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RICCATI EQUATIONS AND NONOSCILLATORY SOLUTIONS OF THIRD ORDER DIFFERENTIAL EQUATIONS

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ABSTRACT. We study the existence of special solutions of second order Riccati type equations. We apply these results to third order linear differential equations with almost constant coefficients. We give new sufficient conditions to know the asymptotic behavior of the logarithmic derivative of a solution y. We recover Poincaré and Perron's results and other asymptotic formulae. Furthermore, we obtain some weaker versions of Levinson and Hartman-Wintner type Theorems.

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1. INTRODUCTION

In this paper, using a Riccati type equation, [1, chap. 6] and [8, 19, 20], we study the classical problem of Poincaré [22] and Perron [17] for a third order scalar linear differential equation

$$y''' + (a_2 + r_2(t))y'' + (a_1 + r_1(t))y' + (a_0 + r_0(t))y = 0, \tag{1.1}$$

where $\lambda_1, \lambda_2, \lambda_3$ are the roots of the polynomial $P(\lambda) = \lambda^3 + a_2\lambda^2 + a_1\lambda + a_0$, $\mathrm{Re}\lambda_1 > \mathrm{Re}\lambda_2 > \mathrm{Re}\lambda_3$ and the locally integrable functions r_k, $k = 0, 1, 2$ are assumed small in some sense, obtaining an explicit and precise formula for the asymptotic behavior of the solutions of type

$$y_i(t) = \exp\left(\int_{t_0}^{t} \widehat{\lambda}_i(s)\, ds\right)$$

and

$$\widehat{\lambda}_i(t) = \lambda_i + \sum_{j=1}^{m} \theta_{ij}(t) + \psi_{i\,m+1}(t), \qquad i = 1, 2, 3, \tag{1.2}$$

where θ_{ij} have explicit expressions in terms of the variable coefficients r_k, $k = 0, 1, 2$ and the error function $\psi_{i\,m+1}$ satisfies an integral equation and it can be estimated by known terms. We have given such a formula with $m = 1$ in Theorems 1, 2, 3 and 4. In Theorem 5, we show a formula (1.2) with $m = 2$. Usually in applications, the

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equations are transformed to equations of Perron-Poincaré type where our results can be applied (see [4, 15, 16, 18, 21]).

The asymptotic theory of a linear differential equation asks for a representation of a fundamental system of solutions in a vicinity of $t = +\infty$. Its importance can hardly be overestimated for several reasons. For its own sake and for the reason that the asymptotic behavior of solution of nonlinear problems require quite often asymptotic integration of a linearized problem. A comprehensive account of the "nonanalytic theory" is given in the texts books [3, 4, 7, 14]. In [2], it is treated also the "analytic theory" which has been largely studied. The others methods are based in the reduction to a first order differential system and in several transformations.

An important feature is given by the fact that for the eigenvalue $\widehat{\lambda}$ which represents the asymptotic formulae (1.2) for the solutions begin always with the original eigenvalue λ of the unperturbed equation. In this way the original "physical meaning" represented by λ is preserved. In quantum mechanics, the physical meaning of the eigenvalues λ are of great importance since they are proportional to the energy levels of a quantum mechanical system.

Given $\gamma \in \mathbb{C}$ with $\mathrm{Re}\gamma > 0$ define the operators

$$G_{\pm}^{\gamma}[r](t) = \int_{t_0}^{\infty} e_{\gamma}^{\pm}(t,s)\, r(s)\, ds,$$

and for $\alpha > 0$

$$\mathcal{L}_{\pm}^{\alpha}[r](t) = \pm \int_{t_0}^{\infty} e_{\alpha}^{\pm}(t,s)\, |r(s)|\, ds,$$

where $e_{\gamma}^{\pm}(t,s) = \pm e^{\mp\gamma(t-s)}$ if $\pm(t-s) \geq 0$ and $e_{\gamma}^{\pm}(t,s) = 0$ otherwise. These operators satisfy the useful inequality

$$\left| G_{\pm}^{\alpha}\left[b\, G_{\pm}^{\beta}[a]\right] \right| \leq \mathcal{L}_{\pm}^{\alpha-\beta}[b]\mathcal{L}_{\pm}^{\beta}[a], \quad \beta < \alpha.$$

Our conditions are given in terms of the functions $\widehat{r}_i = r_0 + \lambda_i r_1 + \lambda_i^2 r_2$, $r_1 + 2\lambda_i r_2$, $i = 1, 2, 3$ and r_2. In this way, Hartman's results [12] are extended, for third order equations, to $\mathcal{G}_i[\widehat{r}_i](t) \to 0$ as $t \to \infty$ and for some explicit $\nu \in (0, 1/2)$,

$$\|\mathcal{L}_{\pm}^{\beta}[r_1 + 2\lambda_i r_2]\|_{\infty} + \|\mathcal{L}_{\pm}^{\beta}[r_2]\|_{\infty} \leq \nu,$$

where $\mathcal{G}_i[r](t) = |G_i[r](t)| + |G_i[r]'(t)|$, G_i is a linear combination of $G_{\pm}^{\gamma}$'s, $\|\cdot\|_{\infty}$ is the supremum norm on $[t_0, \infty)$ and $0 < \beta < \gamma = \min\{\mathrm{Re}(\lambda_1 - \lambda_2), \mathrm{Re}(\lambda_2 - \lambda_3)\}$. The uniform smallness of $\mathcal{L}_{\pm}^{\gamma}[r_1 + 2\lambda_i r_2]$ and $\mathcal{L}_{\pm}^{\gamma}[r_2]$ should be valid in a vicinity of $t = +\infty$. They need not tend to zero as $t \to +\infty$. The terms $\widehat{r}_i$, $r_1 + 2\lambda_i r_2$ and r_2 are not necessarily uniformly small. Moreover, it is possible that $\mathcal{G}_i[r_i](t) \to 0$ as $t \to \infty$ even when r_i is not small. The used method is scalar [8, 19] and does not need a reduction to a first order system, nor any transformation as the usual diagonalization process [2, 4, 5, 6, 7, 11, 18].

Studying similar results for n-order differential equations, Hartman [12, 13] extends Perron's result, assuming

$$\sup_{s\geq t}(1+s-t)^{-1}\int_t^s |r_k(\tau)|\,d\tau \to 0 \text{ as } t\to\infty, \quad k=0,1,2,$$

which is equivalent to $\mathcal{L}^{\alpha}_{\pm}[r_k](t)\to 0$ as $t\to\infty$, $k=0,1,2$ (see Lemma 1) and also to

$$\lim_{t\to\infty}\int_t^{t+1}|r_k(s)|\,ds=0, \quad k=0,1,2.$$

Actually, our extension gives a formula for the solutions y of equation (1.1), which allows several applications. Moreover, the famous theorems of Levinson and Hartman-Wintner and others can be deduced as a by product of ours results. Furthermore, equations with unbounded coefficients are usually reduced to equations (1.1). See [18, 23] and [2, 3, 4, 7]. All the errors in the asymptotic approximation can be estimated. Our results are suitable to deduce the number of linearly independent L^2-solutions, that is the deficiency index. See Gilbert [10] and [15, 16].

2. PRELIMINARIES

We consider a new variable $z=(y'/y)-\lambda$, where $\lambda\in\mathbb{C}$ is a root of P. We will find such a function z with property $z(t)\to 0$ as $t\to\infty$. Also, we want to express a such z as sum of terms known. We will need the following results, see [1, 4, 12].

Lemma 2.1. *Let $\gamma\in\mathbb{C}$, $\alpha=Re\gamma>0$ and let r be a locally integrable function on $[t_0,\infty)$. Consider the functions*

$$r^*(t)=\int_t^{t+1}|r(\tau)|\,d\tau \qquad \text{and} \qquad \overline{r}(t)=\sup_{s\geq t}(1+s-t)^{-1}\int_t^s|r(\tau)|\,d\tau. \tag{2.1}$$

Then the following statements are equivalent: $r^*(t)\to 0$ as $t\to\infty$, $\overline{r}(t)\to 0$ as $t\to\infty$, $\mathcal{L}^{\alpha}_{+}[r](t)\to 0$ as $t\to\infty$ and $\mathcal{L}^{\alpha}_{-}[r](t)\to 0$ as $t\to\infty$. Furthermore, $G^{\gamma}_{\pm}[r](t)\to 0$ as $t\to\infty$ holds if r is conditionally integrable in $[t_0,\infty)$. If $r\in L^p[t_0,\infty)$ for some $p\geq 1$, then $\mathcal{L}^{\alpha}_{\pm}[r](t)\to 0$ as $t\to\infty$ and $\mathcal{L}^{\alpha}_{\pm}[r]\in L^p[t_0,\infty)$.

Note that given a locally integrable function in $[t_0,\infty)$, say r, if either $r(t)\to 0$ as $t\to\infty$ or $r\in L^p[t_0,\infty)$ then $r^*(t)\to 0$ as $t\to\infty$. Also, given $\gamma\in\mathbb{C}$, $\alpha=\mathrm{Re}\gamma>0$, we have $|G^{\gamma}_{\pm}[r](t)|\leq\mathcal{L}^{\alpha}_{\pm}[r](t)$ for all $t\in[t_0,\infty)$.

Lemma 2.2. *Consider $\alpha>0$ and let $\xi:[t_0,\infty)\to I\!R$ be a locally integrable function such that either $\mathcal{L}^{\alpha}_{+}[\xi]$ or $\mathcal{L}^{\alpha}_{-}[\xi]$ is bounded in $[t_0,\infty)$. If $r(t)\to 0$ as $t\to\infty$ then $\mathcal{L}^{\alpha}_{\pm}[\xi r](t)\to 0$ as $t\to\infty$.*

Proof. If $\mathcal{L}^{\gamma}_{\pm}[\xi]$ are bounded then ξ^* is bounded for some $M>0$. So, given $\varepsilon>0$ there exists N such that $|r(t)|<\varepsilon/M$ for all $t\geq N$. Then

$$\int_t^{t+1}|\xi(s)r(s)|\,ds\leq\frac{\varepsilon}{M}\xi^*(t)\leq\varepsilon$$

for all $t \geq N$. Therefore, $(\xi r)^*(t) = \int_t^{t+1} |\xi(s) r(s)|\, ds \to 0$ as $t \to \infty$ and the result follows by the previous Lemma. $\square$

We define $\mathcal{L}_0^\alpha$ as $\mathcal{L}_0^\alpha = \mathcal{L}_+^\alpha + \mathcal{L}_-^\alpha$. Note that $\mathcal{L}_\pm^\alpha[r](t) \to 0$ as $t \to \infty$ if and only if $\mathcal{L}_0^\alpha[r](t) \to 0$ as $t \to \infty$. Let us see an interesting inequality satisfied by these operators.

Lemma 2.3. *If $\beta < \alpha$ then*

(2.2)
$$\mathcal{L}_\pm^\alpha\big[b\mathcal{L}_\pm^\beta[a]\big](t) \leq \mathcal{L}_\pm^{\alpha-\beta}[b](t)\mathcal{L}_\pm^\beta[a](t) \quad \textit{and} \quad \mathcal{L}_0^\alpha\big[b\mathcal{L}_0^\beta[a]\big](t) \leq 2\mathcal{L}_0^{\alpha-\beta}[b](t)\mathcal{L}_0^\beta[a](t).$$

Proof. First, we have

$$\begin{aligned}
\mathcal{L}_+^\alpha\big[b\mathcal{L}_+^\beta[a]\big](t) &= \int_{t_0}^t e^{-\alpha(t-s)}|b(s)| \int_{t_0}^s e^{-\beta(s-\tau)}|a(\tau)|\, d\tau\, ds \\
&= e^{-\alpha t} \int_{t_0}^t \int_\tau^t e^{(\alpha-\beta)s} e^{\beta\tau} |a(\tau)||b(s)|\, ds\, d\tau \\
&\leq e^{-\alpha t} \int_{t_0}^t e^{\beta\tau}|a(\tau)| \int_{t_0}^t e^{(\alpha-\beta)s}|b(s)|\, ds\, d\tau = \mathcal{L}_+^\beta[a]\mathcal{L}_+^{\alpha-\beta}[b].
\end{aligned}$$

Similarly, $\mathcal{L}_-^\alpha\big[b\mathcal{L}_-^\beta[a]\big](t) \leq \mathcal{L}_-^{\alpha-\beta}[b](t)\mathcal{L}_-^\beta[a](t)$. Moreover,

$$\begin{aligned}
\mathcal{L}_0^\alpha\big[b\mathcal{L}_0^\beta[a]\big](t) &= \mathcal{L}_+^\alpha\big[b\mathcal{L}_0^\beta[a]\big](t) + \mathcal{L}_-^\alpha\big[b\mathcal{L}_0^\beta[a]\big](t) \\
&\leq \mathcal{L}_+^\alpha\big[b\mathcal{L}_+^\beta[a]\big](t) + \mathcal{L}_+^\alpha\big[b\mathcal{L}_-^\beta[a]\big](t) + \mathcal{L}_-^\alpha\big[b\mathcal{L}_+^\beta[a]\big](t) + \mathcal{L}_-^\alpha\big[b\mathcal{L}_-^\beta[a]\big](t).
\end{aligned}$$

Now, we have

$$\begin{aligned}
\mathcal{L}_+^\alpha\big[b\mathcal{L}_-^\beta[a]\big](t) &= \int_{t_0}^t e^{-\alpha(t-s)}|b(s)| \int_s^\infty e^{\beta(s-\tau)}|a(\tau)|\, d\tau\, ds \\
&= e^{-\alpha t}\bigg[\int_{t_0}^t e^{-\beta\tau}|a(\tau)| \int_{t_0}^\tau e^{(\alpha+\beta)s}|b(s)|\, ds\, d\tau \\
&\quad + \int_t^\infty e^{-\beta\tau}|a(\tau)| \int_{t_0}^t e^{(\alpha+\beta)s}|b(s)|\, ds\, d\tau\bigg] \\
&= e^{-\alpha t}\int_{t_0}^t e^{-\beta\tau}|a(\tau)|\, e^{(\alpha+\beta)\tau} \int_{t_0}^\tau e^{-(\alpha+\beta)(t-s)}|b(s)|\, ds\, d\tau \\
&\quad + e^{-\alpha t}\int_t^\infty e^{-\beta\tau}|a(\tau)|\, e^{(\alpha+\beta)t} \int_{t_0}^t e^{-(\alpha+\beta)(t-s)}|b(s)|\, ds\, d\tau \\
&= \mathcal{L}_+^\alpha\big[a\mathcal{L}_+^{\alpha+\beta}[b]\big](t) + \mathcal{L}_-^\beta[a](t)\mathcal{L}_+^{\alpha+\beta}[b](t)
\end{aligned}$$

and also $\mathcal{L}_-^\alpha\big[b\mathcal{L}_+^\beta[a]\big](t) = \mathcal{L}_-^\alpha\big[a\mathcal{L}_-^{\alpha+\beta}[b]\big](t) + \mathcal{L}_+^\beta[a](t)\mathcal{L}_-^{\alpha+\beta}[b](t)$. So,

$$\begin{aligned}
\mathcal{L}_0^\alpha\big[b\mathcal{L}_0^\beta[a]\big](t) &\leq \mathcal{L}_+^{\alpha-\beta}[b](t)\mathcal{L}_+^\beta[a](t) + \mathcal{L}_+^\alpha\big[a\mathcal{L}_+^{\alpha+\beta}[b]\big](t) + \mathcal{L}_-^\beta[a](t)\mathcal{L}_+^{\alpha+\beta}[b](t) \\
&\quad + \mathcal{L}_-^\alpha\big[a\mathcal{L}_-^{\alpha+\beta}[b]\big](t) + \mathcal{L}_+^\beta[a](t)\mathcal{L}_-^{\alpha+\beta}[b](t) + \mathcal{L}_-^{\alpha-\beta}[b](t)\mathcal{L}_-^\beta[a](t) \\
&\leq \mathcal{L}_+^\beta[a](t)\big(\mathcal{L}_+^{\alpha-\beta}[b] + \mathcal{L}_-^{\alpha+\beta}[b](t)\big) + \mathcal{L}_-^\beta[a](t)\big(\mathcal{L}_+^{\alpha+\beta}[b] + \mathcal{L}_-^{\alpha-\beta}[b](t)\big) \\
&\quad + \mathcal{L}_+^\alpha\big[a\mathcal{L}_+^{\alpha-\beta}[b]\big](t) + \mathcal{L}_-^\alpha\big[a\mathcal{L}_-^{\alpha-\beta}[b]\big](t)
\end{aligned}$$

$$\leq \mathcal{L}_0^{\beta}[a](t)\mathcal{L}_0^{\alpha-\beta}[b](t) + \mathcal{L}_+^{\beta}[a](t)\mathcal{L}_+^{\alpha-\beta}[b](t) + \mathcal{L}_-^{\beta}[a](t)\mathcal{L}_-^{\alpha-\beta}[b](t)$$
$$\leq 2\mathcal{L}_0^{\beta}[a](t)\mathcal{L}_0^{\alpha-\beta}[b](t).$$

Therefore (2.2) is proved. □

Lemma 2.4. *Suppose that the roots λ_1, λ_2 and λ_3 of P are distinct. Then there are three solutions y_i, $i = 1, 2, 3$ of* (1.1) *such that*

$$y_i(t) = \exp\left(\int_{t_0}^{t} [\lambda_i + z_i(s)]\, ds\right), \quad i = 1, 2, 3, \tag{2.3}$$

where z_i, $i = 1, 2, 3$ satisfy

$$\begin{aligned} &z_i'' + (3\lambda_i + a_2)z_i' + (3\lambda_i^2 + 2a_2\lambda_i + a_1)z_i + r_0(t) + \lambda_i r_1(t) + \lambda_i^2 r_2(t) \\ &\quad + (r_1(t) + 2\lambda_i r_2(t))z_i + r_2(t)z_i' + 3z_i z_i' + (3\lambda_i + a_2 + r_2(t))z_i^2 + z_i^3 = 0. \end{aligned} \tag{2.4}$$

Now, we will study equation (2.4), a second order non-linear Riccati type equation, to know the asymptotic behavior of the solutions of equation (1.1). So, we need to study the non perturbed linear part of equation (2.4), namely, to know the characteristic roots of equation

$$z'' + (3\lambda + a_2)z' + (3\lambda^2 + 2a_2\lambda + a_1)z = 0.$$

Lemma 2.5. *If the roots of P, λ_i, $i = 1, 2, 3$ are distinct, then $\mu = \lambda_j - \lambda_i$, $i \neq j$ satisfy the equation*

$$\mu^2 + (3\lambda_i + a_2)\mu + 3\lambda_i^2 + 2a_2\lambda_i + a_1 = 0.$$

Consider the scalar differential equation

$$z'' + b_1 z' + b_0 z = a(t) + f(t, z, z'), \tag{2.5}$$

where b_1 and b_2 are constant, $f : [t_0, \infty) \times \mathbb{C} \times \mathbb{C} \to \mathbb{R}$ for some $t_0 \in \mathbb{R}$; a and $f(\cdot, x, y)$ (for each $x, y \in \mathbb{C}$) are locally integrable functions on $[t_0, \infty)$. Let $Q(\lambda) = \lambda^2 + b_1\lambda + b_0$ be the characteristic polynomial of associated homogeneous equation, namely, $x'' + b_1 x' + b_0 x = 0$ and let γ_k, $k = 1, 2$ be the roots of Q, with $\gamma_1 \neq \gamma_2$ and $\mathrm{Re}\gamma_k \neq 0$, $k = 1, 2$. Note, there are three situations, depending of values of b_0 and b_1. These are: (i) $\mathrm{Re}\gamma_1, \mathrm{Re}\gamma_2 < 0$; (ii) $\mathrm{Re}\gamma_1 < 0 < \mathrm{Re}\gamma_2$ and (iii) $\mathrm{Re}\gamma_1, \mathrm{Re}\gamma_2 > 0$. For each one of them, we define the Green function by

$$(\gamma_1 - \gamma_2)\, g_+(t, s) = \begin{cases} e^{\gamma_1(t-s)} - e^{\gamma_2(t-s)}, & \text{if } t \geq s \\ 0, & \text{otherwise} \end{cases}, \qquad \text{when } \mathrm{Re}\gamma_1, \mathrm{Re}\gamma_2 < 0,$$

$$(\gamma_1 - \gamma_2)\, g_0(t, s) = \begin{cases} e^{\gamma_1(t-s)}, & \text{if } t \geq s \\ e^{\gamma_2(t-s)}, & \text{otherwise} \end{cases}, \qquad \text{when } \mathrm{Re}\gamma_1 < 0 < \mathrm{Re}\gamma_2,$$

and

$$(\gamma_1-\gamma_2)\, g_-(t,s)=\begin{cases}0, & \text{if } t\geq s\\ e^{\gamma_2(t-s)}-e^{\gamma_1(t-s)}, & \text{otherwise}\end{cases}, \qquad \text{when } \mathrm{Re}\gamma_1,\mathrm{Re}\gamma_2>0.$$

Define, for $n=\pm,0$, the corresponding Green operators G_n and the auxiliary operators $\mathcal{G}_n$ and $\mathcal{L}_n$ by

$$G_n[r](t)=\int_{t_0}^{\infty} g_n(t,s)r(s)\,ds, \qquad \mathcal{G}_n[r](t)=|G_n[r](t)|+|G_n[r]'(t)|$$

$$\text{and} \qquad \mathcal{L}_n[r](t)=\int_{t_0}^{\infty}\left(|g_n(t,s)|+\left|\frac{\partial g_n}{\partial t}(t,s)\right|\right)|r(s)|\,ds,$$

where g_n is the Green function of the equation $x''+b_1x'+b_0x=0$. Note that, taking $\alpha=\min\{|\mathrm{Re}\gamma_1|,|\mathrm{Re}\gamma_2|\}$ we have the following estimates

$$\int_{t_0}^{\infty}|g_n(t,s)||r(s)|\,ds\leq\frac{2}{|\gamma_1-\gamma_2|}\,\mathcal{L}_n^{\alpha}[r](t),$$

$$\int_{t_0}^{\infty}\left|\frac{\partial g_n}{\partial t}(t,s)\right||r(s)|\,ds\leq\frac{|\gamma_1|+|\gamma_2|}{|\gamma_1-\gamma_2|}\,\mathcal{L}_n^{\alpha}[r](t),$$

and adding we obtain $\mathcal{L}_n[r](t)\leq\widetilde{\gamma}\,\mathcal{L}_n^{\alpha}[r](t)$, for $n=\pm,0$, where $\widetilde{\gamma}=\frac{2+|\gamma_1|+|\gamma_2|}{|\gamma_1-\gamma_2|}$.

The following results are related to equation (2.5). Both apply to the three mentioned situations above (so, we omit the index n of the Green functions g_n, Green operators G_n and auxiliary operators $\mathcal{G}_n$ and $\mathcal{L}_n^{\alpha}$, $n=\pm,0$).

Lemma 2.6. *Suppose that in* (2.5) *there exists* $\mathcal{G}[a]$ *and* $\mathcal{G}[a](t)\to0$ *as* $t\to\infty$. *Moreover, assume that* $f(\cdot,0,0)=0$ *and for some constant* $M>0$ *there exists* $\xi_M:[t_0,\infty)\to[0,\infty)$ *such that for all* $t\geq t_0$ *and* $|x_k|+|y_k|\leq M$, $k=1,2$

$$|f(t,x_1,y_1)-f(t,x_2,y_2)|\leq\xi_M(t)(|x_1-x_2|+|y_1-y_2|).$$

If for all $t\geq t_0$

$$\text{(2.6)} \qquad \widetilde{\gamma}\,\mathcal{L}^{\alpha}[\xi_M](t)\leq\varepsilon_0<1, \qquad \widetilde{\gamma}=\frac{2+|\gamma_1|+|\gamma_2|}{|\gamma_1-\gamma_2|}, \quad \alpha=\min\{|Re\gamma_1|,|Re\gamma_2|\}$$

then for every t_0 *such that* $|\mathcal{G}[a](t)|\leq(1-\varepsilon_0)M$ *for all* $t\geq t_0$, *there is a solution* z *of* (2.5), *on* $[t_0,\infty)$, *such that* $z(t),z'(t)\to0$ *as* $t\to\infty$ *and satisfies the integral equation*

$$\text{(2.7)} \qquad z=G[a+f(\cdot,z,z')].$$

Proof. Consider the space $\mathcal{C}_0^1$ defined by

$$\mathcal{C}_0^1[t_0,\infty)=\{x:[t_0,\infty)\to\mathbb{C}\mid x,x' \text{ are continuous and } x(t),x'(t)\to0 \text{ as } t\to\infty\}.$$

Observe that $\mathcal{C}_0^1$ is a Banach space with the norm $\|x\| = \sup_{t\in[t_0,\infty)}\{|x(t)| + |x'(t)|\}$. Define the operator T as

$$Tz(t) = \int_{t_0}^{\infty} g(t,s)\big[a(s) + f(s,z(s),z'(s))\big]\,ds = G[a + f(\cdot,z,z')](t).$$

Note that if $z \in \mathcal{C}_0^1$ then we have Tz and $(Tz)'$ are continuous. Indeed, we have

$$(Tz)'(t) = \int_{t_0}^{\infty} \frac{\partial g}{\partial t}(t,s)\big[a(s) + f(s,z(s),z'(s))\big]\,ds = G[a + f(\cdot,z,z')]'(t).$$

Let $B = \{x \in \mathcal{C}_0^1 \mid \|x\| \le M\}$. Thus, if $z \in B$ then $|f(t,z,z')| \le \xi_M(t)(|z| + |z'|)$. Hence, for $t \ge t_0$

$$|Tz(t)| \le |G[a](t)| + \int_{t_0}^{\infty} |g(t,s)|\,\xi_M(s)\,(|z(s)| + |z'(s)|)\,ds \qquad \text{and}$$

$$|(Tz)'(t)| \le |G[a]'(t)| + \int_{t_0}^{\infty} \left|\frac{\partial g}{\partial t}(t,s)\right|\,\xi_M(s)\,(|z(s)| + |z'(s)|)\,ds.$$

So, $|Tz(t)|+|(Tz)'(t)| \le \mathcal{G}[a](t)+\mathcal{L}[\xi_M(|z|+|z'|)](t) \le \mathcal{G}[a](t)+\widetilde{\gamma}\,\mathcal{L}^{\gamma}[\xi_M(|z|+|z'|)](t)$. Therefore, using Lemma 2.2, we have $\mathcal{L}^{\gamma}[\xi_M(|z|+|z'|)](t) \to 0$ as $t \to \infty$ since $\mathcal{L}^{\gamma}[\xi_M]$ is bounded and we can conclude that $\lim_{t\to\infty} |Tz(t)| + |(Tz)'(t)| = 0$. Thus $Tz \in \mathcal{C}_0^1$. Similarly, for $z_1, z_2 \in B$, we have

$$|Tz_1(t) - Tz_2(t)| + |(Tz_1)'(t) - (Tz_2)'(t)| \le \mathcal{L}[\xi_M](t)\|z_1 - z_2\| \le \widetilde{\gamma}\,\mathcal{L}^{\gamma}[\xi_M](t)\|z_1 - z_2\|.$$

Now, let $t_0 \ge 0$ such that $\mathcal{G}[a](t) \le (1 - \varepsilon_0)M$ for all $t \ge t_0$. Consider $z \in B$, then for $t \ge t_0$

$$|Tz(t)| + |(Tz)'(t)| \le \mathcal{G}[a](t) + M\mathcal{L}[\xi_M](t) \le \mathcal{G}[a](t) + \widetilde{\gamma}\,M\mathcal{L}^{\gamma}[\xi_M](t).$$

Thus, $\|Tz\| \le M$. Therefore, $T : B \to B$ is a contractive operator, namely, for $z_1, z_2 \in B$ we have $\|Tz_1 - Tz_2\| \le \varepsilon_0\|z_1 - z_2\|$. So, there exists a unique $z \in B$ such that $Tz = z$. Then, z is a solution of (2.5) such that $z(t), z'(t) \to 0$ as $t \to \infty$. ☐

Note that if z is the solution given by the previous Lemma and f satisfies $f(t,z(t),z'(t)) \to 0$ as $t \to \infty$, then $a(t) \to 0$ as $t \to \infty$ implies $z''(t) \to 0$ as $t \to \infty$.

Now, we can characterize better the solution given by Lemma 2.6. Indeed, using Lemma 2.3 we can deduce an estimation for the solution of (2.5) given by Lemma 2.6.

Corollary 2.7. *Consider the equation* (2.5), *under the same hypothesis of Lemma* 2.6. *In addition, suppose that*

$$\|\mathcal{L}^{\alpha-\beta}[\xi_M]\|_\infty < \frac{|\gamma_1 - \gamma_2|}{2(2 + |\gamma_1| + |\gamma_2|)} = \frac{1}{2\widetilde{\gamma}}, \tag{2.8}$$

where $0 < \beta < \alpha$. *Then the solution* z *of* (2.5) *given by Lemma* 2.6 *satisfies* z, $z' = O\big(\mathcal{L}^{\beta}[a]\big)$.

Proof. Without loss of generality, suppose $\gamma_1, \gamma_2 \in \mathbb{R}$. For the contractive operator T defined in the proof of the Lemma 2.6, the sequence given by $z_0 = 0$ and $z_{n+1} = Tz_n$ for $n \geq 0$ satisfies $z_n \to z$ as $n \to \infty$ in $\mathcal{C}_0^1$. Now, we will prove that for all $t \geq t_0$ and $n \in \mathbb{N}$

$$|z_n(t)| + |z_n'(t)| \leq \widetilde{\gamma}\, N\, \mathcal{L}^{\beta}[a], \tag{2.9}$$

with K and N satisfying respectively $\|\mathcal{L}^{\alpha-\beta}[\xi_M]\|_\infty \leq K < \frac{1}{2\widetilde{\gamma}}$ and $1 \leq N\,[1 - 2\widetilde{\gamma}K]$. Observe that $N > 0$ since $K < \frac{1}{2\widetilde{\gamma}}$. The induction in (2.9) is clear for $n = 0, 1$. Suppose that (2.9) is true for $n = k$. So, for $n = k+1$, using Lemma 2.3 we have

$$\begin{aligned}
|z_{k+1}(t)| &\leq \frac{2}{|\gamma_1 - \gamma_2|}\left\{\mathcal{L}^{\alpha}[a](t) + \mathcal{L}^{\alpha}[(|z_k| + |z_k'|)\xi_M](t)\right\} \\
&\leq \frac{2}{|\gamma_1 - \gamma_2|}\left\{\mathcal{L}^{\beta}[a](t) + \frac{2N}{|\gamma_1 - \gamma_2|}(2 + |\gamma_1| + |\gamma_2|)\mathcal{L}^{\alpha-\beta}[\xi_M](t)\mathcal{L}^{\beta}[a](t)\right\} \\
&\leq \frac{2}{|\gamma_1 - \gamma_2|}\left\{1 + \frac{2NK}{|\gamma_1 - \gamma_2|}(2 + |\gamma_1| + |\gamma_2|)\right\}\mathcal{L}^{\beta}[a](t).
\end{aligned}$$

Now, for the derivative we have

$$\begin{aligned}
|z_{k+1}'(t)| &\leq \frac{|\gamma_1| + |\gamma_2|}{|\gamma_1 - \gamma_2|}\left\{\mathcal{L}^{\alpha}[a](t) + \mathcal{L}^{\alpha}[(|z_k| + |z_k'|)\xi_M](t)\right\} \\
&\leq \frac{|\gamma_1| + |\gamma_2|}{|\gamma_1 - \gamma_2|}\left\{1 + \frac{2NK}{|\gamma_1 - \gamma_2|}(2 + |\gamma_1| + |\gamma_2|)\right\}\mathcal{L}^{\beta}[a](t).
\end{aligned}$$

So, $|z_{k+1}(t)| + |z_{k+1}'(t)| \leq \widetilde{\gamma}\, N\, \mathcal{L}^{\beta}[a](t)$, and the result follows. $\square$

Note that inequality (2.8) implies the inequality in (2.6) and that this Corollary implies $z, z' \in L^p[t_0, \infty)$ if $a \in L^p[t_0, \infty)$. Define

$$\nu_0 := \frac{|\gamma_1 - \gamma_2|}{2(2 + |\gamma_1| + |\gamma_2|)} \tag{2.10}$$

and note that $\nu_0 < 1/2$.

Remark 2.8. An interesting improvement to Lemma 2.6 and Corollary 1 is possible, namely, if $\theta = G[a]$ then $u = z - \theta$ satisfies equation (2.7) with $f(\cdot, \theta, \theta')$ instead of a and $f(\cdot, u + \theta, u' + \theta') - f(\cdot, \theta, \theta')$ instead of $f(\cdot, z, z')$, that is to say,

$$u = G[f(\cdot, \theta, \theta') + \widehat{f}(\cdot, u, u')], \qquad \widehat{f}(\cdot, u, u') = f(\cdot, u + \theta, u' + \theta') - f(\cdot, \theta, \theta').$$

Then, by Lemma 2.6 and Corollary 1 we have $u = O(\mathcal{L}^{\beta}[f(\cdot, \theta, \theta')])$. Indeed, we can use the same associated function ξ_M. This fact has several applications (see Examples 1,2 and 3), as to obtain formula (1.2) and also Theorems 4-5. Furthermore, it is possible to apply these results to the equation (2.5) with

$$f(t, z, z') = b(t)z + c(t)z' + c_1 z z' + (c_2 + h(t))z^2 + z^3$$

and then to equation (2.4). In this particular case, we have

$$\xi_M(t) = |b(t)| + |c(t)| + 2M(|c_1| + |c_2|) + 2M|h(t)| + 3M^2$$

and taking $\|\mathcal{L}^{\alpha-\beta}[b]\|_\infty + \|\mathcal{L}^{\alpha-\beta}[c]\|_\infty < \nu_0$, where ν_0 is given by (2.10) and h a bounded function, there exists $M > 0$ such that inequality (2.8) is true, since for all $t \geq t_0$ we have

$$\mathcal{L}^{\alpha-\beta}[\xi_M](t) \leq \|\mathcal{L}^{\alpha-\beta}[b]\|_\infty + \|\mathcal{L}^{\alpha-\beta}[c]\|_\infty + \frac{4M}{\alpha-\beta}(|c_1|+|c_2|) + 2M\|\mathcal{L}^{\alpha-\beta}[h]\|_\infty + \frac{6M^2}{\alpha-\beta}.$$

This shows us a condition in terms of coefficients of f in order to satisfy condition (2.8).

3. MAIN RESULTS

In this section, we present the results for the equation (1.1). We use the previous results for the Riccati-type equation. First, denote $\mathcal{B}_V$, the vectorial space of the bounded variation functions on $[t_0,\infty)$, $\mathcal{B}$ the vectorial space of the bounded functions on $[t_0,\infty)$ and for $\alpha,\varepsilon > 0$, $\mathcal{B}_V^{\alpha,\varepsilon}$ defined by $\mathcal{B}_V^{\alpha,\varepsilon} = \{r \in \mathcal{B}_V \mid \|\mathcal{L}_0^{\alpha}[r]\|_\infty < \varepsilon\}$. Moreover,

$$\int_{t_0}^{t} f_1(s)G_{\pm}^{\gamma}[f_2](s)\,ds = \pm\frac{1}{\gamma}\int_{t_0}^{t} f_1(s)f_2(s)\,ds \mp \frac{1}{\gamma}\int_{t_0}^{t} f_1(s)(G_{\pm}^{\gamma}[f_2])'(s)\,ds. \tag{3.1}$$

Now, we will prove a generalization of Perron's theorem [17] for a third order equation. Consider the equation (1.1), where r_0, r_1 and r_2 are locally integrable functions defined on $[0,\infty)$. We will suppose $\mathrm{Re}\lambda_1 > \mathrm{Re}\lambda_2 > \mathrm{Re}\lambda_3$, where λ_i, $i = 1,2,3$ are the roots of $P(\lambda) = \lambda^3 + a_2\lambda^2 + a_1\lambda + a_0$. We denote $\gamma_1 = \lambda_1 - \lambda_2$, $\gamma_2 = \lambda_1 - \lambda_3$, $\gamma_3 = \lambda_2 - \lambda_3$, $\alpha = \min\{\mathrm{Re}\gamma_1, \mathrm{Re}\gamma_3\}$ and $\mathcal{N}(i) = \{1,2,3\} \setminus \{i\}$. Note that $\mathrm{Re}\gamma_k > 0$, $k = 1,2,3$ and by Lemma 2.5 we have that the Green operators associated to homogenous linear part of equations (2.4) are

$$G_1 = (\gamma_2-\gamma_1)^{-1}(G_+^{\gamma_1} - G_+^{\gamma_2}), \quad G_2 = -(\gamma_1+\gamma_3)^{-1}(G_+^{\gamma_3} - G_-^{\gamma_1})$$

$$\text{and} \quad G_3 = (\gamma_2-\gamma_3)^{-1}(G_-^{\gamma_2} - G_-^{\gamma_3}).$$

Following (2.10), these γ_k, $k = 1,2,3$ allows us to define

$$\nu_1 := \frac{|\gamma_1-\gamma_2|}{2(2+|\gamma_1|+|\gamma_2|)}, \quad \nu_2 := \frac{|\gamma_1+\gamma_3|}{2(2+|\gamma_1|+|\gamma_3|)} \quad \text{and} \quad \nu_3 := \frac{|\gamma_2-\gamma_3|}{2(2+|\gamma_2|+|\gamma_3|)}. \tag{3.2}$$

Denote $\widehat{r_i} = r_0 + \lambda_i r_1 + \lambda_i^2 r_2$. Recall, for $i = 1,2,3$, $\mathcal{G}_i[r](t) = |G_i[r](t)| + |G_i[r]'(t)|$. And define $n(1) = +$, $n(2) = 0$ and $n(3) = -$.

Theorem 3.1. *Assume that for all $i = 1,2,3$, it is satisfied $\mathcal{G}_i[\widehat{r_i}](t) \to 0$ as $t \to \infty$, $\|\mathcal{L}_{n(i)}^{\alpha-\beta}[r_1 + 2\lambda_i r_2]\|_\infty + \|\mathcal{L}_{n(i)}^{\alpha-\beta}[r_2]\|_\infty \leq \nu_i$, where $\nu_i > 0$, $i = 1,2,3$ are given by* (3.2) *and $0 < \beta < \alpha$. Then there is a fundamental system of solutions y_i, $i = 1,2,3$ of the equation* (1.1) *such that*

$$\lim_{t\to\infty}\frac{y_i'(t)}{y_i(t)} = \lambda_i, \quad \textit{and} \quad \lim_{t\to\infty}\frac{y_i''(t)}{y_i(t)} = \lambda_i^2. \tag{3.3}$$

Furthermore, as $t \to \infty$

(3.4)

$$y_i(t) = (1+o(1))e^{\lambda_i(t-t_0)} \exp\left(- \prod_{j\in\mathcal{N}(i)} (\lambda_j - \lambda_i)^{-1} \int_{t_0}^{t} [\widehat{r}_i(s) + \widetilde{f}_i(s, z_i(s), z_i'(s))]\, ds \right),$$

where $\widetilde{f}_i(t,z,z') = (r_1(t)+2\lambda_i r_2(t))z + r_2(t)z' + (3\lambda_i + a_2 + r_2(t))z^2 + z^3$, *the functions* $z_i(t), z_i'(t) \to 0$ *as* $t\to\infty$, $i = 1,2,3$, *satisfy* (2.4) *and*

$$z_i, z_i' = O\big(\mathcal{L}^{\beta}_{n(i)}[\widehat{r}_i]\big), \qquad i=1,2,3. \tag{3.5}$$

Moreover, the Wronskian of $\{y_i\}_{i=1,2,3}$ *satisfies*

$$W[y_1,y_2,y_3] = -\gamma_1\gamma_2\gamma_3\, y_1y_2y_3(1+o(1)) \qquad \text{as} \quad t\to\infty.$$

In the particular case, $r_k(t)\to 0$ *as* $t\to\infty$, $k=0,1,2$ *even we have*

$$\lim_{t\to\infty} \frac{y_i'''(t)}{y_i(t)} = \lambda_i^3, \quad i=1,2,3. \tag{3.6}$$

Proof. We apply Lemma 2.6 to the equations (2.4), since we have $\mathcal{G}_i[\widehat{r}_i](t) \to 0$ as $t\to\infty$, $\xi_M(t) = |r_1(t)+2\lambda_i r_2(t)| + |r_2(t)| + 2M(3+|3\lambda_i + a_2|) + 2M|r_2(t)| + 3M^2$ and $\|\mathcal{L}^{\alpha-\beta}_{n(i)}[r_1+2\lambda_i r_2]\|_\infty + \|\mathcal{L}^{\alpha-\beta}_{n(i)}[r_2]\|_\infty \le \nu_i$, $i=1,2,3$. Thus, there are three solutions z_i such that $z_i(t), z_i'(t)\to 0$ as $t\to\infty$, $i=1,2,3$. So, the equation (1.1) have three solutions in the form (2.3). Therefore, (3.3) holds since

$$\lambda_i + z_i(t) = \frac{y_i'(t)}{y_i(t)} \qquad \text{and} \qquad (\lambda_i + z_i(t))^2 + z_i'(t) = \frac{y_i''(t)}{y_i(t)}.$$

Hence, (3.4) follows from (3.1) and the integral equations for z_i,

$$z_i = -G_i\big[\widehat{r}_i + f_i(\cdot, z_i, z_i')\big], \tag{3.7}$$

where $f_i(t,z,z') = \widetilde{f}_i(t,z,z') + 3zz'$ and z_iz_i' is conditionally integrable $i=1,2,3$. By Corollary 1, we have (3.5). Now, $\{y_i\}_{i=1,2,3}$ is a fundamental system of solutions since, by (3.3)

$$\lim_{t\to\infty}\left[\frac{y_2'}{y_2}\frac{y_3''}{y_3} - \frac{y_2''}{y_2}\frac{y_3'}{y_3} + \frac{y_1''}{y_1}\frac{y_3'}{y_3} - \frac{y_1'}{y_1}\frac{y_3''}{y_3} + \frac{y_3'}{y_3}\frac{y_2''}{y_2} - \frac{y_1''}{y_1}\frac{y_2'}{y_2}\right](t) = -\gamma_1\gamma_2\gamma_3 \neq 0$$

and the Wronskian of $\{y_i\}_{i=1,2,3}$ satisfies

$$W[y_1,y_2,y_3] = y_1y_2y_3\left[\frac{y_2'}{y_2}\frac{y_3''}{y_3} - \frac{y_2''}{y_2}\frac{y_3'}{y_3} + \frac{y_1''}{y_1}\frac{y_3'}{y_3} - \frac{y_1'}{y_1}\frac{y_3''}{y_3} + \frac{y_3'}{y_3}\frac{y_2''}{y_2} - \frac{y_1''}{y_1}\frac{y_2'}{y_2}\right].$$

Finally, if $r_k(t)\to 0$ as $t\to\infty$, $k=0,1,2$ then $\mathcal{G}_i[\widehat{r}_i](t)\to 0$ as $t\to\infty$ and $\|\mathcal{L}^{\alpha-\beta}_{n(i)}[r_1+2\lambda_i r_2]\|_\infty + \|\mathcal{L}^{\alpha-\beta}_{n(i)}[r_2]\|_\infty \le \nu_i$, $i=1,2,3$, for t_0 big enough. So, we have (3.3), (3.4), (3.5) and $z_i''(t)\to 0$ as $t\to\infty$, $i=1,2,3$. Therefore, (3.6) follows from

$$(\lambda_i + z_i(t))^3 + 3(\lambda_i+z_i(t))z_i'(t) + z_i''(t) = \frac{y_i'''(t)}{y_i(t)}.$$

□

Theorem 3.1 is a generalization, for third order, of Perron's and Hartman's theorems. Observe that $\mathcal{G}_1[\widehat{r}_1](t) \to 0$ as $t \to \infty$ if and only if $G_+^{\gamma_1}[\widehat{r}_1](t)$ and $G_+^{\gamma_2}[\widehat{r}_1](t) \to 0$ as $t \to \infty$. Analogous statements can be obtained for $\mathcal{G}_2[\widehat{r}_2]$ and $\mathcal{G}_3[\widehat{r}_3]$. Although, $\widehat{r}_i \notin L^p[t_0,\infty)$, $\widehat{r}_i(t)$ does not necessarily tend to zero as $t \to +\infty$, nor for t_0 big enough $\widehat{r}_i$ is uniformly small, the function $\mathcal{G}_i[\widehat{r}_i]$ could be $L^p[t_0,\infty)$ for some $p \geq 1$ (see Example 1, Section 4). Furthermore, if $r_2 \in \mathcal{B}_V$ then the term $r_2 z'$ disappears in the asymptotic formula (3.4). If, in addition $r_1 + 2\lambda_i r_2 \in \mathcal{B}_V$, then the first linear term in $\widetilde{f}$ is simplified since by (3.1)

$$\begin{aligned}\int_{t_0}^{t} &[r_1(s)+2\lambda_i r_2(s)] z_i(s)\,ds \\ &= -\prod_{j\in\mathcal{N}(i)} (\lambda_j - \lambda_i)^{-1} \int_{t_0}^{t} [\widehat{r}_i(s) + f_i(s, z_i(s), z_i'(s))]\,[r_1(s) + 2\lambda_i r_2(s)]\,ds \\ &\quad + c + o(1)\end{aligned}$$

as $t \to \infty$, where c is a constant. So, in (3.4) it will appear only the last integral (see Ths. 2-5 and example 3).

Remark 3.2. Formulae (3.3) and (3.6) have the following error bounds:

$$y_i^{(k)} = \left[\lambda_i^k + O\left(\mathcal{L}^{\beta}_{n(i)}[\widehat{r}_i]\right)\right] y_i, \quad k = 1, 2,$$

$$y_i''' = \left[\lambda_i^3 + O(|\widehat{r}_i|) + O(|r_1 + 2\lambda_i r_2| + |r_2|)\mathcal{L}^{\beta}_{n(i)}[\widehat{r}_i] + O(\mathcal{L}^{\beta}_{n(i)}[\widehat{r}_i])\right] y_i,$$

where $\widehat{r}_i = r_0 + \lambda_i r_1 + \lambda_i^2 r_2$, $i = 1, 2, 3$.

Formula (3.4) can be largely used. In the following results, we take $\nu = \min\{\nu_i \mid i = 1, 2, 3\}$ and denote

$$\theta_i = -G_i[\widehat{r}_i], \quad \widehat{\theta}_i = r_1 + 2\lambda_i r_2 + \theta_i[2(3\lambda_i + a_2 + r_2) + 3\theta_i] + 3(\theta_i + \theta_i'), \quad i = 1, 2, 3. \tag{3.8}$$

Theorem 3.3. *Assume for some $i \in \{1,2,3\}$, $\widehat{r}_i \in L^1[t_0,\infty)$, $\|\mathcal{L}^{\alpha-\beta}_{n(i)}[r_1 + 2\lambda_i r_2]\|_\infty + \|\mathcal{L}^{\alpha-\beta}_{n(i)}[r_2]\|_\infty \leq \nu_i$, where $\nu_i > 0$, $i = 1, 2, 3$ are given by (3.2). Then equation (1.1) has a solution y such that*

$$y(t) = (1 + \varepsilon_i(t)) \exp \lambda_i t,$$

where

$$\varepsilon_i(t) = O\left(\exp\left(\int_t^\infty \mathcal{L}^{\beta}_{n(i)}[\widehat{r}_i](s)\,ds\right) - 1\right).$$

Proof. Applying the same ideas that in the previous result, we know that there exists a solution in the form (2.3), where $z_i, z_i' = O\big(\mathcal{L}^{\beta}_{n(i)}[\widehat{r}_i]\big)$. Since $\widehat{r}_i \in L^1[t_0,\infty)$, we have $z_i \in L^1[t_0,\infty)$,

$$y(t) = c\left[1 + \exp\left(-\int_t^\infty z_i(s)\,ds\right) - 1\right] \exp \lambda_i t,$$

and the result follows. ∎

Theorem 3.4. *Assume for some $i \in \{1,2,3\}$, $\widehat{r}_i, r_1+2\lambda_i r_2 \in L^2[t_0,\infty)$ and $r_2 \in \mathcal{B}_V^{\alpha,\nu}$. Then equation* (1.1) *has a solution y such that*

$$y(t) = (1+o(1))e^{\lambda_i(t-t_0)} \exp\left(-\prod_{j\in\mathcal{N}(i)} (\lambda_j-\lambda_i)^{-1} \int_{t_0}^t \widehat{r}_i(s)\,ds\right).$$

Proof. Since $\widehat{r}_i, r_1+2\lambda_i r_2 \in L^2[t_0,\infty)$ and $r_2 \in \mathcal{B}_V^{\alpha,\nu}$ we have that the hypothesis of Theorem 1 are satisfied. So, there exists a solution satisfying formula (3.4). Moreover, since $z_i, z_i' = O(\mathcal{L}^{\beta}_{n(i)}[\widehat{r}_i])$, by Lemma 2.1, $z_i, z_i' \in L^2[t_0,\infty)$. Hence, using that r_2 is a bounded variation function we can conclude $\int_{t_0}^t \widetilde{f}_i(s, z_i(s), z_i'(s))\,ds = c + o(1)$, and the result follows. $\square$

Recall,

$$\begin{aligned} f_i(t,z,z') &= (r_1(t)+2\lambda_i r_2(t))z + r_2(t)z' + 3zz' + (3\lambda_i + a_2 + r_2(t))z^2 + z^3 \\ &= \widetilde{f}_i(t,z,z') + 3zz'. \end{aligned}$$

Theorem 3.5. *Assume that $f_i(\cdot,\theta_i,\theta_i') \in L^1[t_0,\infty)$ and for all $i=1,2,3$, $\|\mathcal{L}^{\alpha-\beta}_{n(i)}[r_1+2\lambda_i r_2]\|_\infty + \|\mathcal{L}^{\alpha-\beta}_{n(i)}[r_2]\|_\infty \le \nu_i$, where ν_i, $i=1,2,3$ are given by* (3.2)*. Then there exists a fundamental system of solutions y_i, $i=1,2,3$ such that*

$$y_i(t) = (1+o(1))e^{\lambda_i(t-t_0)} \exp\left(-\prod_{j\in\mathcal{N}(i)} (\lambda_j-\lambda_i)^{-1} \int_{t_0}^t \widehat{r}_i(s)\,ds\right).$$

Proof. We omit the index i. If $\widetilde{\theta} := f(\cdot,\theta,\theta')$ and z satisfies (3.7), then $u = z-\theta$ satisfies

$$u = -G[\widetilde{\theta} + \widehat{f}(\cdot,u,u')], \quad \widehat{f}(\cdot,u,u') = f(\cdot,u+\theta,u'+\theta') - \widetilde{\theta}. \tag{3.9}$$

Thus

$$\widehat{f}(\cdot,u,u') = \widehat{\theta}u + r_2u' + 3uu' + [3\lambda + a_2 + r_2 + 3\theta]u^2 + u^3$$

and from Lemma 2.6 and Corollary 1 there exists u satisfying (3.9) with $u, u' = O(\mathcal{L}^{\beta}[\widetilde{\theta}])$. Then, $u, u' \in L^1[t_0,\infty)$ by Lemma 2.1 since $\widetilde{\theta} \in L^1[t_0,\infty)$. Using (3.1) and $z = \theta + u$ the result follows. $\square$

If $r_k \in \mathcal{B}$, $r_k^*(t) \to 0$ as $t\to\infty$, $k=0,1,2$, where r_k^* is given by (2.1) and $\theta_i \in L^1[t_0,\infty)$, for all $i=1,2,3$ then $f(\cdot,\theta_i,\theta_i') \in L^1[t_0,\infty)$, for all $i=1,2,3$ and $\|\mathcal{L}^{\alpha-\beta}_{n(i)}[r_1+2\lambda_i r_2]\|_\infty + \|\mathcal{L}^{\alpha-\beta}_{n(i)}[r_2]\|_\infty \le \nu_i$, where ν_i, $i=1,2,3$ are given by (3.2), for t_0 big enough. Therefore, conditions of Theorem 3.5 are satisfied. Note that the same result is true if for all $i=1,2,3$, $f_i(\cdot,\theta_i,[\theta_i]') \in L^1[t_0,\infty)$, $\|\mathcal{L}^{\alpha-\beta}_{n(i)}[\widehat{\theta}_i]\|_\infty + \|\mathcal{L}^{\alpha-\beta}_{n(i)}[r_2]\|_\infty \le \nu_i$, where ν_i, $i=1,2,3$ are given by (3.2) and $\mathcal{L}^{\alpha}_{n(i)}[\theta_i] \in \mathcal{B}$ holds.

Now, studying equation (3.9) we can deduce the following result which mixes different type of conditions.

Theorem 3.6. *Assume that for all* $i = 1,2,3$, $f_i(\cdot,\theta_i,\theta_i') \in L^p[t_0,\infty)$ *for some* $p \in (1,2]$, $\widehat{\theta}_i \in L^{q_1}[t_0,\infty)$, $r_2 \in L^{q_2}[t_0,\infty) \cup \mathcal{B}_V^{\alpha,\nu}$ *for some* $q_1, q_2 \in [1,q]$ *and* $\theta_i \in \mathcal{B} \cup L^{p'}[t_0,\infty)$, *where* $\frac{1}{p}+\frac{1}{q}=1$ *and* $p' \geq 1$. *Then there exists a fundamental system of solutions* y_i, $i=1,2,3$ *of equation* (1.1) *such that*

(3.10)
$$y_i(t) = (1+o(1))e^{\lambda_i(t-t_0)} \exp\left(-\prod_{j\in\mathcal{N}(i)}(\lambda_j-\lambda_i)^{-1}\int_{t_0}^t \left[\widehat{r}_i(s)+\widetilde{f}_i\big(s,\theta_i(s),\theta_i'(s)\big)\right] ds\right).$$

Proof. Again we omit the index i. Using the same ideas and notations in the previous result, from (3.9) we have now $u, u' = O(\mathcal{L}^\beta[\widetilde{\theta}])$. Then, $u,u' \in L^p[t_0,\infty)$ by Lemma 2.1, since $\widetilde{\theta} \in L^p[t_0,\infty)$. So, by (3.1) and $z = \theta - G[\widetilde{\theta}+\widehat{f}(\cdot,u,u')]$, the result follows. □

For example, Theorem 5 is satisfied if one of the following conditions holds

(a) $r_1, r_2 \in \mathcal{B}$, $\theta_i, \theta_i' \in L^1[t_0,\infty)$, $\widehat{\theta}_i \in L^2[t_0,\infty)$ and $r_2 \in \mathcal{B}_V^{\alpha,\nu}$.
(b) $f_i(\cdot,\theta_i,\theta_i') \in L^2[t_0,\infty)$, $\widehat{\theta}_i \in L^1[t_0,\infty)$, $\theta_i \in \mathcal{B}$ and $r_2 \in L^2[t_0,\infty)$.

All the errors functions can be estimated. The method can be iterated to obtain an expansion as (1.2).

4. EXAMPLES

Now, we will show some examples of the above results. Consider the equation

$$y''' + r_2(t)y'' - (1-r_1(t))y' + r_0(t)y = 0. \tag{4.1}$$

Here, we have $\lambda_1 = 1$, $\lambda_2 = 0$, $\lambda_3 = -1$, $\gamma_1 = \gamma_3 = 1$, $\gamma_2 = 2$. We begin showing two cases where the known results (see Eastham [4]) cannot be applied.

Denote $\mathcal{C}_0 = \{x : [t_0,\infty) \to \mathbb{C} \mid x \text{ are continuous and } x(t) \to 0 \text{ as } t \to \infty\}$.

Example 1. Let $r_2 = 0$, $r_1 = 0$ and $r_0(t) = \cos(t^\beta)$. Note that $r_0 \notin L^p$ for all $p \geq 1$ and its derivatives does not improve its integrable character. Also, note that if $\beta > 1$ then r_0 is conditionally integrable, so for any $\gamma \in \mathbb{C}$ with $\mathrm{Re}\gamma > 0$, $G_\pm^\gamma[r_0](t) \to 0$ as $t \to \infty$. Furthermore, given $p \geq 1$ if $\beta > 1 + 1/p$ then for any $\gamma \in \mathbb{C}$ with $\mathrm{Re}\gamma > 0$, $G_\pm^\gamma[r_0] \in L^p$. So, for any $\beta > 1$, using Remark 2, we can say

$$z_i = -G_i[r_0] + O(\mathcal{L}^\alpha[f_i(\cdot,-G_i[r_0],-G_i[r_0]')]), \qquad 0<\alpha<1$$

where $f_i(t,z,z') = 3zz' + 3\lambda_i z^2 + z^3$. Using formula (3.4) we have that there exists a fundamental system of solutions of equation (4.1) such that

$$y_i(t) = (1+o(1))e^{\lambda_i(t-t_0)} \exp\left(-\prod_{j\in\mathcal{N}(i)}(\lambda_j-\lambda_i)^{-1}\int_{t_0}^t [\cos(s^\beta)+\widetilde{f}_i(s,z_i(s),z_i'(s))]\, ds\right),$$

where $\widetilde{f}_i(t,z,z') = 3\lambda_i z^2 + z^3$. For example:

1. If $\beta > 2$ then $G_i[r_0], G_i[r_0]' \in L^1 \cap \mathcal{C}_0$ and $f_i(\cdot, -G_i[r_0], -G_i[r_0]') \in L^1 \cap \mathcal{C}_0$ for all $i = 1, 2, 3$. Thus, we have $z_i, z_i' \in L^1 \cap \mathcal{C}_0$. Then, there exists a fundamental system of solutions $\{y_i\}_{i=1,2,3}$ of equation (4.1) such that

$$y_i(t) = \left(1 + O\left[\exp\left(\int_t^\infty |\mathcal{L}^\alpha_{n(i)}[r_0](s)|\, ds\right) - 1\right]\right) e^{\lambda_i(t-t_0)}. \tag{4.2}$$

2. If $\beta \in (\frac{3}{2}, 2]$ then for any $\gamma \in \mathbb{C}$ with $\mathrm{Re}\gamma > 0$, $G^\gamma_\pm[r_0] \in L^2 \cap \mathcal{C}_0$. So, $G_i[r_0], G_i[r_0]' \in L^2 \cap \mathcal{C}_0$ and $f_i(\cdot, -G_i[r_0], -G_i[r_0]') \in L^1 \cap \mathcal{C}_0$ for all $i = 1, 2, 3$. Thus, by Theorem 5 we have $z_i, z_i' \in L^2 \cap \mathcal{C}_0$ and there exists a fundamental system of solutions $\{y_i\}_{i=1,2,3}$ of equation (4.1) satisfying

$$y_i(t) = (1 + o(1)) e^{\lambda_i(t-t_0)} \tag{4.3}$$

since r_0 is conditionally integrable. Notice the similarity between (4.2) and (4.3).

3. If $\beta \in (\frac{4}{3}, \frac{3}{2}]$ then for any $\gamma \in \mathbb{C}$ with $\mathrm{Re}\gamma > 0$, $G^\gamma_\pm[r_0] \in L^3 \cap \mathcal{C}_0$. So, $G_i[r_0], G_i[r_0]' \in L^3 \cap \mathcal{C}_0$ and $f_i(\cdot, -G_i[r_0], -G_i[r_0]') \in L^{\frac{3}{2}} \cap \mathcal{C}_0$ for all $i = 1, 2, 3$. Thus, we have $\widehat{f}_i(\cdot, u_i, u_i') \in L^1$, where $u_i = z_i - G_i[r_0]$. Then, by Theorem 5, there exists a fundamental system of solutions $\{y_i\}_{i=1,2,3}$ of equation (4.1) such that

$$y_i(t) = (1 + o(1)) e^{\lambda_i(t-t_0)} \exp\left(-\prod_{j \in \mathcal{N}(i)} (\lambda_j - \lambda_i)^{-1} \int_{t_0}^t \widetilde{f}_i\big(s, G_i[r_0](s), G_i[r_0]'(s)\big)\, ds\right).$$

Although, $G_i[r_0] \in L^p[t_0, \infty)$, linear transformations as

$$y = \exp\left(\lambda t - \int_{t_0}^t G[r_0](s)\, ds\right) \widetilde{y}$$

does not simplify the study of equation (4.1). In example 1, the integrals θ_i of the bad perturbations r_0 help to improve the successive approximations in (1.2). This is not the case in example 2.

Example 2. Let $r_2 = r_1 = 0$ and $r_0(t) = \sin t / \log t$. Note that $r_0(t) \to 0$ as $t \to \infty$, for any $p \geq 1$, $r_0 \notin L^p$ and $r_0' \notin L^1$. Moreover, for any $p \geq 1$, $\theta_i = -G_i[r_0] \notin L^p$, but θ_i are conditionally integrable for every $i = 1, 2, 3$. Again, we cannot apply the known results (see [4, 7]). However, Perron's formulae for the solutions are valid and (3.4) can be used to obtain an asymptotic expression

$$y_i(t) = (1 + o(1)) e^{\lambda_i(t-t_0)} \exp\left(-\prod_{j \in \mathcal{N} i} (\lambda_j - \lambda_i)^{-1} \int_{t_0}^t \left[\frac{\sin s}{\log s} + 3\lambda_i z^2(s) + z^3(s))\right] ds\right),$$

and $z_i, z_i' = O\left(\mathcal{L}^\alpha_{n(i)}[r_0]\right)$, $i = 1, 2, 3$ and $0 < \alpha < 1$. Note that

$$G^\gamma_\pm[r_0](t) = \frac{\pm\gamma \sin t - \cos t}{(1 + \gamma^2) \log t} + \phi_\pm(t),$$

where $\phi_\pm \in L^1$, which shows that the integrals cannot improve. The successive terms $\theta_{i\,k}$ in (1.2), neither. So, we will obtain a series which must approach the characteristic roots $\widehat{\lambda}_i(t)$ of the corresponding variable characteristic polynomial, see [3, 7, 9, 21].

Example 3. Let $r_0(t) = \frac{1}{2\sqrt{t}}$, $r_1(t) = \frac{1}{\sqrt{t+1}}$ and $r_2 = r_0$. We know $r_l \notin L^2$, $l = 0, 1, 2$ but $\widehat{r}_3, r_1 + 2\lambda_3 r_2 \in L^2$, where $\widehat{r}_3 = r_0 - r_1 + r_2$, $r_1 + 2\lambda_3 r_2 = r_1 - 2r_2$. So, for $i = 3$, the hypothesis of Theorem 3 are satisfied and there exists a solution y_3 of equation (4.1) such that

$$y_3(t) = (1 + o(1))e^{-(t-t_0)} \exp\left(-\frac{1}{2}\int_{t_0}^{t}\left[\frac{1}{\sqrt{s}} - \frac{1}{\sqrt{s+1}}\right] ds\right)$$

and we have the formula

$$y_3(t) = (1 + o(1))e^{-t} \exp\left(\sqrt{t+1} - \sqrt{t}\right).$$

On the other hand, note that $\widehat{r}_i \in L^p$ for all $p > 2$, $i = 1, 2$. Hence, for $i = 1, 2$, $\theta_i = G_i[\widehat{r}_i] \in L^p$ for all $p > 2$, $f(\cdot, \theta_i, [\theta_i]') \in L^{q_1}$ for all $q_1 > 1$ and $\widehat{\theta}_i \in L^{q_2}$ for all $q_2 > 2$ in (3.8). Thus, the hypothesis of Theorem 5 are satisfied and we conclude that there exists a solution for each $i = 1, 2$ satisfying (3.10). Moreover, since $r_2[\theta_i]^2 + [\theta_i]^3 \in L^1$ and $r_2[\theta_i]' + 3\theta_i[\theta_i]'$ is conditionally integrable, we have

$$\begin{aligned} y_1(t) =&(1 + o(1))e^{t-t_0}\times \\ &\exp\left(-\frac{1}{2}\int_{t_0}^{t}\left[\frac{1}{\sqrt{s}} + \frac{1}{\sqrt{s+1}} + \left(\frac{1}{\sqrt{s+1}} + \frac{1}{\sqrt{s}}\right)\theta_1(s) + 3[\theta_1]^2(s)\right] ds\right), \end{aligned}$$

and

$$y_2(t) = (1 + o(1)) \exp\left(\int_{t_0}^{t}\left[\frac{1}{2\sqrt{s}} + \frac{1}{\sqrt{s+1}}\theta_2(s)\right] ds\right).$$

Finally, integrating by parts this formula (see (3.1)), we deduce the following asymptotic solutions

$$y_1(t) = (1 + o(1))t^{-1/8}(t+1)^{-1/8}\left(t + \frac{1}{2} + \sqrt{t(t+1)}\right)^{-1/4} e^t \exp\left(-\sqrt{t} - \sqrt{t+1}\right)$$

and

$$y_2(t) = (1 + o(1))\left(t + \frac{1}{2} + \sqrt{t(t+1)}\right)^{1/2} e^{\sqrt{t}}.$$

Since in this case $r_k \in L^3 \cap \mathcal{B}_V$, reducing equation (4.1) to a first order system, and using matrix transformations, the L^3-Hartman-Wintner's or Levinson's theorems can be applied. So, it is possible to approximate eigenvalues up to L^1 perturbation by a simple iteration [1, 2, 7, 10, 18]. Doing this yields asymptotic formulae with simpler calculations and more compact form. In fact, it is possible to obtain

$$y_1(t) = (1 + o(1))t^{-1/2}e^{t-2\sqrt{t}}, \qquad y_2(t) = (1 + o(1))t^{-1/2}e^{\sqrt{t}}$$

$$\text{and} \qquad y_3(t) = (1 + o(1))e^{-t}.$$

This asymptotic formulae are compatible with the previous ones. In fact our formulae can be simplified to obtain the last ones.

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