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# DIFFERENTIABLE PERTURBATIONS OF ORNSTEIN-UHLENBECK OPERATORS

L. MANCA

Dipartimento di Matematica P. e A., Università di Padova,
Via Trieste 63, 35121 Padova, Italy
manca@math.unipd.it

**ABSTRACT.** We prove an extension theorem for a small perturbation of the Ornstein-Uhlenbeck operator $(L, D(L))$ in the space of all uniformly continuous and bounded functions $f : H \to \mathbb{R}$, where $H$ is a separable Hilbert space. We consider a perturbation of the form $N_0\varphi = L\varphi + \langle D\varphi, F\rangle$ where $F : H \to H$ is bounded and Fréchet differentiable with uniformly continuous and bounded differential. Hence, we prove that $N_0$ is essentially $m$-dissipative and its closure in $C_b(H)$ coincides with the infinitesimal generator of a diffusion semigroup associated to a stochastic differential equation in $H$.

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## 1. INTRODUCTION AND SETTING OF THE PROBLEM

Let $H$ be a separable Hilbert space endowed with scalar product $\langle\cdot,\cdot\rangle$ and norm $|\cdot|$. We shall always identify $H$ with its topological dual space $H^*$. $\mathcal{L}(H)$ is the Banach space of all the linear and continuous maps in $H$, endowed with the usual norm $\|\cdot\|_{\mathcal{L}(H)}$. With $C_b(H)$ (resp. $C_b(H;H)$) we denote the Banach space of all uniformly continuous and bounded functions $f : H \to \mathbb{R}$ (resp. $f : H \to H$), endowed with the supremum norm $\|\cdot\|$ (resp. $\|\cdot\|_0$). We also denote by $C_b^1(H)$ (resp. $C_b^1(H;H)$) the space of all $f \in C_b(H)$ (resp. $C_b(H;H)$) that are Fréchet differentiable with differential in $C_b(H;H)$ (resp. with uniformly continuous and bounded differential $Df : H \to \mathcal{L}(H)$). We assume the following

**Hypothesis 1.1.** (i) *$A\colon D(A) \subset H \to H$ is the infinitesimal generator of a strongly continuous semigroup $(e^{tA})_{t\geq 0}$ of type $\mathcal{G}(1,\omega)$, i.e. there exists $\omega \in \mathbb{R}$ such that*

$$\|e^{tA}\|_{\mathcal{L}(H)} \leq e^{\omega t}, \quad t \geq 0; \tag{1.1}$$

(ii) *$Q \in \mathcal{L}(H)$ is self adjoint and positive;*

(iii) *For any $t > 0$ the linear operator $Q_t$, defined as*

$$Q_t x = \int_0^t e^{sA} Q e^{sA^*} x\, ds, \;\; x \in H, \; t \geq 0, \tag{1.2}$$

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*is of trace class.*

(iv) $F \in C_b^1(H;H)$, *and* $K = \sup_{x\in H} \|DF(x)\|_{\mathcal{L}(H)}$.

It is well known (see, for instance, [4]) that thanks to conditions (i)–(iii) it is possible to define the so called *Ornstein-Uhlenbeck* (OU) semigroup $(R_t)_{t\geq 0}$ in $C_b(H)$ by the formula

$$R_t\varphi(x) = \int_0 \varphi(e^{tA}x + y)N_{Q_t}(dy), \quad x \in H, \tag{1.3}$$

where $N_{Q_t}$ is the Gaussian measure on $H$ of mean 0 and covariance operator $Q_t$ (see [4]). It turns out that the OU semigroup in $C_b(H)$ is not a strongly continuous semigroup with respect to the supremum norm, but it is strongly continuous with respect to weaker topologies (See [1], [5], [6], [9]). However, it is possible to define its infinitesimal generator by its resolvent or, in a equivalent way, by means of the approach of the $\pi$-semigroups introduced in [9]

$$\begin{cases} D(L) = \Big\{\varphi \in C_b(H) : \exists g \in C_b(H), \lim_{t\to 0^+} \dfrac{R_t\varphi(x) - \varphi(x)}{t} = g(x), \\ \qquad\qquad x \in H, \sup_{t\in(0,1)} \left\| \dfrac{R_t\varphi - \varphi}{t} \right\| < \infty \Big\} \\ \\ L\varphi(x) = \lim_{t\to 0^+} \dfrac{R_t\varphi(x) - \varphi(x)}{t}, \quad \varphi \in D(L),\, x \in H. \end{cases} \tag{1.4}$$

We are interested in the operator $(N_0, D(N_0))$ defined by

$$N_0\varphi = L\varphi + \mathcal{F}\varphi, \quad \varphi \in D(N_0) = D(L) \cap C_b^1(H),$$

where

$$\mathcal{F}\varphi(x) = \langle D\varphi(x), F(x)\rangle.$$

Now let us consider the stochastic differential equation in $H$

$$\begin{cases} dX(t) = \big(AX(t) + F(X(t))\big)dt + Q^{1/2}dW(t) & t > 0, \\ X(0) = x & x \in H, \end{cases} \tag{1.5}$$

where $(W(t))_{t\geq 0}$ is a cylindrical Wiener process defined on a stochastic basis $(\Omega, \mathcal{G}, (\mathcal{G}_t)_{t\geq 0}, \mathbb{P})$. Since $F \in C_b^1(H;H)$, problem (1.5) has a unique mild solution $\big(X(t,x)\big)_{t\geq 0, x\in H}$ (see [4]), that is for any $x \in H$ the process $\{X(\cdot, x), t \geq 0\}$ is adapted to the filtration $(\mathcal{G}_t)_{t\geq 0}$ and it is continuous in mean square, i.e.

$$\lim_{t\to s} \mathbb{E}\big[|X(t,x) - X(s,x)|^2\big] = 0, \quad \forall s \geq 0.$$

This allows us to define a transition semigroup $(P_t)_{t\geq 0}$ in $C_b(H)$, by setting

$$P_t\varphi(x) = \mathbb{E}\big[\varphi(X(t,x))\big], \quad t \geq 0,\, \varphi \in C_b(H),\, x \in H.$$

The semigroup $(P_t)_{t\geq 0}$ is not strongly continuous in $C_b(H)$. However, it is a $\pi$-semigroup, and we can define its infinitesimal generator $(N, D(N))$ in the same way as for the OU semigroup

$$\begin{cases} D(N) = \Big\{\varphi \in C_b(H) : \exists g \in C_b(H), \lim\limits_{t\to 0^+} \dfrac{P_t\varphi(x) - \varphi(x)}{t} = g(x), \\ \qquad\qquad x \in H, \ \sup\limits_{t\in(0,1)} \left\| \dfrac{P_t\varphi - \varphi}{t} \right\| < \infty \Big\} \\ \\ N\varphi(x) = \lim\limits_{t\to 0^+} \dfrac{P_t\varphi(x) - \varphi(x)}{t}, \quad \varphi \in D(N),\, x \in H. \end{cases}$$

The main result of this paper is the following

**Theorem 1.2.** *Let us assume that Hypothesis* 1.1 *holds. Then, the operator* $(N_0, D(N_0))$, *defined by* $D(N_0) = D(L)\cap C_b^1(H)$ *and* $N_0\varphi = L\varphi+\mathcal{F}\varphi$, $\forall \varphi \in D(N_0)$, *is* $m$*-dissipative in* $C_b(H)$ *and its closure is the operator* $(N, D(N))$.

In [2], it is proved that Theorem 1.2 holds with $F \in C_b^{1,1}(H; H)$, that is $F$ is Fréchet differentiable and its differential $DF : H \to \mathcal{L}(H)$ is Lipschitz continuous.

Perturbations of OU operators as been the object of several papers (see, for instance, $[2, 3, 5\text{–}7, 10]$). Frequently, additional assumptions are taken on the OU operator in order to have $D(L) \subset C_b^1(H)$, see [5], [6].

In order to prove Theorem 1.2 we develope a technique introduced in [2]. The idea is the following: since $F \in C_b^1(H; H)$, there exists a unique solution $\eta(\cdot, x)$ of the abstract Cauchy problem

$$\begin{cases} \dfrac{d}{d\varepsilon}\eta(\varepsilon, x) = F(\eta(\varepsilon, x)), & \varepsilon > 0, \\ \eta(0, x) = x, & x \in H. \end{cases}$$

Then, for any $\varepsilon > 0$ we define the operators $\mathcal{F}_\varepsilon : C_b(H) \to C_b(H)$ and $N_\varepsilon : D(N_\varepsilon) \subset C_b(H) \to C_b(H)$ by setting

$$\mathcal{F}_\varepsilon\varphi(x) = \frac{1}{\varepsilon}\big(\varphi(\eta(\varepsilon, x)) - \varphi(x)\big),$$

$$\begin{cases} D(N_\varepsilon) = D(L) \cap C_b^1(H), \\ N_\varepsilon\varphi = L\varphi + \mathcal{F}_\varepsilon\varphi, & \varphi \in D(N_\varepsilon). \end{cases}$$

By an approximation argument, we are able to prove that the operator $(N_0, D(N_0))$ is $m$-dissipative in $C_b(H)$. Then, by the Lumer-Phillips theorem, it will follow that the closure of $(N_0, D(N_0))$ coincides with the operator $(N, D(N))$.

1.1. **Properties of $\mathcal{F}_\varepsilon$.** The following lemma collects some useful properties of $\eta$.

**Lemma 1.3.** *The following estimates hold*

$$|\eta(t,x)| \le e^{\|F\|_0 t}|x|; \tag{1.6}$$

$$|\eta(t,x)-\eta(t,y)| \le e^{Kt}|x-y|; \tag{1.7}$$

$$|\eta(t,x)-x| \le c\|F\|_0 t \tag{1.8}$$

$$\|\eta_x(t,x)\|_{\mathcal{L}(H)} \le e^{Kt} \tag{1.9}$$

$$\|\eta_x(t,x)-\eta_x(t,y)\|_{\mathcal{L}(H)} \le e^{Kt}\theta_{DF}(e^{Kt}|x-y|), \tag{1.10}$$

*where $\theta_{DF}:\mathbb{R}^+\times\mathbb{R}^+\to\mathbb{R}^+$ is the modulus of continuity of $DF$.*

*Proof.* (1.6), (1.8), (1.9) have been proved in [2, Lemma 2.1].
(1.7). We have

$$|\eta(t,x)-\eta(t,y)| \le |x-y| + \int_0^t \big|F(\eta(s,x))-F(\eta(s,y))|ds \le K\int_0^t |\eta(s,x)-\eta(s,y)|ds.$$

Then (1.7) follows by Gronwall's Lemma.
(1.10). Let $x,y,h\in H$ and set

$$r^h(t) = \eta_x(t,x)\cdot h - \eta_x(t,y)\cdot h = p^h(t,x)-p^h(t,y),$$

where $P^h(t,x)=\eta_x(t,x)\cdot h$ and $p^h(t,y)=\eta_x(t,y)\cdot h$. Then $r^h(t)$ fulfills the following equation

$$\begin{cases} \dfrac{d}{dt}r^h(t) = DF(\eta(t,x))r^h(t) + \big[DF(\eta(t,x))-DF(\eta(t,y))\big]p^h(t,x), \quad t>0 \\ r^h(0)=0. \end{cases}$$

Since $|DF(\eta(t,x))r^h(t)| \le K|r^h(t)|$ it follows that $r^h(t)$ is bounded by

$$|r^h(t)| \le \int_0^t e^{K(t-s)}\big\|DF(\eta(s,x))-DF(\eta(s,y))\big\|_{\mathcal{L}(H)}|p^h(s,x)|ds.$$

By taking into account that $DF : H \to \mathcal{L}(H;H)$ is uniformly continuous and bounded, we denote by $\theta_{DF}$ the modulus of continuity of $DF$. Hence, by (1.7), (1.9) we have

$$|r^h(t)| \le \int_0^t e^{Ks}\theta_{DF}(|\eta(s,x)-\eta(s,y)|)ds|h| \le e^{Kt}\theta_{DF}(e^{Kt}|x-y|)|h|$$

□

**Proposition 1.4.** *For any $\varphi\in C_b^1(H)$ we have*

$$\lim_{\varepsilon\to 0^+}\mathcal{F}_\varepsilon\varphi = \mathcal{F}\varphi \quad \textit{in } C_b(H). \tag{1.11}$$

$$\|\mathcal{F}_\varepsilon\varphi\| \le \|D\varphi\|_0\|F\|_0. \tag{1.12}$$

*Proof.* For all $\varphi \in C_b^1(H)$ we have

$$\begin{aligned}\mathcal{F}_\varepsilon\varphi(x) - \mathcal{F}\varphi(x) &= \frac{1}{\varepsilon}\int_0^\varepsilon \langle D\varphi(\eta(s,x)) - D\varphi(x), F(\eta(s,x))\rangle\, ds \\ &\quad + \frac{1}{\varepsilon}\int_0^\varepsilon \langle D\varphi(x), F(\eta(s,x)) - F(x)\rangle\, ds.\end{aligned}$$

Then by (1.8) we have

$$\begin{aligned}&|\mathcal{F}_\varepsilon\varphi(x) - \mathcal{F}\varphi(x)| \le \\ &\le \frac{1}{\varepsilon}\int_0^\varepsilon (|\theta_{D\varphi}(|\eta(s,x) - x|)\|F\|_0 + \|D\varphi\|_0 K|\eta(s,x) - x|))\, ds \\ &\le \frac{1}{\varepsilon}\int_0^\varepsilon (\theta_{D\varphi}(\|F\|_0 s|)\|F\|_0 + \|D\varphi\|_0 K\|F\|_0 s)\, ds \\ &\le (\theta_{D\varphi}(\|F\|_0\varepsilon|) + \|D\varphi\|_0 K\varepsilon)\, \|F\|_0\end{aligned}$$

where $\theta_{D\varphi}$, is the modulus of continuity of $D\varphi$. This yields (1.11). Moreover, we have

$$\mathcal{F}_\varepsilon\varphi(x) = \frac{1}{\varepsilon}\int_0^\varepsilon \langle D\varphi(\eta(s,x)), F(\eta(s,x))\rangle\, ds$$

that implies (1.12). □

1.2. $m$**-dissipativity of** $N$**.** Given $\varepsilon > 0$ we introduce the following approximating operator

$$N_\varepsilon = L + \mathcal{F}_\varepsilon,\ D(N_\varepsilon) = D(L) \cap C_b^1(H).$$

We have

**Proposition 1.5.** *$N_\varepsilon$ is an essentially $m$-dissipative operator in $C_b(H)$ for any $\varepsilon > 0$. Moreover, for any $f \in C_b^1(H)$ and any $\lambda > \omega + (e^{\varepsilon K} - 1)/\varepsilon$ the operator*

$$R(\lambda, N_\varepsilon) = \big(1 - T_\lambda\big)^{-1} R\left(\lambda + \frac{1}{\varepsilon}, L\right),$$

*where $T_\lambda : C_b(H) \to C_b(H)$ is defined by*

$$T_\lambda\psi(x) = R\left(\lambda + \frac{1}{\varepsilon}, L\right)\left[\frac{1}{\varepsilon}\psi(\eta(\varepsilon,x))\right], \quad x \in H,\ \psi \in C_b(H) \tag{1.13}$$

*maps $C_b^1(H)$ into $D(L) \cap C_b^1(H)$ and*

$$\|DR(\lambda, N_\varepsilon)f\|_0 \le \frac{1}{\lambda - \omega - \frac{e^{K\varepsilon}-1}{\varepsilon}}\|Df\|_0. \tag{1.14}$$

*Proof.* Let $\varepsilon > 0$, $\lambda > 0$, $f \in C_b(H)$. The equation

$$\lambda\varphi_\varepsilon - L\varphi_\varepsilon - \mathcal{F}(\varphi_\varepsilon) = f$$

is equivalent to

$$\left(\lambda + \frac{1}{\varepsilon}\right)\varphi_\varepsilon - L\varphi_\varepsilon - \mathcal{F}(\varphi_\varepsilon) = f + \frac{1}{\varepsilon}\varphi_\varepsilon(\eta(\varepsilon,\cdot))$$

and to

$$\varphi_\varepsilon = R\left(\lambda + \frac{1}{\varepsilon}, L\right) f + T_\lambda \varphi_\varepsilon. \tag{1.15}$$

Since, as we can easily see, for any $\lambda > 0$

$$\|T_\lambda \psi\| \le \frac{1}{1+\lambda\varepsilon}\|\psi\|, \quad \forall \psi \in C_b(H), \tag{1.16}$$

the operator $T_\lambda$ is a contraction in $C_b(H)$ and so equation (1.15) has a unique solution $\varphi_\varepsilon \in C_b(H)$ done by $\varphi_\varepsilon = R(\lambda, N_\varepsilon)f$. Moreover, by (1.13), (1.16) it holds

$$\|\varphi_\varepsilon\| \le \frac{1}{\lambda + \frac{1}{\varepsilon}}\left[\|f\| + \frac{1}{\varepsilon}\|\varphi_\varepsilon\|\right].$$

Consequently,

$$\|\varphi_\varepsilon\| \le \frac{1}{\lambda}\|f\|.$$

Then, $N_\varepsilon$ is $m$-dissipative. Now let $f \in C_b^1(H)$. We recall that for any $\lambda > 0$, $\psi \in C_b(H)$

$$R(\lambda, L)\psi(x) = \int_0^\infty e^{-\lambda t} R_t \psi(x) dt \tag{1.17}$$

and that

$$DR_t\psi(x) = \int_H e^{tA^*} D\psi(e^{tA}x + y) N_{Q_t}(dy).$$

Hence, for any $\lambda > \omega$

$$DR(\lambda, L)\psi(x) = \int_0^\infty \int_H e^{-\lambda t} e^{tA^*} D\psi(e^{tA}x + y) N_{Q_t}(dy) dt \tag{1.18}$$

and so

$$\|DR(\lambda, L)\psi\|_0 \le \frac{1}{\lambda - \omega}\|D\psi\|_0 \tag{1.19}$$

Moreover, as it can be easily seen by (1.18), $DR(\lambda, L)\psi$ is uniformly continuous. Then $R(\lambda, L) : C_b^1(H) \to C_b^1(H)$. Now, in order to prove that $T_\lambda : C_b^1(H) \to C_b^1(H)$ it is sufficient to show that $\psi(\eta(\varepsilon, x)) \in C_b^1(H)$, for any $\psi \in C_b^1(H)$. Indeed, by a standard computation, we have

$$D\psi(\eta(\varepsilon, \cdot))(x) = \eta_x^*(\varepsilon, x) D\psi(\eta(\varepsilon, x)), \quad x \in H.$$

Consequently, by (1.7), (1.10) we have

$$\begin{aligned}
|D\psi(\eta(\varepsilon, \cdot))(x) - D\psi(\eta(\varepsilon, \cdot))(\overline{x})| &\le \|\eta_x^*(\varepsilon, x) - \eta_x^*(\varepsilon, \overline{x})\|_{\mathcal{L}(H)} |D\psi(\eta(\varepsilon, x))| \\
&\quad + \|\eta_x^*(\varepsilon, \overline{x})\|_{\mathcal{L}(H)} |D\psi(\eta(\varepsilon, x)) - D\psi(\eta(\varepsilon, \overline{x}))| \\
&\le e^{\varepsilon K}\theta_{DF}(e^{\varepsilon K}|x - \overline{x}|)\|D\psi\|_0 + e^{\varepsilon K}\theta_{D\psi}(|\eta(\varepsilon, x) - \eta(\varepsilon, \overline{x})|) \\
&\le e^{\varepsilon K}\theta_{DF}(e^{\varepsilon K}|x - \overline{x}|)\|D\psi\|_0 + e^{\varepsilon K}\theta_{D\psi}(e^{\varepsilon K}|x - \overline{x}|),
\end{aligned}$$

for any $x, \overline{x} \in H$. So, $DT_\lambda\psi(\cdot)$ is uniformly continuous. Now we prove that $T_\lambda$ is a contraction in $C_b^1(H)$. By (1.13), (1.17) we have

$$\begin{aligned} T_\lambda\psi(x) &= \frac{1}{\varepsilon}\int_0^\infty e^{-(\lambda+\frac{1}{\varepsilon})t} R_t\psi(\eta(\varepsilon,\cdot))(x)dt \\ &= \frac{1}{\varepsilon}\int_0^\infty \int_H e^{-(\lambda+\frac{1}{\varepsilon})t}\psi(\eta(\varepsilon, e^{tA}x+y))N_{Q_t}(dy)dt \end{aligned}$$

Then

$$DT_\lambda\psi(x) = \frac{1}{\varepsilon}\int_0^\infty \int_H e^{-(\lambda+\frac{1}{\varepsilon})t} e^{tA^*}\eta_x^*(\varepsilon, e^{tA}x+y)D\psi(\eta(\varepsilon, e^{tA}x+y))N_{Q_t}(dy)dt$$

By (1.9) it follows

$$|DT_\lambda\psi(x)| \le \frac{1}{\varepsilon}\int_0^\infty e^{-(\lambda+\frac{1}{\varepsilon}-\omega)t}e^{\varepsilon K}\|D\psi\|_0 dt = \frac{e^{\varepsilon K}}{1+\varepsilon(\lambda-\omega)}\|D\psi\|_0.$$

Therefore, for any $\lambda > \omega + (e^{\varepsilon K}-1)/\varepsilon$ the linear operator $T_\lambda$ is a contraction in $C_b^1(H)$ and its resolvent satisfies

$$(1-T_\lambda)^{-1}(C_b^1(H)) \subset C_b^1(H),$$

$$\|D(1-T_\lambda)^{-1}\psi\|_0 \le \frac{1}{1-\frac{e^{\varepsilon K}}{1+\varepsilon(\lambda-\omega)}}\|D\psi\|_0. \tag{1.20}$$

This implies

$$R(\lambda, N_\varepsilon)(C_b^1(H)) = (1-T_\lambda)^{-1}R\Big(\lambda+\frac{1}{\varepsilon}, L\Big)(C_b^1(H)) \subset C_b^1(H).$$

Then, since $C_b^1(H)$ is dense in $C_b(H)$, it follows that $N_\varepsilon$ is essentially $m$-dissipative. Finally, (1.14) follows by (1.19) and (1.20). □

**Lemma 1.6.** *The operator $N_0$ is dissipative in $C_b(H)$.*

*Proof.* We have to prove that $\|\lambda\varphi - N_0\varphi\| \ge \lambda\|\varphi\|$ for any $\varphi \in D(N_0)$, $\lambda > 0$. So, if $\varphi \in D(L)\cap C_b^1(H)$ and $\lambda > 0$ we set

$$\lambda\varphi - L\varphi - \mathcal{F}\varphi = f.$$

then for any $\varepsilon > 0$ we have

$$\lambda\varphi - N_\varepsilon\varphi = f + \mathcal{F}\varphi - \mathcal{F}_\varepsilon\varphi.$$

It follows

$$\varphi = R(\lambda, N_\varepsilon)(f + \mathcal{F}\varphi - \mathcal{F}_\varepsilon\varphi)$$

and

$$\|\varphi\| \le \frac{1}{\lambda}(\|f\| + \|\mathcal{F}\varphi - \mathcal{F}_\varepsilon\varphi\|)$$

Then by (1.11) it follows

$$\|\varphi\| \le \frac{1}{\lambda}\|f\|.$$

□

Since $N_0$ is dissipative, its closure $\overline{N}_0$ is still dissipative (maybe it is multivalued). By the following theorem follows Theorem 1.2.

**Theorem 1.7.** *$N_0$ is essentially $m$-dissipative.*

*Proof.* Let $f \in C_b^1(H)$, $\varepsilon \in (0,1)$ and $\lambda > \omega + e^K - 1$. We denote by $\varphi_\varepsilon$ the solution of

$$\lambda\varphi_\varepsilon - N_\varepsilon\varphi_\varepsilon = f.$$

By Proposition (1.5) we have $\varphi_\varepsilon \in D(L) \cap C_b^1(H) = D(N_0)$, then $\varphi_\varepsilon$ is solution of

$$\lambda\varphi_\varepsilon - N_0\varphi_\varepsilon = f + \mathcal{F}_\varepsilon\varphi_\varepsilon - \mathcal{F}\varphi_\varepsilon.$$

We claim that $\mathcal{F}_\varepsilon\varphi_\varepsilon - \mathcal{F}\varphi_\varepsilon \to 0$ in $C_b(H)$ as $\varepsilon \to 0^+$. Indeed it holds

$$\begin{aligned}\mathcal{F}_\varepsilon\varphi_\varepsilon(x) - \mathcal{F}\varphi_\varepsilon(x) &= \frac{1}{\varepsilon}\int_0^\varepsilon \left(\langle D\varphi_\varepsilon(\eta(s,x)), F(\eta(s,x))\rangle + \langle D\varphi_\varepsilon(x), F(x)\rangle\right) ds \\ &= \frac{1}{\varepsilon}\int_0^\varepsilon \big(\langle D\varphi_\varepsilon(\eta(s,x)) - D\varphi_\varepsilon(x), F(\eta(s,x))\rangle \\ &\qquad + \langle D\varphi_\varepsilon(x), F(\eta(s,x)) - F(x)\rangle\big)\, ds.\end{aligned}$$

Hence

$$\begin{aligned}&|\mathcal{F}_\varepsilon\varphi_\varepsilon(x) - \mathcal{F}\varphi_\varepsilon(x)| \\ &\quad\le \frac{1}{\varepsilon}\int_0^\varepsilon \left(|D\varphi_\varepsilon(\eta(s,x)) - D\varphi_\varepsilon(x)|\|F\|_0 + \|D\varphi_\varepsilon\|_0|F(\eta(s,x)) - F(x)|\right) ds\end{aligned}$$

By (1.8) we have

$$|F(\eta(s,x)) - F(x)| \le K|\eta(s,x) - x| \le K\|F\|_0 s \le K\|F\|_0\varepsilon.$$

Notice now that since $\varphi_\varepsilon = R(\lambda, N_\varepsilon)f$ and $\varepsilon \in (0,1)$, by (1.14) it follows

$$\|D\varphi_\varepsilon\|_0 \le c_1\|Df\|_0,$$

for all $\varepsilon \in (0,1)$, where $c_1 = (\lambda - \omega - Ke^K)^{-1}$. This also implies

$$\begin{aligned}|D\varphi_\varepsilon(\eta(s,x)) - D\varphi_\varepsilon(x)\|_0 &\le c_1\|Df(\eta(s,x) + \cdot) - Df(x + \cdot)\|_0 \\ &\le c_1|\theta_{Df}(|\eta(s,x) - x|) \le c_1\theta_{Df}(\|F\|_0\varepsilon),\end{aligned}$$

where $\theta_{Df} : \mathbb{R}^+ \to \mathbb{R}^+$ is the modulus of continuity of $Df$. So we find

$$|\mathcal{F}_\varepsilon\varphi_\varepsilon(x) - \mathcal{F}\varphi_\varepsilon(x)| \le c_1\|F\|_0\theta_{Df}(\|F\|_0\varepsilon) + c_1\|Df\|_0 K\|F\|_0\varepsilon.$$

Then $\mathcal{F}_\varepsilon\varphi_\varepsilon - \mathcal{F}\varphi_\varepsilon \to 0$ in $C_b(H)$, as $\varepsilon \to 0^+$. Finally, we have obtained

$$\lim_{\varepsilon\to 0^+}\left[\lambda\varphi_\varepsilon - N_0\varphi_\varepsilon\right] = f$$

in $C_b(H)$. Therefore the closure of the range of $\lambda - N_0$ includes $C_b^1(H)$, which is dense in $C_b(H)$. So, since $N_0$ is dissipative, by the Lumer-Phillips theorem the closure $\overline{N}_0$ of $N_0$ is $m$-dissipative. □

1.3. **Proof of Theorem 1.2.** By Theorem 1.7 the operator $N_0$ is $m$-dissipative in $C_b(H)$. It is also known that if $\varphi \in D(L) \cap C_b^1(H)$, then $N\varphi = L\varphi + \mathcal{F}\varphi$ (see, for instance, [8]) and therefore $(N, D(N))$ is an extension of $(N_0, D(N_0))$. Finally, since the operator $(N, D(N))$ is closed (see Proposition 3.4 in [9]), by the Lumer-Phillips theorem it follows that the closure of $(N_0, D(N_0))$ in $C_b(H)$ coincides with $(N, D(N))$. $\square$

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