d = \Delta_1 \int_{E_1} ||u(t)||dt \text{ for all } t \in [a,b] \setminus E_1.$$

Theorem 3.1 is proved. □

## 4. AN AUXILIARY RESULT FOR THEOREM 1.5

**Lemma 4.1.** *Let $f \in \mathcal{L}_A$ and let $M, \rho, \epsilon$ be positive numbers. Then there exists a neighborhood $\mathcal{U}$ of $f$ in $\mathcal{M}_A$ with the weak topology and $K > 0$ such that for each $g \in \mathcal{U}$ and each $z \in \tilde{H}_{\rho,M} \cap B_X(M)$ there is $(x,u) \in \mathcal{A}(z)$ such that $||u(t)|| \le K$ for almost every $t \in [a,b]$ and $I^g(x,u) \le U^g(z) + \epsilon$.*

*Proof.* We may assume without loss of generality that

$$\epsilon, \rho < 1 < M. \tag{4.1}$$

Recall that $\xi_* : [a,b] \to Y$ is a strongly measurable function which satisfies (1.8). In view of (1.8) there is a number $N_0 > 0$ such that

$$||\xi_*(t)|| \le N_0 \text{ for all } t \in [a,b]. \tag{4.2}$$

By Lemma 2.1 there is $M_1 > 0$ such that

$$U^g(z) \le M_1 \tag{4.3}$$

for each $g \in \mathcal{M}_A$ satisfying

$$(g,f) \in \mathcal{E}_{Aw}(M(D_0+1)(1+(b-a)||B||), 4) \tag{4.4}$$

and each $z \in \tilde{H}_{\rho,M} \cap B_X(M)$. By Lemma 2.2 there is $M_0 > 0$ such that for each $g \in \mathcal{M}_A$ satisfying (4.4), each $z \in \tilde{H}_{\rho,M} \cap B_X(M)$ and each $(x,u) \in \mathcal{A}(z)$ satisfying $I^g(x,u) \le U^g(z) + 1$ the following inequality holds:

$$||x(t)|| \le M_0 \text{ for all } t \in [a,b]. \tag{4.5}$$

We may assume that

$$M_0 > (M+1)(N_0+1)(D_0+1)(1+(b-a)||B||). \tag{4.6}$$

Choose positive numbers $\epsilon_0, \gamma_0$ and a number $N_1 > 1$ such that

$$8\epsilon_0(M_1+4) < \epsilon, \tag{4.7}$$

$$\gamma_0 < 1,\ 32\gamma_0(M_1+2) < b-a, \tag{4.8}$$

$$\gamma_0(M+1)(M_1+1)(D_0+1)(||B||+1) < \rho/4,$$

$$\phi(t)/t \ge \gamma_0^{-1} \text{ for all } t \ge N_1. \tag{4.9}$$

In view of Lemma 2.3 there are

$$\delta_0 \in (0,1),\ N_2 > N_1 + N_0 + M \tag{4.10}$$

such that for each $t \in [a,b]$, each $u \in Y$ satisfying $||u|| \geq N_2$ and each $x_1, x_2 \in X$ satisfying

$$||x_1||, ||x_2|| \leq M_0 + 4,\ ||x_1 - x_2|| \leq \delta_0 \tag{4.11}$$

the following inequality holds:

$$|f(t,x_1,u) - f(t,x_2,u)| \leq \epsilon_0 \min\{f(t,x_1,u),\ f(t,x_2,u)\}. \tag{4.12}$$

Set

$$\Delta_1 = \sup\{f(t,z,u) :\ t \in [a,b],\ z \in B_X(M_0+1),\ u \in B_Y(M_0+1)\} + 1. \tag{4.13}$$

Clearly, $\Delta_1$ is finite. By (A4) there exists

$$\delta_1 \in (0, \delta_0) \tag{4.14}$$

such that for each $t \in [a,b]$, each $x_1, x_2 \in B_X(M_0+4+2N_2)$, $y_1, y_2 \in B_Y(M_0+4+2N_2)$ satisfying

$$||x_1 - x_2||,\ ||y_1 - y_2|| \leq \delta_1 \tag{4.15}$$

the following inequality holds:

$$|f(t,x_1,y_1) - f(t,x_2,y_2)| \leq (32(b-a+1))^{-1}\epsilon. \tag{4.16}$$

Choose a positive number $\gamma_1$ such that

$$\gamma_1 < 1 \text{ and } (M_0+1)32\gamma_1(M_1+1)(D_0+1)(N_2+1)(||B||+1)\rho^{-1} < \delta_1. \tag{4.17}$$

By (1.3) there is a number $K$ such that

$$K > (M_0+1)(M+1)(D_0+1)(1+b-a)(||B||+1)(N_2+1)2 + 32\Delta_1, \tag{4.18}$$

$$\phi(t)/t \geq \gamma_1^{-1} \text{ for all } t \geq K. \tag{4.19}$$

Choose a positive number $\Delta$ such that

$$\Delta < (32(b-a+1))^{-1}\min\{1, \epsilon\} \tag{4.20}$$

and set

$$\mathcal{U} = \{g \in \mathcal{M}_A :\ (f,g) \in \mathcal{E}_{Aw}(K+1, \Delta)\}. \tag{4.21}$$

Assume that

$$g \in \mathcal{U},\ z \in \tilde{H}_{\rho,M} \cap B_X(M). \tag{4.22}$$

In order to prove the proposition it is sufficient to show that there is $(x,u) \in \mathcal{A}(z)$ such that $||u(t)|| \leq K$ for almost every $t \in [a,b]$ and $I^g(x,u) \leq U^g(z) + \epsilon$. There is

$$(x,u) \in \mathcal{A}(z) \tag{4.23}$$

such that

$$I^g(x,u) \le U^g(z) + \epsilon/4. \tag{4.24}$$

We may assume without loss of generality that

$$\mathrm{mes}(\{t \in [a,b] : \ ||u(t)|| \ge K\}) > 0. \tag{4.25}$$

In view of (4.22), (4.21), (4.20), (4.18) and the choice of $M_1$, (4.3) is true. Combined with (4.24) and (4.1) the relation (4.3) implies that

$$I^g(x,u) \le M_1 + 1. \tag{4.26}$$

By (4.22), (4.21), (4.18), (4.20), (4.24) and the choice of $M_0$, the inequality (4.5) holds. Set

$$E_1 = \{t \in [a,b] : \ ||u(t)|| \ge K\}, \ E_2 = \{t \in [a,b] : \ ||u(t)|| \le N_2\},$$

$$E_3 = [a,b] \setminus (E_1 \cup E_2), \tag{4.27}$$

$$d = \int_{E_1} ||u(t)||dt. \tag{4.28}$$

By (4.25), (4.28) and (4.27)

$$d > 0. \tag{4.29}$$

It follows from (4.22) and the definition of $\tilde{H}_{\rho,M}$ (see (1.10)) that there exists a trajectory-control pair $(\tilde{x}, \tilde{u}) \in \mathcal{A}(z)$ such that

$$\tilde{x}(a) = z, \ \tilde{x}(t) \in H_\rho \text{ for all } t \in [a,b], \tag{4.30}$$

$$||\tilde{u}(t)|| \le M, \ t \in [a,b] \text{ (a.e.)}. \tag{4.31}$$

Arguing as in the proof of Theorem 1.1 we can show that it follows from (1.1), (2.1), (4.22), (4.6) and (4.31) that

$$||\tilde{x}(t)|| \le M_0 \text{ for all } t \in [a,b]. \tag{4.32}$$

Arguing as in the proof of Theorem 1.1 (see (3.28)) we can show that (4.28), (4.27), (4.19), (4.26) and (A2) imply that

$$d \le \gamma_1(M_1 + 1). \tag{4.33}$$

In view of (4.27), (4.28) and (4.33)

$$\mathrm{mes}(E_1) \le K^{-1} \int_{E_1} ||u(t)||dt \le K^{-1}d \le K^{-1}\gamma_1(M_1+1). \tag{4.34}$$

Set

$$\alpha = 4D_0||B||d\rho^{-1}. \tag{4.35}$$

By (4.33), (4.17), (4.10), (4.14) and (4.1)

$$\alpha < 1. \tag{4.36}$$

It follows from (4.27), (4.18), (4.10), (4.9), (4.26) and (A2) that

$$\int_{E_1 \cup E_3} ||u(t)||dt \le \int_{E_1 \cup E_3} \gamma_0 \phi(||u(t)||)dt \le \gamma_0 \int_a^b \phi(||u(t)||)dt$$

$$\le \gamma_0 I^g(x,u) \le \gamma_0(M_1+1). \tag{4.37}$$

By (4.27) and (4.37)

$$\text{mes}(E_1 \cup E_3) \le N_2^{-1} \int_{E_1 \cup E_3} ||u(t)||dt \le N_2^{-1}\gamma_0(M_1+1). \tag{4.38}$$

Since the sets $U(t)$, $t \in [a,b]$ and $H$ are convex it follows from (1.1) that $(\alpha\tilde{x} + (1-\alpha)x, \alpha\tilde{u} + (1-\alpha)u)$ is a trajectory-control pair. The inclusions $(\tilde{x},\tilde{u})$, $(x,u) \in \mathcal{A}(z)$ imply that

$$(\alpha\tilde{x} + (1-\alpha)x, \alpha\tilde{u} + (1-\alpha)u) \in \mathcal{A}(z). \tag{4.39}$$

(4.30) and (4.23) imply that

$$\alpha\tilde{x}(t) + (1-\alpha)x(t) \in H_{\alpha\rho} \text{ for all } t \in [a,b]. \tag{4.40}$$

Define

$$v(t) = \xi_*(t),\ t \in E_1,\ v(t) = \alpha\tilde{u}(t) + (1-\alpha)u(t),\ t \in E_2, \tag{4.41}$$

$$v(t) = u(t),\ t \in E_3,$$

$$y(t) = S(t)z + \int_a^t S(t-s)Bv(s)ds,\ t \in [a,b]. \tag{4.42}$$

Since $\xi_* : [a,b] \to Y$ is a strongly measurable function it follows from (4.41) and (4.39) that $v : [a,b] \to Y$ is a strongly measurable function. Relations (4.41), (4.39) and (4.23) imply that

$$v(t) \in U(t),\ t \in [a,b] \text{ (a.e.)}. \tag{4.43}$$

Arguing as in the proof of Theorem 1.1 ( see (3.39)) we can show that

$$||y(t) - (\alpha\tilde{x}(t) + (1-\alpha)x(t))|| \le 3D_0||B||d \text{ for all } t \in [a,b]. \tag{4.44}$$

In view of (4.44) and (4.35)

$$||y(t) - (\alpha\tilde{x}(t) + (1-\alpha)x(t))|| \le (3/4)\alpha\rho \text{ for all } t \in [a,b]. \tag{4.45}$$

Together with (4.40) this inequality implies that

$$y(t) \in H \text{ for all } t \in [a,b]. \tag{4.46}$$

By (4.46), (4.41), (4.42) and (4.43)

$$(y,v) \in \mathcal{A}(z). \tag{4.47}$$

Now we estimate $I^g(y,v) - I^g(x,u)$. By (4.32), (4.5) and (4.35) for each $t \in [a,b]$

$$(4.48)\quad ||x(t) - (\alpha\tilde{x}(t) + (1-\alpha)x(t))|| = \alpha||x(t) - \tilde{x}(t)|| \leq 2\alpha M_0 = 8D_0||B||d\rho^{-1}M_0.$$

Combined with (4.44) this implies that

$$(4.49)\qquad ||y(t) - x(t)|| \leq dD_0||B||(3 + 8M_0\rho^{-1}) \text{ for all } t \in [a,b].$$

Relations (4.49), (4.33), (4.17), (4.14) and (4.10) imply that for all $t \in [a,b]$

$$(4.50)\qquad ||y(t) - x(t)|| \leq 8D_0||B||(1 + M_0\rho^{-1})\gamma_1(M_1 + 1) < \delta_1 < 1.$$

It follows from (4.41), (4.31), (4.27), (4.10), (4.35), (4.33) and (4.17) that for all $t \in E_2$

$$||v(t) - u(t)|| = ||\alpha\tilde{u}(t) + (1-\alpha)u(t) - u(t)|| = \alpha||\tilde{u}(t) - u(t)||$$

$$(4.51)\quad \leq \alpha(M + N_2) \leq 2\alpha N_2 = 8D_0||B||d\rho^{-1}N_2 < 8D_0||B||\rho^{-1}N_2\gamma_1(M_1 + 1) < \delta_1.$$

In view of (4.50) and (4.5)

$$(4.52)\qquad ||y(t)|| \leq ||x(t)|| + 1 \leq M_0 + 1,\ t \in [a,b].$$

By (4.51), (4.27), (4.33) and (4.17) for all $t \in E_2$

$$||v(t)|| \leq ||u(t)|| + 8D_0||B||d\rho^{-1}N_2 \leq N_2(1 + 8D_0||B|d\rho^{-1})$$

$$(4.53)\qquad \leq N_2(1 + \gamma_1(M_1 + 1)8D_0||B||\rho^{-1}) \leq 2N_2.$$

In view of (4.53), (4.41), (4.27), (4.2), (4.18) and (4.10) for all $t \in [a,b]$

$$(4.54)\qquad ||v(t)|| \leq K.$$

Clearly,

$$I^g(x,u) - I^g(y,v) = \sum_{i=1}^{3}[\int_{E_i} g(t,x(t),u(t))dt - \int_{E_i} g(t,y(t),v(t))dt].$$

It follows from (A2), (4.27) and (4.19) that

$$(4.55)\qquad \int_{E_1} g(t,x(t),u(t))dt \geq \int_{E_1} \phi(||u(t)||)dt \geq \gamma_1^{-1}\int_{E_1} ||u(t)||dt.$$

Relations (4.55) and (4.27) imply that

$$(4.56)\qquad \int_{E_1} g(t,x(t),u(t))dt \geq \gamma_1^{-1}K\mathrm{mes}(E_1).$$

In view of (4.52), (4.41), (4.2), (4.6) and (4.13),

$$(4.57)\qquad f(t,y(t),v(t)) \leq \Delta_1 \text{ for all } t \in E_1.$$

By (4.52), (4.41), (4.2), (4.6), (4.18), (4.21), (4.22), (1.5) and (4.57) for all $t \in E_1$

$$(4.58)\qquad g(t,y(t),v(t)) \leq \Delta + f(t,y(t),v(t)) \leq \Delta + \Delta_1.$$

It follows from (4.58), (4.18), (4.20), (4.56) and (4.17) that

$$\int_{E_1} g(t, y(t), v(t))dt \le (\Delta + \Delta_1)\mathrm{mes}(E_1)$$

$$\le 8^{-1}K\mathrm{mes}(E_1) \le 8^{-1}\int_{E_1} g(t, x(t), u(t))dt. \tag{4.59}$$

Let $t \in E_2$. It follows from (4.52), (4.5), (4.54), (4.27), (4.18), (4.22) and (4.21) that

$$|g(t, x(t), u(t)) - g(t, y(t), v(t))| \le |f(t, x(t), u(t)) - f(t, y(t), v(t))|$$

$$+|(g - f)(t, x(t), u(t))| + |(g - f)(t, y(t), v(t))|$$

$$\le |f(t, x(t), u(t)) - f(t, y(t), v(t))| + 2\Delta. \tag{4.60}$$

By (4.52), (4.5), (4.17), (4.51), (4.50) and the choice of $\delta_1$ (see (4.12)),

$$|f(t, y(t), v(t)) - f(t, x(t), u(t))| \le (32(b - a + 1))^{-1}\epsilon. \tag{4.61}$$

Relations (4.60) and (4.61) imply that

$$|g(t, x(t), u(t)) - g(t, y(t), v(t))| \le 2\Delta + (32(b - a + 1))^{-1}\epsilon.$$

Combined with (4.20) this inequality implies that

$$|\int_{E_2} g(t, x(t), u(t))dt - \int_{E_2} g(t, y(t), v(t))dt| \le (b-a)[2\Delta + (32(b-a+1))^{-1}\epsilon] \le 8^{-1}\epsilon. \tag{4.62}$$

Let $t \in E_3$. It follows from (4.41), (4.52), (4.5), (4.27), the choice of $\delta_0, N_2$ (see (4.10), (4.11)), (4.50) and (4.14) that

$$|f(t, x(t), u(t)) - f(t, y(t), v(t))| = |f(t, x(t), u(t)) - f(t, y(t), u(t))|$$

$$\le \epsilon_0 \min\{f(t, x(t), u(t)),\ f(t, y(t), v(t))\}. \tag{4.63}$$

By (4.52), (4.5), (4.18), (4.41), (4.27), (4.22) and (4.21)

$$|g(t, x(t), u(t)) - g(t, y(t), v(t))| \le |f(t, x(t), u(t)) - f(t, y(t), v(t))|$$

$$+|(g-f)(t, x(t), u(t))| + |(g-f)(t, y(t), v(t))| \le |f(t, x(t), u(t)) - f(t, y(t), v(t))| + 2\Delta.$$

Together with (4.63), (4.20), (4.26), (4.27), (4.5), (4.18), (4.20)-(4.22) this inequality implies that

$$|\int_{E_3} g(t, x(t), u(t))dt - \int_{E_3} g(t, y(t), v(t))dt| \le 2\Delta(b - a)$$

$$+\epsilon_0 \int_{E_3} f(t, x(t), u(t))dt$$

$$\le \epsilon/16 + \epsilon_0 \int_{E_3} g(t, x(t), u(t))dt + \epsilon_0 \int_{E_3} |g(t, x(t), u(t)) - f(t, x(t), u(t))|dt$$

$$\le \epsilon/16 + \epsilon_0(M_1 + 1) + \epsilon_0\Delta(b - a) < \epsilon/8. \tag{4.64}$$

By (4.64), (4.59) and (4.62)

$$I^g(x,u) - I^g(y,v) \geq -8^{-1}\epsilon - 8^{-1}\epsilon \geq -\epsilon/4.$$

Together with (4.24) this implies that $I^g(y,v) \leq U^g(z)+\epsilon$. Lemma 4.1 is proved. □

## 5. PROOF OF LEMMA 1.2

In this section we prove the following result which implies Lemma 1.2.

**Lemma 5.1.** *Let $f \in \mathcal{M}_A$ (respectively, $\mathcal{M}_{Ac}$) and let $\epsilon, N$ be positive numbers. Then there is $g \in \mathcal{L}_A$ (respectively, $\mathcal{L}_{Ac}$) such that:*

$$(f,g) \in \mathcal{E}_{As}(N,\epsilon);$$

*if $f \in \mathcal{M}_B$ ($\mathcal{M}_{Bc}$, respectively), then $g \in \mathcal{L}_B$ ($\mathcal{L}_{Bc}$, respectively) and for each $M > 0$ there is $L > 0$ such that*

$$|g(t,x_1,u) - g(t,x_2,u)| \leq L||x_1 - x_2||$$

*for each $t \in [a,b]$, each $u \in Y$ and each $x_1, x_2 \in B_X(M)$.*

In the proof of Lemma 5.1 we use the following simple auxiliary result which is proved in a straightforward manner.

**Lemma 5.2.** *Let $f_1, f_2 : [a,b] \times X \times Y \to [0,\infty)$ be functions which satisfy (A1) and (A3). Then the following assertions hold:*

*1. If (A4) holds with $f = f_i$, $i = 1,2$, then (A4) holds with $f = f_1 + f_2$ and with $f = f_1 f_2$.*

*2. If (B4) holds with $f = f_i$, $i = 1,2$, then (B4) holds with $f = f_1 + f_2$ and with $f = f_1 f_2$.*

*Proof of Lemma 5.1.* Consider a function $\tilde{\phi} : [0,\infty) \to [0,\infty)$ such that for each integer $i \geq 0$

$$\tilde{\phi}(i) = \phi(i+1),\ \tilde{\phi}(\alpha i + (1-\alpha)(i+1)) = \alpha\tilde{\phi}(i) + (1-\alpha)\tilde{\phi}(i+1) \text{ for all } \alpha \in [0,1].$$

Clearly, the function $\tilde{\phi} : [0,\infty) \to [0,\infty)$ is increasing and Lipschitzian on all bounded subsets of $[0,\infty)$, $\tilde{\phi}(t) \geq \phi(t)$ for all $t \geq 0$ and $\lim_{t\to\infty} \tilde{\phi}(t)/t = \infty$.

Let $f \in \mathcal{M}_A$ and let $\epsilon, N > 0$. Set

$$\psi(t) = 1,\ t \in [0, N+1],\ \psi(t) = 0,\ t \in [N+2, \infty),$$

$$\psi(t) = N + 2 - t,\ t \in (N+1, N+2),$$

$$g(t,x,u) = \psi(||x||)\psi(||u||)f(t,x,u) + (1 - \psi(||x||)\psi(||u||))[||x|| + ||u|| + \tilde{\phi}(||u||) + 1],$$

$$(t,x,u) \in [a,b] \times X \times Y.$$

Clearly $g$ satisfies (A1)-(A3). It is not difficult to see that (B4) holds for each of the following functions

$$(t,x,u) \to \psi(||x||),\ (t,x,u) \to \psi(||u||),\ (t,x,u) \to ||x||,\ (t,x,u) \to ||u||,$$

$$(t,x,u) \to \tilde{\phi}(||u||),\ (t,x,u) \in [a,b] \times X \times Y.$$

Together with Lemma 5.2 and the definition of $g$ this implies that (A4) holds for $g$ and if $f$ satisfies (B4), then (B4) holds for $g$. Clearly (A5) holds for $g$. Thus $g \in \mathcal{L}_A$. Evidently $(f,g) \in \mathcal{E}_{As}(N,\epsilon)$. Clearly, if $f \in \mathcal{M}_{Ac}$, then $g \in \mathcal{L}_{Ac}$.

Assume that $f \in \mathcal{M}_B$. We have already shown that $g \in \mathcal{M}_B$. Let $M > 0$. Since (B4) holds for $g$ there is $L > 0$ such that for each $t \in [a,b]$, each $x_1, x_2 \in B_X(M+N+4)$ and each $y_1, y_2 \in B_Y(M+N+4)$ the inequality

$$|g(t,x_1,y_1) - g(t,x_2,y_2)| \le L(||x_1 - x_2|| + ||y_1 - y_2||)$$

holds. Assume that

$$t \in [a,b],\ u \in Y,\ x_1, x_2 \in B_X(M).$$

There are two cases: $||u|| \ge N+4$; $||u|| < N+4$. Assume that $||u|| \ge N+4$. By this inequality, the definition of $g$ and the definition of $\psi$

$$|g(t,x_1,u) - g(t,x_2,u)| = |||x_1|| - ||x_2||| \le ||x_1 - x_2||.$$

Assume that $||u|| < N+4$. Then it follows from the choice of $L$ that

$$|g(t,x_1,u) - g(t,x_2,u)| \le L||x_1 - x_2||.$$

Clearly, in both cases

$$|g(t,x_1,u) - g(t,x_2,u)| \le (L+1)||x_1 - x_2||.$$

This completes the proof of Lemma 5.1.

## 6. **PROOFS OF THEOREMS 1.3-1.5**

*Proof of Theorem 1.5.* By Lemma 1.2, $\mathcal{L}_A$ is an everywhere dense subset of $\mathcal{M}_A$ with the strong topology and $\mathcal{L}_{Ac}$ is an everywhere dense subset of $\mathcal{M}_{Ac}$ with the strong topology.

Let $f \in \mathcal{L}_A$ and $n$ be a natural number. By Lemma 4.1 there exist $K(f,n) > 0$ and an open neighborhood $\mathcal{U}(f,n)$ of $f$ in $\mathcal{M}_A$ with the weak topology such that the following property holds:

(P1) If $g \in \mathcal{U}(f,n)$ and $z \in \tilde{H}_{1/n,n} \cap B_X(n)$, then there is $(x,u) \in \mathcal{A}(z)$ such that $||u(t)|| \le K(f,n)$ for almost every $t \in [a,b]$ and $I^g(x,u) \le U^g(z) + 1/n$.

Define

$$\mathcal{F}_A = \cap_{n=1}^{\infty} \cup \{\mathcal{U}(f,n) :\ f \in \mathcal{L}_A\},\ \mathcal{F}_{Ac} = \cap_{n=1}^{\infty} \cup \{\mathcal{U}(f,n) \cap \mathcal{M}_{Ac} :\ f \in \mathcal{L}_{Ac}\}.$$

Clearly $\mathcal{F}_A$ is a countable intersection of open (in the weak topology) everywhere dense (in the strong topology) subsets of $\mathcal{M}_A$ and $\mathcal{F}_{Ac}$ is a countable intersection of open (in the weak topology) everywhere dense (in the strong topology) subsets of $\mathcal{M}_{Ac}$. Let

$$g \in \mathcal{F}_A,\ z \in \cup\{\tilde{H}_{\rho,M} :\ \rho, M > 0\}, \epsilon > 0. \tag{6.1}$$

Choose a natural number $m$ such that

$$z \in \tilde{H}_{1/m,m} \cap B_X(M),\ \epsilon > 1/m. \tag{6.2}$$

By (6.1) and the definition of $\mathcal{F}_A$ there is $f_m \in \mathcal{L}_A$ such that $g \in \mathcal{U}(f_m, m)$. It follows from this inclusion, (P1) and (6.1) that there is $(x, u) \in \mathcal{A}(z)$ such that $||u(t)|| \le K(f_m, m)$ for almost every $t \in [a, b]$ and

$$I^g(x, u) \le U^g(z) + m^{-1} < U^g(z) + \epsilon.$$

Since $\epsilon$ is an arbitrary positive number we conclude that

$$\inf\{I^g(y, v) :\ (y, v) \in \mathcal{A}_L(z)\} \le U^g(z) = \inf\{I^g(y, v) :\ (y, v) \in \mathcal{A}(z)\}.$$

This completes the proof of Theorem 1.5.

*Proof of Theorem 1.4.* By Lemma 1.2, $\mathcal{L}_B$ is an everywhere dense subset of $\mathcal{M}_B$ and $\mathcal{L}_{Bc}$ is an everywhere dense subset of $\mathcal{M}_{Bc}$. Let $f \in \mathcal{L}_B$ and $n$ be a natural number. By Theorem 1.1 there exists $K(f, n) > 0$ such that the following property holds:

(P2) If $g \in \mathcal{M}_B$ satisfies $(f, g) \in \mathcal{E}_B(K(f, n), 8n)$, $z \in \tilde{H}_{1/n,n} \cap B_X(n)$ and if $(x, u) \in \mathcal{A}(z)$ satisfies

$$\mathrm{mes}(\{t \in [a, b] :\ ||u(t)|| \ge K(f, n)\}) > 0,$$

then there exists $(y, v) \in \mathcal{A}(z)$ such that $I^g(y, v) < I^g(x, u)$,

$$||v(t)|| \le K(f, n) \text{ for almost every } t \in [a, b].$$

Denote by $\mathcal{U}(f, n)$ an open neighborhood of $f$ in $\mathcal{M}_B$ such that

$$\{g \in \mathcal{M}_B :\ (f, g) \in \mathcal{E}_B(K(f, n), n)\} \subset \mathcal{U}(f, n)$$

$$\subset \{g \in \mathcal{M}_B :\ (f, g) \in \mathcal{E}_B(K(f, n), 2n)\}. \tag{6.3}$$

Define

$$\mathcal{F}_B = \cap_{n=1}^{\infty} \cup \{\mathcal{U}(f, n) :\ f \in \mathcal{L}_B\},\ \mathcal{F}_{Bc} = \cap_{n=1}^{\infty} \cup \{\mathcal{U}(f, n) \cap \mathcal{M}_{Bc} :\ f \in \mathcal{L}_{Bc}\}.$$

Clearly $\mathcal{F}_B$ is a countable intersection of open everywhere dense subsets of $\mathcal{M}_B$ and $\mathcal{F}_{Bc}$ is a countable intersection of open everywhere dense subsets of $\mathcal{M}_{Bc}$.

Let

$$g \in \mathcal{F}_B \text{ and } M, q, \rho > 0. \tag{6.4}$$

Choose a natural number

$$m > \max\{M, 1/\rho, 4q\}.$$

In view of (6.4) and the definition of $\mathcal{F}_B$ there is $f_m \in \mathcal{L}_B$ such that

$$g \in \mathcal{U}(f_m, m). \tag{6.5}$$

Assume that

$$h \in \mathcal{M}_B,\ (g, h) \in \mathcal{E}_B(K(f_m, m), q),\ z \in \tilde{H}_{\rho,M} \cap B_X(M), \tag{6.6}$$

$$(x, u) \in \mathcal{A}(z),\ \ \text{mes}(\{t \in [a, b] :\ ||u(t)|| \geq K(f_m, m)\}) > 0.$$

It follows from (6.6), (6.5) and (6.3) that

$$(h, f_m) \in \mathcal{E}_B(K(f_m, m), 3m).$$

Clearly, $z \in \tilde{H}_{1/m,m} \cap B_X(m)$. Now by property (P2) there is $(y, v) \in \mathcal{A}(z)$ such that $I^g(y, v) < I^g(x, u)$,

$$||v(t)|| \leq K(f_m, m) \text{ for almost every } t \in [a, b].$$

Theorem 1.4 is proved.

*Proof of Theorem 1.3.* By Lemma 1.2, $\mathcal{L}_B$ is an everywhere dense subset of $\mathcal{M}_B$ and $\mathcal{L}_{Bc}$ is an everywhere dense subset of $\mathcal{M}_{Bc}$. Let $f \in \mathcal{M}_B$. By Theorem 1.1 there is $K(f) > 0$ such that the following property holds:

(P3) If $g \in \mathcal{M}_B$ satisfies $(f, g) \in \mathcal{E}_K(K(f), 8q)$, $z \in \tilde{H}_{\rho,M} \cap B_X(M)$ and if $(x, u) \in \mathcal{A}(z)$ satisfies $\text{mes}(\{t \in [a, b] :\ ||u(t)|| \geq K(f)\}) > 0$, then there exists $(y, v) \in \mathcal{A}(z)$ such that $I^g(y, v) < I^g(x, u)$ and $||v(t)|| \leq K(f)$ for almost every $t \in [a, b]$.

Denote by $\mathcal{U}(f)$ an open neighborhood of $f$ in $\mathcal{M}_B$ such that

$$\{g \in \mathcal{M}_B :\ (f, g) \in \mathcal{E}_B(K(f), q)\} \subset \mathcal{U}(f)$$

$$\subset \{g \in \mathcal{M}_B :\ (f, g) \in \mathcal{E}_B(K(f), 2q)\}. \tag{6.7}$$

Define

$$\mathcal{F} = \cup\{\mathcal{U}(f) :\ f \in \mathcal{L}_B\},\ \mathcal{F}_c = \mathcal{M}_{Bc} \cap [\cup\{\mathcal{U}(f) :\ f \in \mathcal{L}_{Bc}\}].$$

Clearly, $\mathcal{F}$ is a n open everywhere dense subset of $\mathcal{M}_B$ and $\mathcal{F}_c$ is an open everywhere dense subset of $\mathcal{M}_{Bc}$. Let $g \in \mathcal{F}_B$. By this inclusion and the definition of $\mathcal{F}$ there is $f \in \mathcal{L}_B$ such that

$$g \in \mathcal{U}(f). \tag{6.8}$$

Assume that

$$h \in \mathcal{M}_B,\ (g, h) \in \mathcal{E}_B(K(f), q),\ z \in \tilde{H}_{\rho,M} \cap B_X(M), \tag{6.9}$$

$$(x, u) \in \mathcal{A}(z),\ \ \text{mes }(\{t \in [a, b] :\ ||u(t)|| \geq K(f)\}) > 0.$$

It follows from (6.7)-(6.9) that $(h, f) \in \mathcal{E}_B(K(f), 3q)$. Now by property (P3) and (6.9) there is $(y, v) \in \mathcal{A}(z)$ such that $I^g(y, v) < I^g(x, u)$ and $||v(t)|| \le K(f)$ for almost every $t \in [a, b]$. Theorem 1.4 is proved.

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