Dynamic Systems and Applications 17 (2008) 503-514

APPROXIMATION METHODS OF SOLUTIONS FOR EQUILIBRIUM PROBLEM IN HILBERT SPACES

SHIH-SEN CHANG, YEOL JE CHO, AND JONG KYU KIM

Department of Mathematics, Yibin University, Yibin, Sichuan 644007, China
sszhang_1@yahoo.com.cn
Department of Mathematics Education and the RINS, Gyeongsang National University, Chinju 660-701, Korea yjcho@gsnu.ac.kr
Department of Mathematics Education, Kyungnam University
Masan 631-701, Korea jongkyuk@kyungnam.ac.kr

ABSTRACT. The purpose of this paper is, by using viscosity approximation methods, to find a common element of the set of solutions of an equilibrium problem and the set of fixed point of a nonexpansive mappings in a Hilbert space and to prove, under suitable conditions, some strong convergence theorems for approximating a solution of the problem under consideration.

AMS (MOS) Subject Classification. 47H09, 47H10

1. INTRODUCTION PRELIMINARIES

Throughout this paper, we always assume that H is a real Hilbert space, C is a nonempty closed convex subset of H and $\phi : C \times C \to R$ is a real functional with $\phi(x,x) = 0$ for all $x \in C$. The "so called" *equilibrium problem for functional* ϕ is to finding a point $x^* \in C$ such that

$$\phi(x^*, y) \geq 0, \quad \forall y \in C. \tag{1.1}$$

Denote the set of solutions of the equilibrium problem (1.1) by $EP(\phi)$.

This equilibrium problem contains fixed point problem, optimization problem, variational inequality problem and Nash equilibrium problem as its special cases (see, for example, Blum and Oetti [2]).

Recently, Antipin and Flam [1], Blum and Oettli [2], Moudafi [7], Moudafi et al. [8], Combettes and Hirstoaga [4] introduced and studied iterative schemes of finding the best approximation to the initial data when $EP(\phi)$ is nonempty and proved some strong convergence theorems in Hilbert spaces.

Very recently, Takahashi and Takahashi [10] introduced a new iterative scheme by using the viscosity approximation methods (see Moudafi [7] and Xu [11]) for finding a common element of the set of solutions of the equilibrium problem (1.1) and the set

Received April 14, 2007

of fixed points of a nonexpansive mapping in a Hilbert space and proved some strong convergence theorems. Their results extend and improve the corresponding results given in [4, 7].

Motivated and inspired by Combettes-Histoaga [4], Takahashi and Takahashi [10], the purpose of this paper is, by using viscosity approximation methods, to find a common element of the set of solutions of equilibrium problem (1.1) and the set of fixed points of a nonexpamsive mappings in Hilbert spaces and to establish some strong convergence theorems.

For this purpose, first, we recall some definitions, lemmas and notations.

In the sequel, we use $x_n \rightharpoonup x$ and $x_n \to x$ to denote the weak convergence and strong convergence of the sequence $\{x_n\}$ in H, respectively.

In a Hilbert space H, for any $x \in H$, there exists a unique nearest point in C, denoted by $P_C(x)$, such that

$$||x - P_C x|| \leq ||x - y||, \ \ \forall y \in C.$$

Such a mapping P_C from H onto C is called the *metric projection.* We know that P_C is nonexpansive. Further, for any $x \in H$ and $z \in C$,

$$z = P_C(x) \Leftrightarrow \langle x - z, \ z - y \rangle \geq 0, \ \ \forall y \in C.$$

For solving the equilibrium problem (1.1) for the functional $\phi : C \times C \to R$, let us assume that ϕ satisfies the following conditions:

(A1) $\phi(x, x) = 0, \ \ \forall x \in C$;
(A2) ϕ is monotone, i.e.,

$$\phi(x, y) + \phi(y, x) \leq 0, \ \ \forall x, y \in C;$$

(A3) for any $x, y, z \in C$ the functional $x \mapsto \phi(x, y)$ is upper-hemicontinuous, i.e.,

$$\limsup_{t \to 0+} \phi(tz + (1 - t)x, \ y) \leq \phi(x, y), \ \ \forall x, \ y \ z \in C;$$

(A4) $y \mapsto \phi(x, y)$ is convex and lower semi-continuous.

The following lemmas will be needed in proving our main results:

Lemma 1.1. ([2]) *Let H be a real Hilbert space, C be a nonempty closed convex subset of H, $\phi : C \times C \to R$ be a functional satisfying the conditions* (A1)–(A4)*. Then, for any given $x \in H$ and $r > 0$, there exists $z \in C$ such that*

$$\phi(z, y) + \frac{1}{r}\langle y - z, \ z - x \rangle \geq 0, \ \ \forall y \in C.$$

Lemma 1.2. ([4]) *Let all the conditions in Lemma* 1.1 *are satisfied. For any* $r > 0$ *and* $x \in C$*, define a mapping* $T_r : H \to C$ *as follows:*

$$T_r(x) = \{z \in C : \phi(z, y) + \frac{1}{r}\langle y - z,\ z - x\rangle \geq 0,\ \ \forall y \in C\},\ \ \forall x \in H.$$

Then the following holds:

(1) T_r *is single-valued;*

(2) T_r *is firmly nonexpansive, i.e.,*

$$||T_r x - T_r y||^2 \leq \langle T_r x - T_r y,\ x - y\rangle,\ \ \forall x, y \in H,$$

and so $||T_r x - T_r y|| \leq ||x - y||,\ \ \forall x, y \in H.$

(3) $F(T_r) = EP(\phi),\ \ \forall r > 0;$

(4) $EP(\phi)$ *is a closed and convex set.*

Lemma 1.3. ([9]) *Let* X *be a Banach space,* $\{x_n\}$, $\{y_n\}$ *be two bounded sequences in* X *and* $\{\beta_n\}$ *be a sequence in* $[0, 1]$ *satisfying*

$$0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1.$$

Suppose that $x_{n+1} = \beta_n x_n + (1 - \beta_n)y_n,\ \ \forall n \geq 0$ *and*

$$\limsup_{n\to\infty}\{||y_{n+1} - y_n|| - ||x_{n+1} - x_n||\} \leq 0,$$

then $\lim_{n\to\infty} ||y_n - x_n|| = 0.$

Lemma 1.4. ([6]) *Let* $\{a_n\}$ *and* $\{b_n\}$ *be two nonnegative real sequences satisfying the following condition:*

$$a_{n+1} \leq (1 - \lambda_n)a_n + b_n,\ \ \forall n \geq n_0,$$

where n_0 *is some nonnegative integer,* $\{\lambda_n\}$ *is a sequence in (0, 1) with* $\sum_{n=0}^{\infty} \lambda_n = \infty$ *and* $b_n = \circ(\lambda_n)$*. Then* $\lim_{n\to\infty} a_n = 0.$

Lemma 1.5. ([5]) *Let* X *be a uniformly convex Banach space,* C *be a nonempty closed convex subset of* X *and* $T : C \to X$ *be a nonexpansive mapping with a fixed point. Then* $I - T$ *is demiclosed in the sense that if* $\{x_n\}$ *is a sequence in* C *and if* $x_n \rightharpoonup x$ *and* $(I - T)x_n \to 0$*, then* $(I - T)x = 0.$

Lemma 1.6. ([3]) *Let* E *be a real Banach space,* $J : E \to 2^{E^*}$ *be the normalized duality mapping and* x, y *be any given points in* E*. Then the following conclusion holds:*

$$||x + y||^2 \leq ||x||^2 + 2\langle y,\ j(x + y)\rangle,\ \ \forall j(x + y) \in J(x + y).$$

Especially, if $E = H$ *is a real Hilbert space, then*

$$||x + y||^2 \leq ||x||^2 + 2\langle y,\ x + y\rangle,\ \ \forall x, y \in H.$$

2. MAIN RESULTS

In this section, we shall prove our main theorems in this paper:

Theorem 2.1. *Let H be a real Hilbert space, C be a nonempty closed convex subset of H, $\phi : C \times C \to R$ be a functional satisfying the conditions* (A1)–(A4), *$T : C \to H$ be a nonexpansive mapping with $F(T)\bigcap EP(\phi) \neq \emptyset$ and $f : H \to H$ be a contraction mapping with a contractive constant $\alpha \in (0,1)$. Let $\{\alpha_n\}$, $\{\beta_n\}$ be two sequences in $[0,1]$ and $\{r_n\} \subset (0,\infty)$ be a real sequence satisfying the following conditions:*

(i) $\alpha_n \to 0$; $\sum_{n=0}^{\infty} \alpha_n = \infty$; $|1 - \frac{\alpha_n}{\alpha_{n+1}}| \to 0$;
(ii) *There exist a, $b \in (0,1)$ such that $a \le \beta_n \le b$ for all $n \ge 0$;*
(iii) *$0 < r < r_n$ for all $n \ge 0$ and $|r_n - r_{n+1}| \to 0$ as $n \to \infty$.*

where r, a and b are some positive constants. For any $x_0 \in H$, let $\{x_n\}$ and $\{u_n\}$ be the sequences defined by

$$\begin{cases} \phi(u_n, y) + \dfrac{1}{r_n}\langle y - u_n\, u_n - x_n\rangle \ge 0, \quad \forall y \in C, \\ x_{n+1} = \beta_n x_n + (1-\beta_n) z_n, \\ z_n = \alpha_n f(u_n) + (1-\alpha_n) T u_n, \quad \forall n \ge 0. \end{cases} \tag{2.1}$$

Then $x_n \to x^* \in F(T)\bigcap EP(\phi)$, where $x^* = P_{F(T)\cap EP(\phi)} f(x^*)$.

Proof. We divide the proof into six steps:

(I) We first prove that the mapping $P_{F(T)\cap EP(\phi)} f : H \to C$ has a unique fixed point.

In fact, since $f : H \to H$ is a contraction and $P_{F(T)\cap EP(\phi)} : H \to F(T) \cap EP(\phi)$ is also a contraction, we have

$$||P_{F(T)\cap EP(\phi)} f(x) - P_{F(T)\cap EP(\phi)} f(y)|| \le \alpha ||x-y||, \quad \forall x, y \in H.$$

Therefore, there exists a unique $x^* \in C$ such that $x^* = P_{F(T)\cap EP(\phi)} f(x^*)$.

(II) Now we prove that the sequences $\{x_n\}$ and $\{u_n\}$ are bounded in H and C, respectively.

In fact, from the definition of T_r in Lemma 1.2, we know that $u_n = T_{r_n} x_n$. Therefore, for any $p \in F(T)\bigcap EP(\phi)$, we have

$$||u_n - p|| = ||T_{r_n} x_n - T_{r_n} p|| \le ||x_n - p||. \tag{2.2}$$

Therefore, it follows from (2.1) and (2.2) that

$$\begin{aligned} &||x_{n+1} - p|| \\ &\le \beta_n ||x_n - p|| + (1-\beta_n)||z_n - p|| \\ &\le \beta_n ||x_n - p|| + (1-\beta_n)||\alpha_n (f(u_n) - p) + (1-\alpha_n)(T u_n - p)|| \end{aligned}$$

$$\begin{aligned}
&\leq \beta_n||x_n - p|| + (1-\beta_n)\{\alpha_n||f(u_n) - f(p)|| \\
&\quad + \alpha_n||f(p) - p|| + (1-\alpha_n)||Tu_n - p||\} \\
&\leq \beta_n||x_n - p|| + (1-\beta_n)\alpha_n\alpha||u_n - p|| \\
&\quad + (1-\beta_n)\alpha_n||f(p) - p|| + (1-\beta_n)(1-\alpha_n)||u_n - p|| \\
&\leq \beta_n||x_n - p|| + (1-\beta_n)(1-\alpha_n(1-\alpha))||x_n - p|| + (1-\beta_n)\alpha_n||f(p) - p|| \\
&\leq \max\{||x_n - p||,\ \frac{||f(p) - p||}{1-\alpha}\} \\
&\leq \cdots \\
&\leq \max\{||x_0 - p||,\ \frac{||f(p) - p||}{1-\alpha}\}.
\end{aligned}$$

This implies that $\{x_n\}$ is a bounded sequence in H. By (2.2), we know that $\{u_n\}$ is a bounded sequence in C and so $\{Tu_n\}$, $\{f(u_n)\}$, $\{z_n\}$ all are bounded sequences in H. Let

$$M = \sup_{n\geq 0}\{||u_n - x_n|| + ||x_n - y||^2 + ||f(u_n)|| + ||T(u_n)||\}, \tag{2.3}$$

where $y \in H$ is some given point.

(III) Now, we make an estimation for $\{||u_{n+1} - u_n||\}$.

By the definition of T_r, $u_n = T_{r_n}x_n$ and $u_{n+1} = T_{r_{n+1}}x_{n+1}$. Hence we have

$$\phi(u_{n+1}, y) + \frac{1}{r_{n+1}}\langle y - u_{n+1},\ u_{n+1} - x_{n+1}\rangle \geq 0, \quad \forall y \in C, \tag{2.4}$$

$$\phi(u_n, y) + \frac{1}{r_n}\langle y - u_n,\ u_n - x_n\rangle \geq 0, \quad \forall y \in C. \tag{2.5}$$

Take $y = u_{n+1}$ in (2.5) and $y = u_n$ in (2.4). Then, adding the resulting inequalities and noting the condition (A2), we have

$$\langle u_{n+1} - u_n,\ \frac{u_n - x_n}{r_n} - \frac{u_{n+1} - x_{n+1}}{r_{n+1}}\rangle \geq 0$$

and hence

$$\langle u_{n+1} - u_n,\ u_n - u_{n+1} + u_{n+1} - x_n - \frac{r_n}{r_{n+1}}(u_{n+1} - x_{n+1})\rangle \geq 0.$$

This implies that

$$\begin{aligned}
||u_{n+1} - u_n||^2 &\leq \langle u_{n+1} - u_n,\ x_{n+1} - x_n + (1 - \frac{r_n}{r_{n+1}})(u_{n+1} - x_{n+1})\rangle \\
&\leq ||u_{n+1} - u_n||\{||x_{n+1} - x_n|| + |1 - \frac{r_n}{r_{n+1}}| \cdot ||u_{n+1} - x_{n+1}||\}
\end{aligned}$$

Thus, by the condition (iii), we have

$$\begin{aligned}
||u_{n+1} - u_n|| &\leq ||x_{n+1} - x_n|| + |1 - \frac{r_n}{r_{n+1}}|||u_{n+1} - x_{n+1}|| \\
&\leq ||x_{n+1} - x_n|| + \frac{1}{r}|r_{n+1} - r_n| \cdot M.
\end{aligned} \tag{2.6}$$

(IV) Now we prove that $||Tu_n - u_n|| \to 0$.

In fact, it follows from (2.1) and (2.6) that

$$\begin{aligned}
&||z_{n+1} - z_n|| \\
&= ||\alpha_{n+1} f(u_{n+1}) + (1-\alpha_{n+1})Tu_{n+1} - \alpha_n f(u_n) - (1-\alpha_n)Tu_n|| \\
&= ||\alpha_{n+1} f(u_{n+1}) - \alpha_{n+1} f(u_n) + \alpha_{n+1} f(u_n) - \alpha_n f(u_n) \\
&\quad + (1-\alpha_{n+1})Tu_{n+1} - (1-\alpha_{n+1})Tu_n + (1-\alpha_{n+1})Tu_n - (1-\alpha_n)Tu_n|| \\
&\le \alpha_{n+1}||f(u_{n+1}) - f(u_n)|| + 2|\alpha_n - \alpha_{n+1}|M + (1-\alpha_{n+1})||Tu_{n+1} - Tu_n|| \\
&\le \alpha_{n+1}\alpha||u_{n+1} - u_n|| + 2|\alpha_n - \alpha_{n+1}|M + (1-\alpha_{n+1})||u_{n+1} - u_n|| \\
&\le ||u_{n+1} - u_n|| + 2|\alpha_n - \alpha_{n+1}|M \\
&\le ||x_{n+1} - x_n|| + \frac{1}{r}|r_{n+1} - r_n||M + 2|1 - \frac{\alpha_n}{\alpha_{n+1}}|M
\end{aligned}$$

It follows from the conditions (i) and (iii) that

$$\limsup_{n\to\infty}\{||z_{n+1} - z_n|| - ||x_{n+1} - x_n||\} \le 0.$$

By virtue of Lemma 1.3, we obtain that

$$||x_n - z_n|| \to 0 \quad \text{as} \ \ n \to \infty. \tag{2.7}$$

From (2.1) and (2.7), we have

$$||x_{n+1} - x_n|| = (1-\beta_n)||x_n - z_n|| \to 0, \quad \text{as} \ \ n \to \infty. \tag{2.8}$$

It follows from (2.6), (2.8) and the condition (iii) that

$$||u_{n+1} - u_n|| \to 0, \quad \text{as} \ \ n \to \infty. \tag{2.9}$$

Since $\alpha_n \to 0$ and $\{z_n\}$, $\{f(u_n)\}$, $\{Tu_n\}$ all are bounded, from (2.7), we have

$$\begin{aligned}
||x_n - Tu_n|| &\le ||x_n - z_n|| + ||z_n - Tu_n|| \\
&\le ||x_n - z_n|| + \alpha_n||f(u_n) - Tu_n|| \to 0.
\end{aligned} \tag{2.10}$$

Furthermore, for any $p \in F(T)\bigcap EP(\phi)$, we have

$$\begin{aligned}
||u_n - p||^2 &= ||T_{r_n}x_n - T_{r_n}p||^2 \\
&\le \langle T_{r_n}x_n - T_{r_n}p,\ x_n - p\rangle \\
&= \langle u_n - p,\ x_n - p\rangle \\
&= \frac{1}{2}\{||u_n - p||^2 + ||x_n - p||^2 - ||x_n - u_n||^2\}.
\end{aligned}$$

Hence we have

$$||u_n - p||^2 \le ||x_n - p||^2 - ||x_n - u_n||^2. \tag{2.11}$$

From the convexity of function $x \mapsto ||x||^2$ and (2.11), we have

$$\begin{aligned} ||x_{n+1} - p||^2 &= ||\beta_n x_n + (1-\beta_n) z_n - p||^2 \\ &\le \beta_n ||x_n - p||^2 + (1-\beta_n)||z_n - p||^2 \\ &\le \beta_n ||x_n - p||^2 + (1-\beta_n)\{\alpha_n ||f(u_n) - p||^2 \\ &\quad + (1-\alpha_n)||Tu_n - p||^2\} \\ &\le \beta_n ||x_n - p||^2 + (1-\beta_n)\alpha_n ||f(u_n) - p||^2 \\ &\quad + (1-\alpha_n)(1-\beta_n)||u_n - p||^2 \\ &\le \beta_n ||x_n - p||^2 + (1-\beta_n)\alpha_n ||f(u_n) - p||^2 + (1-\beta_n)||u_n - p||^2 \\ &\le \beta_n ||x_n - p||^2 + (1-\beta_n)\alpha_n ||f(u_n) - p||^2 \\ &\quad + (1-\beta_n)\{||x_n - p||^2 - ||x_n - u_n||^2\} \end{aligned}$$

and so

$$\begin{aligned} &(1-\beta_n)||x_n - u_n||^2 \\ &\le ||x_n - p||^2 - ||x_{n+1} - p||^2 + (1-\beta_n)\alpha_n ||f(u_n) - p||^2 \\ &\le (||x_n - p|| - ||x_{n+1} - p||)(||x_n - p|| + ||x_{n+1} - p||) + \alpha_n ||f(u_n) - p||^2 \\ &\le (||x_n - x_{n+1}||)(||x_n - p|| + ||x_{n+1} - p||) + \alpha_n ||f(u_n) - p||^2. \end{aligned}$$

Since $\alpha_n \to 0$, $\{x_n\}$ and $\{f(u_n)\}$ are bounded and $||x_n - x_{n+1}|| \to 0$, we have

$$||x_n - u_n|| \to 0 \quad \text{as} \quad n \to \infty \tag{2.12}$$

and so

$$||Tu_n - u_n|| \le ||Tu_n - x_n|| + ||x_n - u_n|| \to 0. \tag{2.13}$$

The desired conclusion is proved.

(V) Now, we prove that

$$\limsup_{n\to\infty} \langle f(x^*) - x^*,\ x_n - x^* \rangle \le 0,$$

where $x^* = P_{F(T)\cap EP(\phi)} f(x^*)$.

In fact, we can choose a subsequence $\{x_{n_j}\} \subset \{x_n\}$ such that

$$\lim_{n_j\to\infty} \langle f(x^*) - x^*,\ x_{n_j} - x^*,\ \rangle = \limsup_{n\to\infty} \langle f(x^*) - x^*,\ x_n - x^* \rangle. \tag{2.14}$$

Since $\{u_{n_j}\}$ is bounded, without loss of generality, we can assume that $u_{n_j} \rightharpoonup w \in C$. By (2.13), $||Tu_n - u_n|| \to 0$ and hence $||Tu_{n_j} - u_{n_j}|| \to 0$. It follows from the demiclosed principle (see Lemma 1.4) that $Tw = w$ and $Tu_{n_j} \rightharpoonup w$.

Next, we prove that $w \in F(T) \cap EP(\phi)$. It is sufficient to prove that $w \in EP(\phi)$. In fact, since $u_n = T_{r_n} x_n$, we have

$$\phi(u_n, y) + \frac{1}{r_n}\langle y - u_n,\ u_n - x_n \rangle \ge 0, \quad \forall y \in C.$$

It follows from the condition (A2) that

$$\frac{1}{r_n}\langle y-u_n,\ u_n-x_n\rangle \geq \phi(y,u_n)$$

and so

$$\langle y-u_{n_j},\ \frac{u_{n_j}-x_{n_j}}{r_{n_j}}\rangle \geq \phi(y,u_{n_j}). \tag{2.15}$$

Since $\frac{||u_{n_j}-x_{n_j}||}{r_{n_j}} \leq \frac{||u_{n_j}-x_{n_j}||}{r} \to 0$ and $u_{n_j} \rightharpoonup w$, by virtue of the condition (A4), we have

$$\liminf_{n_j\to\infty} \phi(y,u_{n_j}) \leq \lim_{n_j\to\infty}\langle y-u_{n_j},\ \frac{u_{n_j}-x_{n_j}}{r_{n_j}}\rangle = 0,$$

that is,

$$\phi(y,w) \leq 0, \quad \forall y \in C. \tag{2.16}$$

For any $t \in (0,1)$ and $y \in C$, let $y_t = ty+(1-t)w$. Then $y_t \in C$ and so we have $\phi(y_t,w) \leq 0$. It follows from the conditions (A1), (A4) and (2.16) that

$$\begin{aligned}
0 &= \phi(y_t,y_t)\\
&\leq t\phi(y_t,y)+(1-t)\phi(y_t,w)\\
&\leq t\phi(y_t,y).
\end{aligned}$$

This implies that $\phi(y_t,y) \geq 0$ for all $t \in (0,1)$. Letting $t \to 0^+$, by the condition (A3), we have

$$\phi(w,y) \geq 0, \quad \forall y \in C.$$

This shows that $w \in EP(\phi)$ and so $w \in F(T)\cap EP(\phi)$.

Since $x^* = P_{F(T)\cap EP(\phi)}f(x^*)$, $u_{n_j} \rightharpoonup w$ and $||u_n - x_n|| \to 0$ (see (2.12)), we have

$$\begin{aligned}
\limsup_{n\to\infty}\langle f(x^*)-x^*,\ x_n-x^*\rangle &= \lim_{n_j\to\infty}\langle f(x^*)-x^*,\ x_{n_j}-x^*\rangle\\
&= \lim_{n_j\to\infty}\langle f(x^*)-x^*,\ u_{n_j}-(u_{n_j}-x_{n_j})-x^*\rangle\\
&= \langle f(x^*)-x^*,\ w-x^*\rangle \leq 0.
\end{aligned} \tag{2.17}$$

The desired conclusion is proved.

(VI) Finally, we prove that $x_n \to x^*$ as $n \to \infty$.

In fact, it follows form (2.1) and Lemma 1.6 that

$$\begin{aligned}
||z_n-x^*||^2 &= ||\alpha_n(f(u_n)-x^*)+(1-\alpha_n)(Tu_n-x^*)||^2\\
&\leq (1-\alpha_n)^2||Tu_n-x^*||^2+2\alpha_n\langle f(u_n)-x^*,\ z_n-x^*\rangle\\
&\leq (1-\alpha_n)^2||u_n-x^*||^2+2\alpha_n\langle f(u_n)-f(x^*)+f(x^*)-x^*,\ z_n-x^*\rangle\\
&\leq (1-\alpha_n)^2||u_n-x^*||^2+2\alpha_n\alpha||u_n-x^*||\cdot||z_n-x^*||\\
&\quad +2\alpha_n\langle f(x^*)-x^*,\ z_n-x^*\rangle\\
&\leq (1-\alpha_n)^2||u_n-x^*||^2+\alpha_n\alpha\{||u_n-x^*||^2+||z_n-x^*||^2\}
\end{aligned}$$

$$+ 2\alpha_n\langle f(x^*) - x^*,\ z_n - x^*\rangle$$

and so, from (2.2),

$$\begin{aligned} ||z_n - x^*||^2 &\le \frac{(1-\alpha_n)^2 + \alpha_n\alpha}{1-\alpha_n\alpha}||u_n - x^*||^2 + \frac{2\alpha_n\langle f(x^*) - x^*,\ z_n - x^*\rangle}{1-\alpha_n\alpha} \\ &\le \frac{(1-\alpha_n)^2 + \alpha_n\alpha}{1-\alpha_n\alpha}||x_n - x^*||^2 + \frac{2\alpha_n\langle f(x^*) - x^*,\ z_n - x^*\rangle}{1-\alpha_n\alpha}. \end{aligned} \tag{2.18}$$

Since $\alpha_n \to 0$, for any $\varepsilon > 0$, there exists a nonnegative integer n_0 such that $1-\alpha\alpha_n > \frac{1}{2}$ for all $n \ge n_0$. Note that

$$\begin{aligned} \frac{(1-\alpha_n)^2 + \alpha_n\alpha}{1-\alpha_n\alpha} &\le \frac{1-\alpha_n+\alpha_n^2}{1-\alpha_n\alpha} \\ &\le (1-\alpha_n(1-\alpha)) + \frac{\alpha_n^2}{1-\alpha_n\alpha} \\ &\le (1-\alpha_n(1-\alpha)) + 2\alpha_n^2, \quad \forall n \ge n_0. \end{aligned} \tag{2.19}$$

Thus, substituting (2.19) into (2.18) and noting (2.3), we have

$$\begin{aligned} ||z_n - x^*||^2 \le (1-\alpha_n(1-\alpha))||x_n - x^*||^2 + 2\alpha_n^2 M \\ + \frac{2\alpha_n\langle f(x^*) - x^*,\ z_n - x^*\rangle}{1-\alpha_n\alpha}, \quad \forall n \ge n_0, \end{aligned} \tag{2.20}$$

where $M = \sup_{n\ge 0}||x_n - x^*||^2$. And so, from (2.1), (2.20) and the convexity of $x \mapsto ||x||^2$, we have

$$\begin{aligned} ||x_{n+1} - x^*||^2 &\le \beta_n||x_n - x^*||^2 + (1-\beta_n)||z_n - x^*||^2 \\ &\le \beta_n||x_n - x^*||^2 + (1-\beta_n)\{(1-\alpha_n(1-\alpha))||x_n - x^*||^2 \\ &\quad + 2\alpha_n^2 M + \frac{2\alpha_n\langle f(x^*) - x^*,\ z_n - x^*\rangle}{1-\alpha_n\alpha}\} \\ &\le (1-(1-\beta_n)\alpha_n(1-\alpha))||x_n - x^*||^2 \\ &\quad + 2\alpha_n^2 M + (1-\beta_n)\frac{2\alpha_n\langle f(x^*) - x^*,\ z_n - x^*\rangle}{1-\alpha_n\alpha} \\ &\le (1-(1-b)\alpha_n(1-\alpha))||x_n - x^*||^2 \\ &\quad + 2\alpha_n^2 M + (1-\beta_n)\frac{2\alpha_n\langle f(x^*) - x^*,\ z_n - x^*\rangle}{1-\alpha_n\alpha}, \quad \forall n \ge n_0 \end{aligned} \tag{2.21}$$

From (2.7), since we have $||x_n - z_n|| \to 0$, it follows from (2.17) that

$$\begin{aligned} &\limsup_{n\to\infty}\langle f(x^*) - x^*,\ z_n - x^*\rangle \\ &= \limsup_{n\to\infty}\langle f(x^*) - x^*,\ z_n - x_n + x_n - x^*\rangle \\ &= \limsup_{n\to\infty}\langle f(x^*) - x^*,\ x_n - x^*\rangle \le 0. \end{aligned} \tag{2.22}$$

Let

$$\gamma_n = \max\{0,\ \ \langle f(x^*) - x^*,\ z_n - x^*\rangle\}.$$

Then $\gamma_n \ge 0$.

Next, we prove that

$$\gamma_n \to 0. \tag{2.23}$$

In fact, it follows from (2.22) that for any given $\varepsilon > 0$, there exists $n_1 \geq n_0$ such that

$$\langle f(x^*) - x^*,\ z_n - x^* \rangle < \varepsilon.$$

and so we have

$$0 \leq \gamma_n < \varepsilon \quad \text{as} \quad n \to \infty.$$

By the arbitrariness of $\varepsilon > 0$, we get $\gamma_n \to 0$. By virtue of $\{\gamma_n\}$, we can rewrite (2.21) as follows:

$$\begin{aligned} &||x_{n+1} - x^*||^2 \\ &\leq (1 - (1-b)\alpha_n(1-\alpha))||x_n - x^*||^2 + 2\alpha_n^2 M + 4\alpha_n\gamma_n, \quad \forall n \geq n_0. \end{aligned} \tag{2.24}$$

Therefore, taking $a_n = ||x_n - x^*||^2$, $\lambda_n = (1-b)\alpha_n(1-\alpha)$ and $b_n = 2\alpha_n^2 M + 4\alpha_n\gamma_n$, by Lemma 1.4 and the conditions (i)–(iii), the sequence $x_n \to x^*$ as $n \to \infty$. This completes the proof.

From Theorem 2.1, we can obtain the following:

Theorem 2.2. *Let H be a real Hilbert space, C be a nonempty closed convex subset of H, $T : C \to H$ be a nonexpansive mapping with $F(T) \neq \emptyset$ and $f : H \to H$ be a contraction mapping with a contractive constant $\alpha \in (0,1)$. Let $\{\alpha_n\}$, $\{\beta_n\}$ be two sequences in $[0,1]$ satisfying the following conditions:*

(i) $\alpha_n \to 0$; $\sum_{n=0}^{\infty} \alpha_n = \infty$; $|1 - \frac{\alpha_n}{\alpha_{n+1}}| \to 0$;
(ii) *There exist $a,\ b \in (0,1)$ such that $a \leq \beta_n \leq b$ for all $n \geq 0$.*

For any $x_0 \in H$, let $\{x_n\}$ be the sequences defined by

$$\begin{cases} x_{n+1} = \beta_n x_n + (1-\beta_n) z_n, \\ z_n = \alpha_n f(u_n) + (1-\alpha_n)T(u_n), \quad \forall n \geq 0, \end{cases} \tag{2.25}$$

where $u_n = P_C x_n$ for all $n \geq 0$ and P_C is the metric projection from H onto C. Then $x_n \to x^* \in F(T)$ as $n \to \infty$, where $x^* = P_{F(T)} f(x^*)$.

Proof. Taking $\phi(x,y) = 0$ for all $x, y \in C$ and $r_n = 1$ for all $n \geq 1$ in Theorem 2.1, then we have

$$\langle y - u_n,\ u_n - x_n \rangle \geq 0, \quad \forall y \in C.$$

This implies that $u_n = P_C x_n$. Therefore, the conclusion of Theorem 2.2 can be obtained from Theorem 2.1 immediately.

Theorem 2.3. *Let H be a real Hilbert space, C be a nonempty closed convex subset of H, $\phi : C \times C \to R$ be a functional satisfying the conditions* (A1)–(A4) *such that $EP(\phi) \neq \emptyset$ and $f : H \to H$ be a contraction mapping with a contractive constant $\alpha \in (0,1)$. Let $\{\alpha_n\}$, $\{\beta_n\}$ be two sequences in $[0,1]$ and $\{r_n\} \subset (0,\infty)$ be a real sequence satisfying the following conditions:*

(i) $\alpha_n \to 0$; $\sum_{n=0}^{\infty} \alpha_n = \infty$; $|1 - \frac{\alpha_n}{\alpha_{n+1}}| \to 0$;

(ii) *There exist* $a,\ b \in (0,1)$ *such that* $a \leq \beta_n \leq b$ *for all* $n \geq 0$;

(iii) $0 < r < r_n$ *for all* $n \geq 0$ *and* $|r_n - r_{n+1}| \to 0$ *as* $n \to \infty$.

For any $x_0 \in H$, *let* $\{x_n\}$ *and* $\{u_n\}$ *be the sequences defined by*

$$\begin{cases} \phi(u_n, y) + \dfrac{1}{r_n}\langle y - u_n\ u_n - x_n\rangle \geq 0, \quad \forall y \in C, \\ x_{n+1} = \beta_n x_n + (1-\beta_n) z_n, \\ z_n = \alpha_n f(x_n) + (1-\alpha_n) u_n, \quad \forall n \geq 0. \end{cases} \tag{2.26}$$

Then $x_n \to x^* \in EP(\phi)$ *as* $n \to \infty$, *where* $x^* = P_{EP(\phi)} f(x^*)$.

Proof. Taking $T = I$ in Theorem 2.1, then $F(T) = H$ and so $P_{F(T)\cap EP(\phi)} = P_{EP(\phi)}$. Therefore, the conclusion of Theorem 2.3 can be obtained from Theorem 2.1.

ACKNOWLEDGEMENTS: The first author was supported by the Natural Science foundation of Yibin University (No 2005-Z003) and the third author was supported by the Korea Research Foundation Grant funded by the Korea Government (MOEHRD, Basic Research Promotion Fund) (KRF-2006-311-C00201).

REFERENCES

[1] A. S. Antipin and S. Flam, Equilibrium programming using proximal-like algorithms, *Math. Program*, 78(1):29–41, 1997.

[2] E. Blum and S. Oettli, From optimization and variational inequalities to equilibrium problems, *The Math. Student*, 63(1-4):123–145, 1994.

[3] S. S. Chang, Some results for asymptotically pseudo-contractive mappings and asymptotically nonexpansive mappings, *Proc. Amer. Math. Soc.*, 129:845–853, 2001.

[4] P. L. Combettes and S. A. Hirstoaga, Equilibrium programming in Hilbert spaces, *J. Nonlinear Convex Anal.*, 6:117–135, 2005.

[5] K. Goebel and W. A. Kirk, Topics in Metric Fixed Point Theory,*Cambridge Studies in Advanced Mathematics*, 28: Cambridge Univ., Press, 1990.

[6] L. S. Liu, Ishikawa and Manniterative processes with errors for nonlinear strongly accretive mappings in Banach spaces, *J. Math. Anal. Appl.*, 194:114–125, 1995.

[7] A. Moudafi, Viscosity methods for fixed points problems, *J. Math. Anal. Appl.*, 241:46–55, 2000.

[8] A. Moudafi and M. Thera, Proximal and dynamical approaches to equilibrium problems, *Lecture Notes in Econom. and Math. Systems*, Springer-Verlag, Berlin, 1990, 187–201.

[9] T. Suzuki, Strong convergence of Krasnoselskii and Mann's type sequences for one parameter nonexpansive semigroups without Bochner integrals, *J. Math. Anal. Appl.*, 305:227–229, 2005.

[10] S. Takahashi and W. Takahashi, Viscosity approximation methods for Equilibrium problems and fixed point problems in Hilbert spaces, *J. Math. Anal. Appl.*, 2006, doi:10.1016/j.jmaa.2006.08.036.

[11] H. K. Xu, Viscosity approximation methods for nonexpansive mappings, *J. Math. Anal. Appl.*, 298:279–291, 2004.