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OSCILLATION OF SECOND-ORDER NONLINEAR DYNAMIC EQUATIONS ON TIME SCALES

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ABSTRACT. By using the functions of the form $H(t,s)$ and a generalized Riccati technique, we establish new Kamenev-type and interval oscillation criteria for second-order nonlinear dynamic equations on time scales of the form

$$\big(p(t)x^{\Delta}(t)\big)^{\Delta} + f(t, x(\sigma(t))) = 0.$$

The obtained interval oscillation criteria can be applied to equations with forcing term. Two examples are included to show the significance of the results.

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1. INTRODUCTION

In this paper, we study the second order nonlinear dynamic equation

$$\big(p(t)x^{\Delta}(t)\big)^{\Delta} + f(t, x(\sigma(t))) = 0 \tag{1.1}$$

on a time scale $\mathbb{T}$, where $p \in C_{rd}(\mathbb{T}, (0,\infty))$, $f \in C(\mathbb{T}\times\mathbb{R}, \mathbb{R})$.

For convenience, we recall some concepts related to time scales. More details can be found in [1, 2].

Definition 1.1. A time scale is an arbitrary nonempty closed subset of the set $\mathbb{R}$ of real numbers with the topology and ordering inherited from $\mathbb{R}$. Let $\mathbb{T}$ be a time scale, for $t \in \mathbb{T}$ the forward jump operator is defined by $\sigma(t) := \inf\{s \in \mathbb{T} : s > t\}$, the backward jump operator by $\rho(t) := \sup\{s \in \mathbb{T} : s < t\}$, and the graininess function by $\mu(t) := \sigma(t) - t$, where $\inf\emptyset := \sup\mathbb{T}$ and $\sup\emptyset := \inf\mathbb{T}$. If $\sigma(t) > t$, t is said to be right-scattered; otherwise, it is right-dense. If $\rho(t) < t$, t is said to be left-scattered; otherwise, it is left-dense. The set $\mathbb{T}^{\kappa}$ is defined as follows: If $\mathbb{T}$ has a left-scattered maximum m, then $\mathbb{T}^{\kappa} = \mathbb{T} - \{m\}$; otherwise, $\mathbb{T}^{\kappa} = \mathbb{T}$.

Definition 1.2. For a function $f : \mathbb{T} \to \mathbb{R}$ and $t \in \mathbb{T}^{\kappa}$, we define the delta-derivative $f^{\Delta}(t)$ of $f(t)$ to be the number (provided it exists) with the property that given any

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$\varepsilon > 0$, there is a neighborhood U of t (i.e., $U = (t-\delta, t+\delta) \cap \mathbb{T}$ for some δ) such that

$$\left|[f(\sigma(t)) - f(s)] - f^{\Delta}(t)[\sigma(t) - s]\right| \leq \varepsilon|\sigma(t) - s| \quad \text{for all } s \in U.$$

We say that f is delta-differentiable (or in short: differentiable) on $\mathbb{T}^{\kappa}$ provided $f^{\Delta}(t)$ exists for all $t \in \mathbb{T}^{\kappa}$.

It is easily seen that if f is continuous at $t \in \mathbb{T}$ and t is right-scattered, then f is differentiable at t with

$$f^{\Delta}(t) = \frac{f(\sigma(t)) - f(t)}{\mu(t)}.$$

Moreover, if t is right-dense then f is differential at t iff the limit

$$\lim_{s \to t} \frac{f(t) - f(s)}{t - s}$$

exists as a finite number. In this case

$$f^{\Delta}(t) = \lim_{s \to t} \frac{f(t) - f(s)}{t - s}.$$

In addition, if $f^{\Delta} \geq 0$, then f is nondecreasing. A useful formula is

$$f^{\sigma}(t) = f(t) + \mu(t)f^{\Delta}(t), \text{ where } f^{\sigma}(t) := f(\sigma(t)). \tag{1.2}$$

We will make use of the following product and quotient rules for the derivative of the product fg and the quotient f/g (where $gg^{\sigma} \neq 0$) of two differentiable functions f and g:

$$(fg)^{\Delta} = f^{\Delta}g + f^{\sigma}g^{\Delta} = fg^{\Delta} + f^{\Delta}g^{\sigma}, \tag{1.3}$$

$$\left(\frac{f}{g}\right)^{\Delta} = \frac{f^{\Delta}g - fg^{\Delta}}{gg^{\sigma}}. \tag{1.4}$$

Definition 1.3. Let $f : \mathbb{T} \to \mathbb{R}$ be a function, f is called right-dense continuous (rd-continuous) if it is continuous at right-dense points in $\mathbb{T}$ and its left-sided limits exist (finite) at left-dense points in $\mathbb{T}$. A function $F : \mathbb{T} \to \mathbb{R}$ is called an antiderivative of f provided $F^{\Delta}(t) = f(t)$ holds for all $t \in \mathbb{T}^{k}$. By the antiderivative, the Cauchy integral of f is defined as $\int_a^b f(s)\Delta s = F(b) - F(a)$, and $\int_a^{\infty} f(s)\Delta s = \lim_{t\to\infty} \int_a^t f(s)\Delta s$.

Let $C_{rd}(\mathbb{T}, \mathbb{R})$ denote the set of all rd-continuous functions mapping $\mathbb{T}$ to $\mathbb{R}$. It is shown in [2] that every rd-continuous function has an antiderivative.

An integration by parts formula is

$$\int_a^b f(t)g^{\Delta}(t)\Delta t = [f(t)g(t)]\Big|_a^b - \int_a^b f^{\Delta}(t)g^{\sigma}(t)\Delta t, \tag{1.5}$$

and another useful formula is

$$\int_{\rho(b)}^b f(t)\Delta t = f(\rho(b))(b - \rho(b)). \tag{1.6}$$

Without loss of generality, we assume throughout that $0 \in \mathbb{T}$ and $\sup \mathbb{T} = \infty$ since we are interested in extending oscillation criteria for the corresponding differential and difference equations, namely

$$(p(t)x'(t))' + f(t, x(t)) = 0$$

with $\mathbb{T} = \mathbb{R}_+ := [0, \infty)$, and

$$\Delta(p_n \Delta x_n) + f(n, x_{n+1}) = 0$$

with $\mathbb{T} = \mathbb{N}_0$, the set of nonnegative integers.

A solution $x(t)$ of Eq. (1.1) is said to have a generalized zero at $t^* \in \mathbb{T}$ if $x(t^*)x(\sigma(t^*)) \leq 0$, and it is said to be nonoscillatory on $\mathbb{T}$ if there exists $t_0 \in \mathbb{T}$ such that $x(t)x(\sigma(t)) > 0$ for $t > t_0$. Otherwise, it is oscillatory.

The theory of time scales, which has recently received a lot of attention, was introduced by Stefan Hilger in his PhD thesis [8] in 1988 (supervised by Bernd Aulbach) in order to unify continuous and discrete analysis. In recent years, there has been much research activity concerning the oscillation and nonoscillation of solutions of dynamic equation on time scales, e.g., see [2–7, 9–11, 15, 16] and the references therein. In Dosly and Hilger [5], the authors considered the second-order dynamic equation

$$(p(t)x^{\Delta}(t))^{\Delta} + q(t)x(\sigma(t)) = 0, \tag{1.7}$$

and gave necessary and sufficient conditions for the oscillation of all solutions on unbounded time scales. In Del Medico and Kong [4], the authors employed the following Riccati transformation

$$u(t) = \frac{p(t)x^{\Delta}(t)}{x(t)} \tag{1.8}$$

and gave the conditions for oscillation of Eq. (1.7) on a measure chain. And in Yang [14], the author considered the oscillation of solutions of the differential equation

$$(p(t)x'(t))' + q(t)f(x(t)) = g(t). \tag{1.9}$$

In this paper, we shall use a generalized Riccati transformation which is more general than (1.8) and was used in [12, 13] for nonlinear differential equations, and establish new Kamenev-type as well as interval oscillation criteria for Eq. (1.1) in Sections 2 and 3, respectively. The obtained interval oscillation criteria can be applied to such equations with forcing term as (1.9). Finally in Section 4, two examples are included to show the significance of the results.

For simplicity, throughout this paper, we denote $(a, b) \bigcap \mathbb{T}$ by (a, b), where $a, b \in \mathbb{R}$, and $[a, b], [a, b), (a, b]$ are denoted similarly.

2. KAMENEV-TYPE CRITERIA

In this section we establish Kamenev-type criteria for oscillation of Eq. (1.1). We assume throughout this section that:

(C1) $f \in C(\mathbb{T} \times \mathbb{R}, \mathbb{R})$, $uf(t,u) > 0$ for $u \neq 0$, and there exists a function $q \in C_{rd}(\mathbb{T}, \mathbb{R})$ such that $uf(t,u) \geq q(t)u^2$.

Our approach to oscillation problems of Eq. (1.1) is based largely on the application of the Riccati transformation. Now, we give the first lemma.

Lemma 2.1. *Assume x(t) is a solution of Eq.* (1.1) *satisfies* $x(t) > 0$ *for* $t \in [t_0, \infty)$ *with* $t_0 \in \mathbb{T}$. *For* $t \in [t_0, \infty)$, *define*

$$u(t) = \frac{A(t)p(t)x^{\Delta}(t)}{x(t)} + B(t), \tag{2.1}$$

where $A \in C^1_{rd}(\mathbb{T}, \mathbb{R}_+ \backslash \{0\})$, $B \in C^1_{rd}(\mathbb{T}, \mathbb{R})$, *and* $A(t)p(t) \pm \mu(t)B(t) > 0$ *for* $t \in [t_0, \infty)$, *then u(t) satisfies*

$$\mu(t)u(t) - \mu(t)B(t) + A(t)p(t) > 0 \tag{2.2}$$

and

(2.3)

$$u^{\Delta}(t) + \Phi_1(t) + \frac{A(t)u^2(t) - [(A^{\sigma}(t) + A(t))B(t) + A^{\Delta}(t)A(t)p(t)]u(t) + A^{\sigma}(t)B^2(t)}{A(t)[\mu(t)u(t) - \mu(t)B(t) + A(t)p(t)]} \leq 0,$$

where $\Phi_1(t) = A^{\sigma}(t)\left(q(t) - \left(\frac{B(t)}{A(t)}\right)^{\Delta}\right)$, $A^{\sigma}(t) = A(\sigma(t))$, $t \in [t_0, \infty)$.

Proof. First,

$$\begin{aligned} \mu u - \mu B + Ap &= \mu \frac{Apx^{\Delta}}{x} + \mu B - \mu B + Ap \\ &= Ap\frac{x^{\sigma} - x}{x} + Ap = Ap\frac{x^{\sigma}}{x} > 0, \end{aligned}$$

i.e., (2.2) holds. Then differentiating (2.1) and using (1.1), it follows that

$$\begin{aligned} u^{\Delta} &= A^{\Delta}\left(\frac{px^{\Delta}}{x}\right) + A^{\sigma}\left(\frac{px^{\Delta}}{x}\right)^{\Delta} + B^{\Delta} \\ &= \frac{A^{\Delta}}{A}(u - B) + A^{\sigma}\frac{(px^{\Delta})^{\Delta}x - p(x^{\Delta})^2}{xx^{\sigma}} + B^{\Delta} \\ &= \frac{A^{\Delta}}{A}u + B^{\Delta} - \frac{A^{\Delta}}{A}B - \frac{A^{\sigma}f}{x^{\sigma}} - A^{\sigma}\frac{p(x^{\Delta})^2}{xx^{\sigma}} \\ &\leq \frac{A^{\Delta}}{A}u + A^{\sigma}\left(\frac{B}{A}\right)^{\Delta} - A^{\sigma}q - A^{\sigma}\frac{p(x^{\Delta})^2}{xx^{\sigma}} \\ &= \frac{A^{\Delta}}{A}u - \Phi_1 - \frac{A^{\sigma}}{A}\frac{(u-B)^2}{\mu u - \mu B + Ap} \end{aligned}$$

$$= \frac{1}{A(\mu u-\mu B+Ap)}\left(-Au^2+[(A^\sigma+A)B+A^\Delta Ap]u-A^\sigma B^2\right)-\Phi_1,$$

i.e., (2.3) holds. Lemma 2.1 is proved. □

Let $D_0=\{s\in\mathbb{T}:s\geq 0\}$; $D=\{(t,s)\in\mathbb{T}^2:t\geq s\geq 0\}$. For any function $f(t,s)$: $\mathbb{T}^2\to\mathbb{R}$, denote by f_1^Δ and f_2^Δ the partial derivatives of f with respect to t and s, respectively. For $E\subset\mathbb{R}$, denote by $L_{loc}(E)$ the space of functions which are integrable on any compact subset of E. Define

$$\begin{aligned}
&(\mathscr{A},\mathscr{B})=\{(A,B):A(s)\in C^1_{rd}(D_0,\mathbb{R}_+\backslash\{0\}),B(s)\in C^1_{rd}(D_0,\mathbb{R}),\\
&\qquad A(s)p(s)\pm\mu(s)B(s)>0,s\in D_0\};\\
&\mathscr{H}^*=\{H(t,s)\in C^1(D,\mathbb{R}_+):H(t,t)=0,H(t,s)>0,H_2^\Delta(t,s)\leq 0,t>s\geq 0\};\\
&\mathscr{H}_*=\{H(t,s)\in C^1(D,\mathbb{R}_+):H(t,t)=0,H(t,s)>0,H_1^\Delta(t,s)\geq 0,t>s\geq 0\};\\
&\mathscr{H}=\mathscr{H}^*\bigcap\mathscr{H}_*.
\end{aligned}$$

These function classes will be used throughout this paper. Now, we are in a position to give our first result.

Theorem 2.1. *Assume that there exist* $(A,B)\in(\mathscr{A},\mathscr{B})$ *and* $H\in\mathscr{H}^*$ *such that* $M_1(t,\cdot)\in L([0,\rho(t)])$ *and for any* $t_0\in\mathbb{T}$,

$$\limsup_{t\to\infty}\frac{1}{H(t,t_0)}\Bigg[\int_{t_0}^t H(t,\sigma(s))\Phi_1(s)\Delta s-\int_{t_0}^{\rho(t)}M_1(t,s)\Delta s \\ +H_2^\Delta(t,\rho(t))(A(\rho(t))p(\rho(t))-\mu(\rho(t))B(\rho(t)))\chi_{t-\rho(t)}\Bigg]=\infty, \tag{2.4}$$

where Φ_1 *is defined as before,* $\chi:\mathbb{R}_+\to\mathbb{R}$ *with* $\chi_t=\begin{cases}0, & t=0,\\ 1, & t\in(0,\infty),\end{cases}$ *and*

$$M_1(t,s)\triangleq\frac{(H(t,s)A(s)B(s)+H(t,\sigma(s))A^\sigma(s)B(s)+A(s)p(s)(H(t,s)A(s))^{\Delta_s})^2}{4H(t,\sigma(s))A(s)\min\{A(s)(A(s)p(s)-\mu(s)B(s)),A^\sigma(s)(A(s)p(s)+\mu(s)B(s))\}}.$$

Then Eq. (1.1) *is oscillatory.*

Proof. Assume Eq. (1.1) is not oscillatory. Without loss of generality we may assume there exists $t_0\in[0,\infty)$ such that $x(t)>0$ for $t\in[t_0,\infty)$. Let $u(t)$ be defined by (2.1). Then by Lemma 2.1, (2.2) and (2.3) hold.

For simplicity in the following, we let $H_\sigma=H(t,\sigma(s)),H=H(t,s),H_2^\Delta=H_2^\Delta(t,s)$, and omit the arguments in the integrals. For $s\in\mathbb{T}$,

$$H_\sigma-H=H_2^\Delta\mu. \tag{2.5}$$

Multiplying (2.3), where t is replaced by s, by H_σ, and integrating it with respect to s from t_0 to t with $t\in\mathbb{T}$ and $t\geq\sigma(t_0)$, we obtain

$$\int_{t_0}^t H_\sigma\Phi_1\Delta s\leq-\int_{t_0}^t\left(H_\sigma u^\Delta+H_\sigma\frac{Au^2-[(A^\sigma+A)B+A^\Delta Ap]u+A^\sigma B^2}{A(\mu u-\mu B+Ap)}\right)\Delta s.$$

Noting that $H(t,t)=0$, by the integration by parts formula we have

$$
\begin{aligned}
(2.6)\quad &\int_{t_0}^{t} H_\sigma \Phi_1 \Delta s \\
&\le H(t,t_0)u(t_0)+\int_{t_0}^{t}\left(H_2^\Delta u - H_\sigma \frac{Au^2-[(A^\sigma+A)B+A^\Delta Ap]u+A^\sigma B^2}{A(\mu u-\mu B+Ap)}\right)\Delta s \\
&\le H(t,t_0)u(t_0)+\int_{t_0}^{t}\left(H_2^\Delta u - H_\sigma \frac{Au^2-[(A^\sigma+A)B+A^\Delta Ap]u}{A(\mu u-\mu B+Ap)}\right)\Delta s \\
&= H(t,t_0)u(t_0)+\int_{\rho(t)}^{t} H_2^\Delta u\Delta s \\
&\quad +\int_{t_0}^{\rho(t)}\left(H_2^\Delta u - H_\sigma \frac{Au^2-[(A^\sigma+A)B+A^\Delta Ap]u}{A(\mu u-\mu B+Ap)}\right)\Delta s.
\end{aligned}
$$

Since $H_2^\Delta \le 0$ on D, from (2.2) we see that for $t \ge \sigma(t_0)$,

$$
\begin{aligned}
(2.7)\quad \int_{\rho(t)}^{t} H_2^\Delta u\Delta s &= H_2^\Delta(t,\rho(t))u(\rho(t))(t-\rho(t)) \\
&= H_2^\Delta(t,\rho(t))u(\rho(t))\mu(\rho(t))\chi_{t-\rho(t)} \\
&\le -H_2^\Delta(t,\rho(t))(A(\rho(t))p(\rho(t))-\mu(\rho(t))B(\rho(t)))\chi_{t-\rho(t)}.
\end{aligned}
$$

For $t\ge\sigma(t_0)$, $s\in[t_0,\rho(t))$, and $u(s)\le 0$,

$$
\begin{aligned}
&H_2^\Delta u - H_\sigma \frac{Au^2-[(A^\sigma+A)B+A^\Delta Ap]u}{A(\mu u-\mu B+Ap)} \\
&= \frac{1}{A(\mu u-\mu B+Ap)}\left(-AHu^2+[HAB+H_\sigma A^\sigma B+Ap(HA)^\Delta]u\right) \\
&= -\frac{Hu^2}{\mu u-\mu B+Ap}+\frac{HAB+H_\sigma A^\sigma B+Ap(HA)^\Delta}{A(Ap-\mu B)}u \\
&\quad -\frac{HAB+H_\sigma A^\sigma B+Ap(HA)^\Delta}{A(Ap-\mu B)}\frac{\mu u^2}{\mu u-\mu B+Ap} \\
&= -\frac{H_\sigma A^\sigma(Ap+\mu B)u^2}{A(Ap-\mu B)(\mu u-\mu B+Ap)}+\frac{HAB+H_\sigma A^\sigma B+Ap(HA)^\Delta}{A(Ap-\mu B)}u \\
&\le -\frac{H_\sigma A^\sigma(Ap+\mu B)u^2}{A(Ap-\mu B)^2}+\frac{HAB+H_\sigma A^\sigma B+Ap(HA)^\Delta}{A(Ap-\mu B)}u \\
&= -\frac{H_\sigma A^\sigma(Ap+\mu B)}{A(Ap-\mu B)^2}\left[u-\frac{(Ap-\mu B)(HAB+H_\sigma A^\sigma B+Ap(HA)^\Delta)}{2H_\sigma A^\sigma(Ap+\mu B)}\right]^2 \\
&\quad +\frac{(HAB+H_\sigma A^\sigma B+Ap(HA)^\Delta)^2}{4H_\sigma A^\sigma(Ap+\mu B)A} \\
&\le \frac{(HAB+H_\sigma A^\sigma B+Ap(HA)^\Delta)^2}{4H_\sigma A\min\{A(Ap-\mu B),A^\sigma(Ap+\mu B)\}}=M_1.
\end{aligned}
$$

Therefore, for all $t\ge\sigma(t_0)$, $s\in[t_0,\rho(t))$, we have

$$
(2.8)\qquad H_2^\Delta u - H_\sigma \frac{Au^2-[(A^\sigma+A)B+A^\Delta Ap]u}{A(\mu u-\mu B+Ap)} \le M_1.
$$

Then from (2.6), (2.7) and (2.8) we obtain that for $t \in \mathbb{T}$ and $t > \sigma(t_0)$,

$$\begin{aligned} \int_{t_0}^{t} H_\sigma \Phi_1 \Delta s \le H(t,t_0)u(t_0) + \int_{t_0}^{\rho(t)} M_1 \Delta s \\ -H_2^\Delta(t,\rho(t))(A(\rho(t))p(\rho(t)) - \mu(\rho(t))B(\rho(t)))\chi_{t-\rho(t)}. \end{aligned} \tag{2.9}$$

Hence

$$\frac{1}{H(t,t_0)}\Bigg[\int_{t_0}^{t} H(t,\sigma(s))\Phi_1(s)\Delta s - \int_{t_0}^{\rho(t)} M_1(t,s)\Delta s$$
$$+H_2^\Delta(t,\rho(t))(A(\rho(t))p(\rho(t)) - \mu(\rho(t))B(\rho(t)))\chi_{t-\rho(t)}\Bigg] \le u(t_0) < \infty,$$

which contradicts (2.4) and completes the proof. □

In the sequel we define

$$\mathbb{T}_1 = \{s \in \mathbb{T} : s \text{ is right-dense}\} \text{ and } \mathbb{T}_2 = \{s \in \mathbb{T} : s \text{ is right-scattered}\}. \tag{2.10}$$

Note that this result does not apply to the case where all points in $\mathbb{T}$ are right-dense.

Theorem 2.2. *Let $(A,B) \in (\mathscr{A},\mathscr{B}), H \in \mathscr{H}_*, M_2(\cdot,t) \in L_{loc}([\sigma(t),\infty))$, and $\mathbb{T}_1, \mathbb{T}_2$ be defined by (2.10). Then Eq. (1.1) is oscillatory provided there exists $\{t_n\}_{n=1}^\infty \subset \mathbb{T}_2$, $t_n \to \infty$, such that for any $t_0 \in \mathbb{T}$, one of the following holds:*

(i) $\lim_{n\to\infty} N(t_n,t_0) = \infty$ *and*

$$\limsup_{n\to\infty} \frac{1}{N(t_n,t_0)}\left[\int_{t_0}^{t_n} H(\sigma(s),t_0)\Phi_1(s)\Delta s - \int_{\sigma(t_0)}^{t_n} M_2(s,t_0)\Delta s\right] = \infty; \tag{2.11}$$

(ii) $\limsup_{n\to\infty} N(t_n,t_0) = \infty$ *and*

$$\lim_{n\to\infty} \frac{1}{N(t_n,t_0)}\left[\int_{t_0}^{t_n} H(\sigma(s),t_0)\Phi_1(s)\Delta s - \int_{\sigma(t_0)}^{t_n} M_2(s,t_0)\Delta s\right] = \infty; \tag{2.12}$$

(iii) $\limsup_{n\to\infty} N(t_n,t_0) < \infty$ *and*

$$\limsup_{n\to\infty} \left[\int_{t_0}^{t_n} H(\sigma(s),t_0)\Phi_1(s)\Delta s - \int_{\sigma(t_0)}^{t_n} M_2(s,t_0)\Delta s\right] = \infty, \tag{2.13}$$

where $N(t,s) = H(t,s)(A(t)p(t) - \mu(t)B(t))/\mu(t)$, Φ_1 is defined as before, and

$$M_2(s,t) \triangleq \frac{(H(s,t)A(s)B(s) + H(\sigma(s),t)A^\sigma(s)B(s) + A(s)p(s)(H(s,t)A(s))^{\Delta_s})^2}{4H(s,t)A(s)\min\{A(s)(A(s)p(s) - \mu(s)B(s)), A^\sigma(s)(A(s)p(s) + \mu(s)B(s))\}}.$$

Proof. Assume that Eq. (1.1) is not oscillatory. Without loss of generality we may assume there exists $t_0 \in [0,\infty)$ such that $x(t) > 0$ for $t \in [t_0,\infty)$. Let $u(t)$ be defined by (2.1). Then by Lemma 2.1, (2.2) and (2.3) hold. For simplicity in the following, we let $H'_\sigma = H(\sigma(s),t_0), H' = H(s,t_0), H_1^\Delta = H_1^\Delta(s,t_0)$, and omit the arguments in the integrals. Multiplying (2.3), where t is replaced by s, by H'_σ, and integrating it with respect to s from t_0 to t and then using the integration by parts formula we have that

$$\begin{aligned}
(2.14)\int_{t_0}^{t} & H'_\sigma \Phi_1 \Delta s \\
&\leq -\int_{t_0}^{t}\left(H'_\sigma u^\Delta + H'_\sigma \frac{Au^2-[(A^\sigma+A)B+A^\Delta Ap]u+A^\sigma B^2}{A(\mu u-\mu B+Ap)}\right)\Delta s \\
&= -H(t,t_0)u(t)+\int_{t_0}^{t}\left(H_1^\Delta u - H'_\sigma \frac{Au^2-[(A^\sigma+A)B+A^\Delta Ap]u+A^\sigma B^2}{A(\mu u-\mu B+Ap)}\right)\Delta s \\
&\leq -H(t,t_0)u(t)+\left(\int_{t_0}^{\sigma(t_0)}+\int_{\sigma(t_0)}^{t}\right)\left(H_1^\Delta u - H'_\sigma \frac{Au^2-[(A^\sigma+A)B+A^\Delta Ap]u}{A(\mu u-\mu B+Ap)}\right)\Delta s.
\end{aligned}$$

For $s \in [t_0, t)$,

$$H'_\sigma - H_1^\Delta \mu = H'. \tag{2.15}$$

Hence

$$\begin{aligned}
(2.16)\qquad & \int_{t_0}^{\sigma(t_0)}\left(H_1^\Delta u - H'_\sigma \frac{Au^2-[(A^\sigma+A)B+A^\Delta Ap]u}{A(\mu u-\mu B+Ap)}\right)\Delta s \\
&= \mu(t_0)\left(H_1^\Delta u - H'_\sigma \frac{Au^2-[(A^\sigma+A)B+A^\Delta Ap]u}{A(\mu u-\mu B+Ap)}\right)\Big|_{s=t_0} \\
&= \frac{[-AH'u^2+(H'AB+H'_\sigma A^\sigma B+Ap(H'A)^\Delta)u]\mu}{A(\mu u-\mu B+Ap)}\Big|_{s=t_0} \\
&= \frac{(H'_\sigma A^\sigma B+Ap(H'A)^\Delta)u\mu}{A(\mu u-\mu B+Ap)}\Big|_{s=t_0} \\
&\leq \left(p(H'A)^\Delta + \frac{H'_\sigma A^\sigma B}{A}\right)\Big|_{s=t_0}\chi_{\mu(t_0)} \\
&= \left[p(t_0)H_1^\Delta(t_0,t_0)A^\sigma(t_0)+\frac{H(\sigma(t_0),t_0)A^\sigma(t_0)B(t_0)}{A(t_0)}\right]\chi_{\mu(t_0)}.
\end{aligned}$$

Furthermore, for $t \geq t_0$, $s \in [\sigma(t_0), t)$, and $u(s) \leq 0$,

$$\begin{aligned}
H_1^\Delta u & - H'_\sigma \frac{Au^2-[(A^\sigma+A)B+A^\Delta Ap]u}{A(\mu u-\mu B+Ap)} \\
&= \frac{1}{A(\mu u-\mu B+Ap)}\left(-AH'u^2+[H'AB+H'_\sigma A^\sigma B+Ap(H'A)^\Delta]u\right) \\
&= -\frac{H'u^2}{\mu u-\mu B+Ap}+\frac{H'AB+H'_\sigma A^\sigma B+Ap(H'A)^\Delta}{A(Ap-\mu B)}u \\
&\quad -\frac{H'AB+H'_\sigma A^\sigma B+Ap(H'A)^\Delta}{A(Ap-\mu B)}\frac{\mu u^2}{\mu u-\mu B+Ap} \\
&= -\frac{H'_\sigma A^\sigma(Ap+\mu B)u^2}{A(Ap-\mu B)(\mu u-\mu B+Ap)}+\frac{H'AB+H'_\sigma A^\sigma B+Ap(H'A)^\Delta}{A(Ap-\mu B)}u \\
&\leq -\frac{H'_\sigma A^\sigma(Ap+\mu B)u^2}{A(Ap-\mu B)^2}+\frac{H'AB+H'_\sigma A^\sigma B+Ap(H'A)^\Delta}{A(Ap-\mu B)}u
\end{aligned}$$

$$
\begin{aligned}
&= -\frac{H'_\sigma A^\sigma (Ap+\mu B)}{A(Ap-\mu B)^2}\left[u - \frac{(Ap-\mu B)(H'AB + H'_\sigma A^\sigma B + Ap(H'A)^\Delta)}{2H'_\sigma A^\sigma (Ap+\mu B)}\right]^2 \\
&\quad + \frac{(H'AB + H'_\sigma A^\sigma B + Ap(H'A)^\Delta)^2}{4H'_\sigma A^\sigma (Ap+\mu B)A} \\
&\le \frac{(H'AB + H'_\sigma A^\sigma B + Ap(H'A)^\Delta)^2}{4H'A\min\{A(Ap-\mu B), A^\sigma(Ap+\mu B)\}} = M_2.
\end{aligned}
$$

For $t \ge t_0$, $s \in [\sigma(t_0), t)$, and $u(s) > 0$,

$$
\begin{aligned}
&H_1^\Delta u - H'_\sigma \frac{Au^2 - [(A^\sigma + A)B + A^\Delta Ap]u}{A(\mu u - \mu B + Ap)} \\
&\quad = -\frac{H'_\sigma A^\sigma (Ap+\mu B)u^2}{A(Ap-\mu B)(\mu u - \mu B + Ap)} + \frac{H'AB + H'_\sigma A^\sigma B + Ap(H'A)^\Delta}{A(Ap-\mu B)} u \\
&\quad = -\frac{H'}{\mu u - \mu B + Ap}\left[u - \frac{H'AB + H'_\sigma A^\sigma B + Ap(H'A)^\Delta}{2H'A}\right]^2 \\
&\qquad + \frac{(H'AB + H'_\sigma A^\sigma B + Ap(H'A)^\Delta)^2}{4H'A^2(\mu u - \mu B + Ap)} \\
&\quad \le \frac{(H'AB + H'_\sigma A^\sigma B + Ap(H'A)^\Delta)^2}{4H'A\min\{A(Ap-\mu B), A^\sigma(Ap+\mu B)\}} = M_2.
\end{aligned}
$$

Hence, for all $t \ge t_0$, $s \in [\sigma(t_0), t)$, we have

$$
H_1^\Delta u - H'_\sigma \frac{Au^2 - [(A^\sigma + A)B + A^\Delta Ap]u}{A(\mu u - \mu B + Ap)} \le M_2. \tag{2.17}
$$

From (2.14), (2.16) and (2.17), we have

$$
\begin{aligned}
\int_{t_0}^t H'_\sigma \Phi_1 \Delta s &\le -H(t,t_0)u(t) + \int_{\sigma(t_0)}^t M_2(s,t_0)\Delta s \\
&\quad + \left[p(t_0)H_1^\Delta(t_0,t_0)A^\sigma(t_0) + \frac{H(\sigma(t_0),t_0)A^\sigma(t_0)B(t_0)}{A(t_0)}\right]\chi_{\mu(t_0)}.
\end{aligned} \tag{2.18}
$$

For $t \in \mathbb{T}_2$, (2.2) implies

$$
-H(t,t_0)u(t) \le H(t,t_0)\frac{A(t)p(t) - \mu(t)B(t)}{\mu(t)} = N(t,t_0).
$$

Hence

$$
\begin{aligned}
\int_{t_0}^t H(\sigma(s),t_0)\Phi_1(s)\Delta s &\le N(t,t_0) + \int_{\sigma(t_0)}^t M_2(s,t_0)\Delta s \\
&\quad + \left[p(t_0)H_1^\Delta(t_0,t_0)A^\sigma(t_0) + \frac{H(\sigma(t_0),t_0)A^\sigma(t_0)B(t_0)}{A(t_0)}\right]\chi_{\mu(t_0)}.
\end{aligned} \tag{2.19}
$$

Assume condition (i) holds. Let $t = t_n$ in (2.19). Then we obtain

$$
\begin{aligned}
&\frac{1}{N(t_n,t_0)}\left[\int_{t_0}^{t_n} H(\sigma(s),t_0)\Phi_1(s)\Delta s - \int_{\sigma(t_0)}^{t_n} M_2(s,t_0)\Delta s\right] \\
&\quad \le 1 + \frac{\chi_{\mu(t_0)}}{N(t_n,t_0)}\left[p(t_0)H_1^\Delta(t_0,t_0)A^\sigma(t_0) + \frac{H(\sigma(t_0),t_0)A^\sigma(t_0)B(t_0)}{A(t_0)}\right].
\end{aligned}
$$

Taking the lim sup as $n \to \infty$ on both sides, we have

$$\limsup_{n\to\infty} \frac{1}{N(t_n,t_0)} \left[\int_{t_0}^{t_n} H(\sigma(s),t_0)\Phi_1(s)\Delta s - \int_{\sigma(t_0)}^{t_n} M_2(s,t_0)\Delta s\right] < \infty,$$

which contradicts (2.11).

The conclusions with conditions (ii) and (iii) can be proved similarly. We omit the details. □

When $(A,B) = (1,0)$, Theorems 2.1 and 2.2 above reduce to the following two corollaries, respectively.

Corollary 2.1. *Assume that there exists $H \in \mathscr{H}^*$ such that for any $t_0 \in \mathbb{T}$,*

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \Bigg[\int_{t_0}^{t} H(t,\sigma(s))q(s)\Delta s - \int_{t_0}^{\rho(t)} \frac{\left(H_2^{\Delta}(t,s)\right)^2}{4H(t,\sigma(s))} p(s)\Delta s + H_2^{\Delta}(t,\rho(t))p(\rho(t))\chi_{t-\rho(t)} \Bigg] = \infty, \tag{2.20}$$

where $\chi : \mathbb{R}_+ \to \mathbb{R}$ satisfies that

$$\chi_t = \begin{cases} 0, & t = 0, \\ 1, & t \in (0,\infty). \end{cases}$$

Then Eq. (1.1) *is oscillatory.*

Corollary 2.2. *Let $H \in \mathscr{H}_*$ and let $\mathbb{T}_1, \mathbb{T}_2$ be defined by* (2.10)*. Then Eq.* (1.1) *is oscillatory provided there exists $\{t_n\}_{n=1}^{\infty} \subset \mathbb{T}_2$, $t_n \to \infty$, such that for any $t_0 \in \mathbb{T}$, one of the following holds:*

(i) $\lim_{n\to\infty} H(t_n,t_0)p(t_n)/\mu(t_n) = \infty$ *and*

$$\limsup_{n\to\infty} \frac{\mu(t_n)}{H(t_n,t_0)p(t_n)} \left[\int_{t_0}^{t_n} H(\sigma(s),t_0)q(s)\Delta s - \int_{\sigma(t_0)}^{t_n} \frac{\left(H_1^{\Delta}(s,t_0)\right)^2}{4H(s,t_0)} p(s)\Delta s\right] = \infty;$$

(ii) $\limsup_{n\to\infty} H(t_n,t_0)p(t_n)/\mu(t_n) = \infty$ *and*

$$\lim_{n\to\infty} \frac{\mu(t_n)}{H(t_n,t_0)p(t_n)} \left[\int_{t_0}^{t_n} H(\sigma(s),t_0)q(s)\Delta s - \int_{\sigma(t_0)}^{t_n} \frac{\left(H_1^{\Delta}(s,t_0)\right)^2}{4H(s,t_0)} p(s)\Delta s\right] = \infty;$$

(iii) $\limsup_{n\to\infty} H(t_n,t_0)p(t_n)/\mu(t_n) < \infty$ *and*

$$\limsup_{n\to\infty} \left[\int_{t_0}^{t_n} H(\sigma(s),t_0)q(s)\Delta s - \int_{\sigma(t_0)}^{t_n} \frac{\left(H_1^{\Delta}(s,t_0)\right)^2}{4H(s,t_0)} p(s)\Delta s\right] = \infty.$$

3. INTERVAL CRITERIA WITH FORCING TERM

In this section, we establish interval criteria for oscillation of Eq. (1.1). First, we give two lemmas.

Lemma 3.1. *Assume that there exist* $c_1 < b_1 < c_2 < b_2$, $\gamma \geq 1$, *functions* $q, g \in C_{rd}(\mathbb{T}, \mathbb{R})$ *such that* $q(t) \geq 0$ *and* $q(t) \not\equiv 0$ *for* $t \in [c_1, b_1] \bigcup [c_2, b_2]$,

$$g(t) \begin{cases} \leq 0, & t \in [c_1, b_1], \\ \geq 0, & t \in [c_2, b_2], \end{cases}$$

and

$$\frac{f(t,y)}{y} \geq q(t)|y|^{\gamma-1} - \frac{g(t)}{y} \tag{3.1}$$

for all $t \in [c_1, b_1] \bigcup [c_2, b_2]$ *and* $y \neq 0$. *If* $x(t)$ *is a solution of Eq.* (1.1) *such that* $x(t) > 0$ *on* $[c_1, \sigma(b_1)]$ *(or* $x(t) < 0$ *on* $[c_2, \sigma(b_2)]$*), define* $u(t)$ *as in* (2.1) *on* $[c_i, b_i], i = 1, 2$. *Then for any* $(A, B) \in (\mathscr{A}, \mathscr{B}), H \in \mathscr{H}^*$, *and* $M_1(t, \cdot) \in L([0, \rho(t)])$, *we have*

$$\begin{aligned} \int_{c_i}^{b_i} H(b_i, \sigma(s))\Phi_2(s)\Delta s \leq H(b_i, c_i)u(c_i) + \int_{c_i}^{\rho(b_i)} M_1(b_i, s)\Delta s \\ -H_2^{\Delta}(b_i, \rho(b_i))(A(\rho(b_i))p(\rho(b_i)) - \mu(\rho(b_i))B(\rho(b_i)))\chi_{b_i - \rho(b_i)}, \quad i = 1, 2, \end{aligned} \tag{3.2}$$

where $\Phi_2(s) = A^{\sigma}(s)\left(\gamma(\gamma-1)^{(1-\gamma)/\gamma}[q(s)]^{\frac{1}{\gamma}}|g(s)|^{1-\frac{1}{\gamma}} - \left(\frac{B(s)}{A(s)}\right)^{\Delta}\right)$ *for* $\gamma > 1$, *and* $\Phi_2(s) = \Phi_1(s)$ *for* $\gamma = 1$. M_1 *is defined as before.*

Proof. Suppose that $x(t)$ is a solution of Eq. (1.1) such that $x(t) > 0$ on $[c_1, \sigma(b_1)]$.

(i) $\gamma > 1$. Note that $g(t) \leq 0$ on $[c_1, b_1]$, differentiating $u(t)$ we have

$$\begin{aligned} u^{\Delta} &= \frac{A^{\Delta}}{A}(u - B) - \frac{A^{\sigma}}{x^{\sigma}}f(t, x^{\sigma}) - \frac{A^{\sigma}p(x^{\Delta})^2}{xx^{\sigma}} + B^{\Delta} \\ &\leq \frac{A^{\Delta}}{A}u + A^{\sigma}\left(\frac{B}{A}\right)^{\Delta} - A^{\sigma}\left[\frac{|g|}{x^{\sigma}} + q(x^{\sigma})^{\gamma-1}\right] - \frac{A^{\sigma}p(x^{\Delta})^2}{xx^{\sigma}}. \end{aligned}$$

From Hölder inequality we have

$$\begin{aligned} &\frac{|g|}{x^{\sigma}} + q(x^{\sigma})^{\gamma-1} \\ &= \frac{\gamma-1}{\gamma}\left\{\left(\frac{\gamma}{\gamma-1}\right)^{(\gamma-1)/\gamma}\left[\frac{|g|}{x^{\sigma}}\right]^{1-\frac{1}{\gamma}}\right\}^{\frac{\gamma}{\gamma-1}} + \frac{1}{\gamma}\left\{\gamma^{1/\gamma}[q]^{\frac{1}{\gamma}}(x^{\sigma})^{1-\frac{1}{\gamma}}\right\}^{\gamma} \\ &\geq \gamma(\gamma-1)^{(1-\gamma)/\gamma}[q]^{\frac{1}{\gamma}}|g|^{1-\frac{1}{\gamma}}. \end{aligned}$$

Therefore,

$$\begin{aligned} u^{\Delta} &\leq \frac{A^{\Delta}}{A}u + A^{\sigma}\left(\frac{B}{A}\right)^{\Delta} - A^{\sigma}[q]^{\frac{1}{\gamma}}|g|^{1-\frac{1}{\gamma}} - \frac{A^{\sigma}p(x^{\Delta})^2}{xx^{\sigma}} \\ &= \frac{A^{\Delta}}{A}u - \frac{A^{\sigma}(u-B)^2}{A(\mu u - \mu B + Ap)} - \Phi_2. \end{aligned}$$

That is, for $\gamma > 1$,

$$u^\Delta(t) + \Phi_2(t) + \frac{A(t)u^2(t) - [(A^\sigma(t) + A(t))B(t) + A^\Delta(t)A(t)p(t)]u(t) + A^\sigma(t)B^2(t)}{A(t)(\mu(t)u(t) - \mu(t)B(t) + A(t)p(t))} \leq 0. \tag{3.3}$$

(ii) For $\gamma = 1$, we have

$$\begin{aligned} u^\Delta &\leq \frac{A^\Delta}{A}u + A^\sigma\left(\frac{B}{A}\right)^\Delta - A^\sigma\left[\frac{|g|}{x^\sigma} + q\right] - \frac{A^\sigma p(x^\Delta)^2}{xx^\sigma} \\ &\leq \frac{A^\Delta}{A}u - \frac{A^\sigma p(x^\Delta)^2}{xx^\sigma} + A^\sigma\left[\left(\frac{B}{A}\right)^\Delta - q\right]. \end{aligned}$$

Then (3.3) also holds.

From (i) and (ii) above, we see that (3.3) holds for $\gamma \geq 1$. Following the proof of Theorem 2.1, from (3.3) we have

$$\begin{aligned} &\int_{t_0}^t H_\sigma \Phi_2 \Delta s \leq H(t,t_0)u(t_0) + \int_{t_0}^{\rho(t)} M_1 \Delta s \\ &-H_2^\Delta(t,\rho(t))(A(\rho(t))p(\rho(t)) - \mu(\rho(t))B(\rho(t)))\chi_{t-\rho(t)}, \end{aligned}$$

then letting $t_0 = c_1, t = b_1$, we get

$$\begin{aligned} &\int_{c_1}^{b_1} H(b_1,\sigma(s))\Phi_2(s)\Delta s \leq H(b_1,c_1)u(c_1) + \int_{c_1}^{\rho(b_1)} M_1(b_1,s)\Delta s \\ &-H_2^\Delta(b_1,\rho(b_1))(A(\rho(b_1))p(\rho(b_1)) - \mu(\rho(b_1))B(\rho(b_1)))\chi_{b_1-\rho(b_1)}. \end{aligned} \tag{3.4}$$

If $x(t) < 0$ on $[c_2, \sigma(b_2)]$, then we see that $g(t) \geq 0$ on $[c_2, b_2]$ and

$$\begin{aligned} u^\Delta &= \frac{A^\Delta}{A}(u - B) - \frac{A^\sigma}{x^\sigma}f(t,x^\sigma) - \frac{A^\sigma p(x^\Delta)^2}{xx^\sigma} + B^\Delta \\ &\leq \frac{A^\Delta}{A}u + A^\sigma\left(\frac{B}{A}\right)^\Delta - A^\sigma\left[\frac{g}{|x^\sigma|} + q|x^\sigma|^{\gamma-1}\right] - \frac{A^\sigma p(x^\Delta)^2}{xx^\sigma}. \end{aligned}$$

Following the steps in case (1), we have

$$\begin{aligned} &\int_{c_2}^{b_2} H(b_2,\sigma(s))\Phi_2(s)\Delta s \leq H(b_2,c_2)u(c_2) + \int_{c_2}^{\rho(b_2)} M_1(b_2,s)\Delta s \\ &-H_2^\Delta(b_2,\rho(b_2))(A(\rho(b_2))p(\rho(b_2)) - \mu(\rho(b_2))B(\rho(b_2)))\chi_{b_2-\rho(b_2)}. \end{aligned} \tag{3.5}$$

From (3.4) and (3.5), we see that (3.2) holds for $\gamma \geq 1$. The proof is complete. □

Lemma 3.2. *Assume that there exist $a_1 < c_1 < a_2 < c_2$, $\gamma \geq 1$, functions $q, g \in C_{rd}(\mathbb{T},\mathbb{R})$ such that $q(t) \geq 0$ and $q(t) \not\equiv 0$ for $t \in [a_1,c_1]\bigcup[a_2,c_2]$ and*

$$g(t)\begin{cases} \leq 0, & t \in [a_1,c_1], \\ \geq 0, & t \in [a_2,c_2], \end{cases}$$

and (3.1) *holds for all* $t \in [a_1, c_1] \bigcup [a_2, c_2]$ *and* $y \neq 0$. *If* $x(t)$ *is a solution of Eq.* (1.1) *such that* $x(t) > 0$ *on* $[a_1, \sigma(c_1)]$ *(or* $x(t) < 0$ *on* $[a_2, \sigma(c_2)]$*), define* $u(t)$ *as in* (2.1) *on* $[a_i, c_i], i = 1, 2$. *Then for any* $(A, B) \in (\mathscr{A}, \mathscr{B}), H \in \mathscr{H}_*, M_2(\cdot, t) \in L_{loc}([\sigma(t), \infty))$, *we have*

$$\int_{a_i}^{c_i} H(\sigma(s), a_i)\Phi_2(s)\Delta s \leq -H(c_i, a_i)u(c_i) + \int_{\sigma(a_i)}^{c_i} M_2(s, a_i)\Delta s + \left[p(a_i)H_1^{\Delta}(a_i, a_i)A^{\sigma}(a_i) + \frac{H(\sigma(a_i), a_i)A^{\sigma}(a_i)B(a_i)}{A(a_i)}\right]\chi_{\mu(a_i)}, \tag{3.6}$$

where Φ_2 *and* M_2 *are defined as before.*

Proof. By (3.3), following the proof of Theorem 2.2 we have

$$\int_{t_0}^{t} H(\sigma(s), t_0)\Phi_1(s)\Delta s \leq -H(t, t_0)u(t) + \int_{\sigma(t_0)}^{t} M_2(s, t_0)\Delta s + \left[p(t_0)H_1^{\Delta}(t_0, t_0)A^{\sigma}(t_0) + \frac{H(\sigma(t_0), t_0)A^{\sigma}(t_0)B(t_0)}{A(t_0)}\right]\chi_{\mu(t_0)}.$$

and then letting $t_0 = a_i, t = c_i$ we get (3.6). ∎

Theorem 3.1. *Assume that the following two conditions hold:*

(C2) *For any* $T \geq t_0$, *there exist* $T \leq a_1 < b_1 \leq a_2 < b_2$, $\gamma \geq 1$, *functions* $q, g \in C_{rd}(\mathbb{T}, \mathbb{R})$ *such that* $q(t) \geq 0$ *and* $q(t) \not\equiv 0$ *for* $t \in [a_1, b_1] \bigcup [a_2, b_2]$,

$$g(t) \begin{cases} \leq 0, & t \in [a_1, b_1], \\ \geq 0, & t \in [a_2, b_2], \end{cases}$$

and (3.1) *holds for all* $t \in [a_1, b_1] \bigcup [a_2, b_2]$ *and* $y \neq 0$;

(C3) *There exist* $c_i \in (a_i, b_i), i = 1, 2$, $(A, B) \in (\mathscr{A}, \mathscr{B}), H \in \mathscr{H}$, $M_1(t, \cdot) \in L([0, \rho(t)]), M_2(\cdot, t) \in L_{loc}([\sigma(t), \infty))$ *such that*

$$\begin{aligned} &\frac{1}{H(c_i, a_i)}\left[\int_{a_i}^{c_i} H(\sigma(s), a_i)\Phi_2(s)\Delta s - \int_{\sigma(a_i)}^{c_i} M_2(s, a_i)\Delta s\right] \\ &+\frac{1}{H(b_i, c_i)}\left[\int_{c_i}^{b_i} H(b_i, \sigma(s))\Phi_2(s)\Delta s - \int_{c_i}^{\rho(b_i)} M_1(b_i, s)\Delta s\right] \\ &> \frac{H_2^{\Delta}(b_i, \rho(b_i))}{H(b_i, c_i)}\left(A(\rho(b_i))p(\rho(b_i)) - \mu(\rho(b_i))B(\rho(b_i))\right)\chi_{b_i - \rho(b_i)} \\ &-\frac{1}{H(c_i, a_i)}\left[p(a_i)H_1^{\Delta}(a_i, a_i)A^{\sigma}(a_i) + \frac{H(\sigma(a_i), a_i)A^{\sigma}(a_i)B(a_i)}{A(a_i)}\right]\chi_{\mu(a_i)}, i = 1, 2. \end{aligned} \tag{3.7}$$

Then Eq. (1.1) *is oscillatory.*

Proof. Suppose that $x(t)$ is a nonoscillatory solution of Eq. (1.1) which is eventually positive, say $x(t) > 0$ when $t \geq T \geq t_0$ for some T depending on the solution $x(t)$. From the assumption (C2), we can choose $a_1, b_1 \geq T$ so that $g(t) \leq 0$ on the interval $I = [a_1, b_1]$ with $a_1 < b_1$. From Lemmas 3.1 and 3.2 we see that (3.2) and (3.6) hold for $i = 1$. By dividing (3.2) and (3.6) by $H(b_1, c_1)$ and $H(c_1, a_1)$, respectively, and then adding them, we have

$$\begin{aligned}
&\frac{1}{H(c_i,a_i)}\left[\int_{a_i}^{c_i} H(\sigma(s),a_i)\Phi_2(s)\Delta s - \int_{\sigma(a_i)}^{c_i} M_2(s,a_i)\Delta s\right]\\
&+\frac{1}{H(b_i,c_i)}\left[\int_{c_i}^{b_i} H(b_i,\sigma(s))\Phi_2(s)\Delta s - \int_{c_i}^{\rho(b_i)} M_1(b_i,s)\Delta s\right]\\
&\le \frac{H_2^{\Delta}(b_i,\rho(b_i))}{H(b_i,c_i)}\left(A(\rho(b_i))p(\rho(b_i)) - \mu(\rho(b_i))B(\rho(b_i))\right)\chi_{b_i-\rho(b_i)}\\
&-\frac{1}{H(c_i,a_i)}\left[p(a_i)H_1^{\Delta}(a_i,a_i)A^{\sigma}(a_i) + \frac{H(\sigma(a_i),a_i)A^{\sigma}(a_i)B(a_i)}{A(a_i)}\right]\chi_{\mu(a_i)},
\end{aligned}$$

which contradicts the assumption (3.7) with $i = 1$.

When $x(t)$ is eventually negative, we choose $a_2, b_2 \ge T$ so that $g(t) \ge 0$ on $[a_2, b_2]$ to reach a similar contradiction. Hence every solution of Eq. (1.1) has at least one generalized zero in (a_1, b_1) or (a_2, b_2).

Pick a sequence $\{T_j\} \subset \mathbb{T}$ such that $T_j \ge T$ and $T_j \to \infty$ as $j \to \infty$. By assumption, for each $j \in \mathbb{N}$ there exists $a_j, b_j, c_j \in \mathbb{R}$ such that $T_j \le a_j < c_j < b_j$ and (3.7) holds, where a, b, c are replaced by a_j, b_j, c_j, respectively. Hence every solution $x(t)$ has at least one generalized zero $t_j \in (a_j, b_j)$. Noting that $t_j > a_j \ge T_j, j \in \mathbb{N}$, we see that every solution has arbitrarily large generalized zeros. Thus, Eq. (1.1) is oscillatory. □

Corollary 3.1. *Assume that* (C3) *holds and*

(C4) *There exist* $c_i \in (a_i, b_i), i = 1, 2$, $(A, B) \in (\mathscr{A}, \mathscr{B}), H \in \mathscr{H}$, $M_1(t,\cdot) \in L([0, \rho(t)])$, $M_2(\cdot, t) \in L_{loc}([\sigma(t), \infty))$ *such that*

$$\begin{aligned}
(3.8)\qquad &\int_{a_i}^{c_i} H(\sigma(s),a_i)\Phi_2(s)\Delta s - \int_{\sigma(a_i)}^{c_i} M_2(s,a_i)\Delta s\\
&+\left[p(a_i)H_1^{\Delta}(a_i,a_i)A^{\sigma}(a_i) + \frac{H(\sigma(a_i),a_i)A^{\sigma}(a_i)B(a_i)}{A(a_i)}\right]\chi_{\mu(a_i)} > 0,
\end{aligned}$$

and

$$\begin{aligned}
(3.9)\qquad &\int_{c_i}^{b_i} H(b_i,\sigma(s))\Phi_2(s)\Delta s - \int_{c_i}^{\rho(b_i)} M_1(b_i,s)\Delta s\\
&-H_2^{\Delta}(b_i,\rho(b_i))(A(\rho(b_i))p(\rho(b_i)) - \mu(\rho(b_i))B(\rho(b_i)))\chi_{b_i-\rho(b_i)} > 0,
\end{aligned}$$

where Φ_2, M_1, M_2 *are defined as before. Then Eq.* (1.1) *is oscillatory.*

Proof. By (3.8) and (3.9) we get (3.7). Therefore, Eq. (1.1) is oscillatory by Theorem 3.1. □

When $q \in C_{rd}(\mathbb{T}, \mathbb{R}_+), g(t) \equiv 0, \gamma = 1$, we have the following corollary.

Corollary 3.2. *Assume that* (C1) *holds and there exist* $(A,B)\in(\mathscr{A},\mathscr{B}), H\in\mathscr{H}$*,* $M_1(t,\cdot)\in L([0,\rho(t)]), M_2(\cdot,t)\in L_{loc}([\sigma(t),\infty))$ *such that for any* $l\in\mathbb{T}$

$$\limsup_{t\to\infty}\Big\{\int_l^t H(\sigma(s),l)\Phi_1(s)\Delta s-\int_{\sigma(l)}^t M_2(s,l)\Delta s \\ -\Big[p(l)H_1^{\Delta}(l,l)A^{\sigma}(l)+\frac{H(\sigma(l),l)A^{\sigma}(l)B(l)}{A(l)}\Big]\chi_{\mu(l)}\Big\}>0 \tag{3.10}$$

and

$$\limsup_{t\to\infty}\Big[\int_l^t H(t,\sigma(s))\Phi_1(s)\Delta s-\int_l^{\rho(t)} M_1(t,s)\Delta s \\ +H_2^{\Delta}(t,\rho(t))(A(\rho(t))p(\rho(t))-\mu(\rho(t))B(\rho(t)))\chi_{t-\rho(t)}\Big]>0. \tag{3.11}$$

Then Eq. (1.1) *is oscillatory.*

Proof. When (C1) holds, it follows that (C2) holds for $g(t)\equiv 0,\gamma=1$. Now $\Phi_1(s)=\Phi_2(s)$. For any $T\ge t_0$, let $a_1=T$. In (3.10) we choose $l=a_1$. Then there exists $c_1>a_1$ such that

$$\int_{a_1}^{c_1} H(\sigma(s),a_1)\Phi_1(s)\Delta s-\int_{\sigma(a_1)}^{c_1} M_2(s,a_1)\Delta s \\ -\Big[p(a_1)H_1^{\Delta}(a_1,a_1)A^{\sigma}(a_1)+\frac{H(\sigma(a_1),a_1)A^{\sigma}(a_1)B(a_1)}{A(a_1)}\Big]\chi_{\mu(a_1)}>0. \tag{3.12}$$

In (3.11) we choose $l=c_1$. Then there exists $b_1>c_1$ such that

$$\int_{c_1}^{b_1} H(b_1,\sigma(s))\Phi_1(s)\Delta s-\int_{c_1}^{\rho(b_1)} M_1(t,s)\Delta s \\ +H_2^{\Delta}(b_1,\rho(b_1))(A(\rho(b_1))p(\rho(b_1))-\mu(\rho(b_1))B(\rho(b_1)))\chi_{b_1-\rho(b_1)}>0. \tag{3.13}$$

Combining (3.12) and (3.13) we obtain (3.7) with $i=1$.

Next, choose $l=a_2=b_1$. Then there exists $c_2>a_2$ such that

$$\int_{a_2}^{c_2} H(\sigma(s),a_2)\Phi_1(s)\Delta s-\int_{\sigma(a_2)}^{c_2} M_2(s,a_2)\Delta s \\ -\Big[p(a_2)H_1^{\Delta}(a_2,a_2)A^{\sigma}(a_2)+\frac{H(\sigma(a_2),a_2)A^{\sigma}(a_2)B(a_2)}{A(a_2)}\Big]\chi_{\mu(a_2)}>0. \tag{3.14}$$

In (3.11) we choose $l=c_2$. Then there exists $b_2>c_2$ such that

$$\int_{c_2}^{b_2} H(b_2,\sigma(s))\Phi_1(s)\Delta s-\int_{c_2}^{\rho(b_2)} M_1(t,s)\Delta s \\ +H_2^{\Delta}(b_2,\rho(b_2))(A(\rho(b_2))p(\rho(b_2))-\mu(\rho(b_2))B(\rho(b_2)))\chi_{b_2-\rho(b_2)}>0. \tag{3.15}$$

Combining (3.14) and (3.15) we obtain (3.7) with $i=2$. The conclusion thus follows from Theorem 3.1. ∎

4. EXAMPLES

In this section, we will show the application of our oscillation criteria in two examples. We first give an example to show Corollary 2.1.

Example 4.1. Consider the equation

$$\Big(p(t)x^{\Delta}(t)\Big)^{\Delta} + t^2\left[x^2(\sigma(t))+1\right]x(\sigma(t)) = 0, \tag{4.1}$$

where $p \in C_{rd}(\mathbb{T},(0,1])$, $t \in \mathbb{T}$.

We choose $q(s) = s^2$ and $H(t,s) = (t-s)^2$.

(1) $\mathbb{T} = \mathbb{R}_+$,

$$\begin{aligned}
&\limsup_{t\to\infty} \frac{1}{H(t,t_0)}\Bigg[\int_{t_0}^{t} H(t,\sigma(s))q(s)\Delta s - \int_{t_0}^{\rho(t)} \frac{\big(H_2^{\Delta}(t,s)\big)^2}{4H(t,\sigma(s))}p(s)\Delta s \\
&\qquad + H_2^{\Delta}(t,\rho(t))p(\rho(t))\chi_{t-\rho(t)}\Bigg] \\
&\geq \limsup_{t\to\infty} \frac{1}{(t-t_0)^2}\int_{t_0}^{t}\left[(t-s)^2s^2-1\right]ds \\
&= \limsup_{t\to\infty} \frac{1}{(t-t_0)^2}\left[\frac{t^5}{30} - \frac{t_0^3t^2}{3} - \left(1+\frac{t_0^4}{2}\right)t + t_0 - \frac{t_0^5}{5}\right] \\
&\geq \limsup_{t\to\infty}\left[\frac{t^3}{30} - \frac{t_0^3}{3} - \left(1+\frac{t_0^4}{2}\right)\frac{1}{t} + \left(t_0-\frac{t_0^5}{5}\right)\frac{1}{t^2}\right] \\
&= \infty.
\end{aligned}$$

That is, (2.20) holds. By Corollary 2.1 we get that Eq. (4.1) is oscillatory.

(2) $\mathbb{T} = \mathbb{N}_0$,

$$\begin{aligned}
&\limsup_{t\to\infty} \frac{1}{H(t,t_0)}\Bigg[\int_{t_0}^{t} H(t,\sigma(s))q(s)\Delta s - \int_{t_0}^{\rho(t)} \frac{\big(H_2^{\Delta}(t,s)\big)^2}{4H(t,\sigma(s))}p(s)\Delta s \\
&\qquad + H_2^{\Delta}(t,\rho(t))p(\rho(t))\chi_{t-\rho(t)}\Bigg] \\
&\geq \limsup_{n\to\infty} \frac{1}{(n-l)^2}\left[\sum_{k=l}^{n-1}(n-k-1)^2k^2 - \sum_{k=l}^{n-2}\frac{(2(n-k)-1)^2}{4(n-k-1)^2} - 1\right] \\
&\geq \limsup_{n\to\infty} \sum_{k=l}^{n-2}\left(\frac{k^2}{n^2} - \frac{1}{n} - \frac{1}{4n^2(n-k-1)^2}\right) \\
&= \infty.
\end{aligned}$$

That is, (2.20) holds. By Corollary 2.1 we get that Eq. (4.1) is oscillatory.

(3) $\mathbb{T} = \{2^n, n \in \mathbb{N}_0\}$,

$$\begin{aligned}
&\limsup_{t\to\infty} \frac{1}{H(t,t_0)}\Bigg[\int_{t_0}^{t} H(t,\sigma(s))q(s)\Delta s - \int_{t_0}^{\rho(t)} \frac{\big(H_2^{\Delta}(t,s)\big)^2}{4H(t,\sigma(s))}p(s)\Delta s \\
&\qquad + H_2^{\Delta}(t,\rho(t))p(\rho(t))\chi_{t-\rho(t)}\Bigg]
\end{aligned}$$

$$\geq \limsup_{t\to\infty} \frac{1}{(t-t_0)^2}\left[\int_{t_0}^{t}(t-2s)^2 s^2\Delta s - \int_{t_0}^{\frac{t}{2}} \frac{(((t-s)^2)^{\Delta_s})^2}{4(t-2s)^2}\Delta s - \frac{t}{2}\right]$$
$$= \limsup_{n\to\infty} \frac{1}{(2^n-2^l)^2}\left[\sum_{k=l}^{n-1}(2^n-2^{k+1})2^{2k}2^k - \sum_{k=l}^{n-2} 2^k\left(1+\frac{2^k}{2^{n+1}-2^{k+2}}\right)^2\right]$$
$$= \limsup_{n\to\infty} \frac{1}{(2^n-2^l)^2}\sum_{k=l}^{n-2}\left[2^{n+3k}-2^{4k+1}-2^k-\frac{2^{k+1}}{2^{n+1}-2^{k+2}}-\frac{2^{3k}}{(2^{n+1}-2^{k+2})^2}\right]$$
$$\geq \limsup_{n\to\infty} \frac{1}{(2^n-2^l)^2}\sum_{k=l}^{n-2}\left[2^{n+3k}-2^{4k+1}-2^k-\frac{2^{k+1}}{2^n}-\frac{2^{3k}}{2^{2n}}\right]$$
$$\geq \limsup_{n\to\infty} \frac{1}{(2^n-2^l)^2}\sum_{k=l}^{n-2}\left[2^{n+3k}-2^{4k+1}-2^k-2^{k+1}-2^k\right]$$
$$= \limsup_{n\to\infty} \frac{1}{(2^n-2^l)^2}\sum_{k=l}^{n-2}2^{3k}\left[2^n-2^{k+1}-2^{-2k+2}\right]$$
$$\geq \limsup_{n\to\infty} \frac{\sum_{k=l}^{n-2}2^{3k}}{(2^n-2^l)^2} = \limsup_{n\to\infty}\frac{2^{n+3l-6}}{7} = \infty.$$

That is, (2.20) holds. By Corollary 2.1 we get that Eq. (4.1) is oscillatory.

The second example illustrates Corollary 3.1.

Example 4.2. Consider the equation

$$(4.2)\qquad \left(p(t)x^{\Delta}(t)\right)^{\Delta} + q(t)x(\sigma(t))\left[2+\cos x(\sigma(t)) + \frac{\sin^2 x(\sigma(t))}{1+e^{\cos x(\sigma(t))}}\right] + \sin\frac{\pi}{12}t = 0,$$

where $p \in C_{rd}(\mathbb{T},(0,1])$, $t\in\mathbb{T}$, and

$$q(t) = \begin{cases} 1, & t\in[24n,24n+6]\bigcup[24n+12,24n+18], \\ \cos\frac{\pi}{3}(t-6), & t\in(24n+6,24n+12)\bigcup(24n+18,24n+24),\ n\in\mathbb{N}_0. \end{cases}$$

For any $T>0$, there exists $n\in\mathbb{N}_0$ such that $24n>T$. Let $\gamma=1$, $a_1=24n$, $b_1=24n+6$, $c_1=24n+3$, $a_2=24n+12$, $b_2=24n+18$, $c_2=24n+15$, $(A,B)=(1,0), H(t,s)=(t-s)^2$.

(1) $\mathbb{T}=\mathbb{R}_+$,

$$\int_{a_1}^{c_1} H(\sigma(s),a_1)\Phi_2(s)\Delta s - \int_{\sigma(a_1)}^{c_1} M_2(s,a_1)\Delta s$$
$$+\left[p(a_1)H_1^{\Delta}(a_1,a_1)A^{\sigma}(a_1)+\frac{H(\sigma(a_1),a_1)A^{\sigma}(a_1)B(a_1)}{A(a_1)}\right]\chi_{\mu(a_1)}$$
$$\geq \int_{24n}^{24n+3}\left[(s-24n)^2-1\right]ds = 6>0.$$

$$\int_{a_2}^{c_2} H(\sigma(s),a_2)\Phi_2(s)\Delta s - \int_{\sigma(a_2)}^{c_2} M_2(s,a_2)\Delta s$$
$$+\left[p(a_2)H_1^{\Delta}(a_2,a_2)A^{\sigma}(a_2)+\frac{H(\sigma(a_2),a_2)A^{\sigma}(a_2)B(a_2)}{A(a_2)}\right]\chi_{\mu(a_2)}$$

$$\geq \int_{24n+12}^{24n+15} [(s-24n-12)^2-1]\,ds = 6 > 0.$$

$$\int_{c_1}^{b_1} H(b_1,\sigma(s))\Phi_2(s)\Delta s - \int_{c_1}^{\rho(b_1)} M_1(b_1,s)\Delta s$$
$$-H_2^{\Delta}(b_1,\rho(b_1))(A(\rho(b_1))p(\rho(b_1)) - \mu(\rho(b_1))B(\rho(b_1)))\chi_{b_1-\rho(b_1)}$$
$$\geq \int_{24n+3}^{24n+6} [(24n+6-s)^2-1]\,ds = 6 > 0.$$

$$\int_{c_2}^{b_2} H(b_2,\sigma(s))\Phi_2(s)\Delta s - \int_{c_2}^{\rho(b_2)} M_1(b_2,s)\Delta s$$
$$-H_2^{\Delta}(b_2,\rho(b_2))(A(\rho(b_2))p(\rho(b_2)) - \mu(\rho(b_2))B(\rho(b_2)))\chi_{b_2-\rho(b_2)}$$
$$\geq \int_{24n+15}^{24n+18} [(24n+18-s)^2-1]\,ds = 6 > 0.$$

Hence (3.8) and (3.9) hold, by Corollary 3.1 we have that Eq. (4.2) is oscillatory.

(2) $\mathbb{T} = \mathbb{N}_0$,

$$\int_{a_1}^{c_1} H(\sigma(s),a_1)\Phi_2(s)\Delta s - \int_{\sigma(a_1)}^{c_1} M_2(s,a_1)\Delta s$$
$$+\left[p(a_1)H_1^{\Delta}(a_1,a_1)A^{\sigma}(a_1) + \frac{H(\sigma(a_1),a_1)A^{\sigma}(a_1)B(a_1)}{A(a_1)}\right]\chi_{\mu(a_1)}$$
$$\geq \sum_{k=24n}^{24n+2}(k+1-24n)^2 - \sum_{k=24n+1}^{24n+2}\frac{[(k+1-24n)^2-(k-24n)^2]^2}{4(k-24n)^2} + p(24n)$$
$$\geq \frac{163}{16} > 0.$$

$$\int_{a_2}^{c_2} H(\sigma(s),a_2)\Phi_2(s)\Delta s - \int_{\sigma(a_2)}^{c_2} M_2(s,a_2)\Delta s$$
$$+\left[p(a_2)H_1^{\Delta}(a_2,a_2)A^{\sigma}(a_2) + \frac{H(\sigma(a_2),a_2)A^{\sigma}(a_2)B(a_2)}{A(a_2)}\right]\chi_{\mu(a_2)}$$
$$\geq \sum_{k=24n+12}^{24n+14}(k+1-24n-12)^2$$
$$-\sum_{k=24n+13}^{24n+14}\frac{[(k+1-24n-12)^2-(k-24n-12)^2]^2}{4(k-24n-12)^2} + p(24n+12)$$
$$\geq \frac{163}{16} > 0.$$

$$\int_{c_1}^{b_1} H(b_1,\sigma(s))\Phi_2(s)\Delta s - \int_{c_1}^{\rho(b_1)} M_1(b_1,s)\Delta s$$
$$-H_2^{\Delta}(b_1,\rho(b_1))(A(\rho(b_1))p(\rho(b_1)) - \mu(\rho(b_1))B(\rho(b_1)))\chi_{b_1-\rho(b_1)}$$

$$\geq \sum_{k=24n+3}^{24n+5} (24n+6-k-1)^2$$
$$- \sum_{k=24n+3}^{24n+4} \frac{[(24n+6-k-1)^2 - (24n+6-k)^2]^2}{4(24n+6-k-1)^2} - 1$$
$$= \frac{3}{16} > 0.$$

$$\int_{c_2}^{b_2} H(b_2, \sigma(s))\Phi_2(s)\Delta s - \int_{c_2}^{\rho(b_2)} M_1(b_2, s)\Delta s$$
$$-H_2^{\Delta}(b_2, \rho(b_2))(A(\rho(b_2))p(\rho(b_2)) - \mu(\rho(b_2))B(\rho(b_2)))\chi_{b_2-\rho(b_2)}$$
$$\geq \sum_{k=24n+15}^{24n+17} (24n+18-k-1)^2$$
$$- \sum_{k=24n+15}^{24n+16} \frac{[(24n+18-k-1)^2 - (24n+18-k)^2]^2}{4(24n+18-k-1)^2} - 1$$
$$= \frac{3}{16} > 0.$$

Hence (3.8) and (3.9) hold, by Corollary 3.1 we have that Eq. (4.2) is oscillatory.

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