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EXISTENCE OF SOLUTIONS FOR BOUNDARY VALUE PROBLEMS ON INFINITE INTERVALS

İSMAİL YASLAN

Department of Mathematics, Pamukkale University, Denizli 20070, Turkey

ABSTRACT. In this paper, the Schauder fixed point theorem is used to investigate the existence of solutions of the boundary value problems (BVP) for second-order nonlinear differential equations on infinite intervals.

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1. INTRODUCTION

We consider the second order nonlinear differential equation

$$-[p(x)y']' + q(x)y = f(x, y, Ty), \quad 0 \le x < \infty, \tag{1.1}$$

where $y = y(x)$ is a desired solution, and

$$(Ty)(x) = \int_0^\infty K(x, s)y(s)ds,$$

$K \in C[\mathbf{R}^+ \times \mathbf{R}^+, \mathbf{R}^+]$ and $\mathbf{R}^+$ is the set of all nonnegative numbers.

For convenience, let us list some conditions.

(C1) The coefficients $p(x)$ and $q(x)$ are real-valued measurable functions on $[0, \infty)$ such that

$$\int_0^b \frac{dx}{|p(x)|} < \infty, \qquad \int_0^b |q(x)|dx < \infty$$

for each finite positive number b. Moreover, the functions $p(x)$ and $q(x)$ are such that all solutions of the second order linear differential equation

$$-[p(x)y']' + q(x)y = 0, \quad 0 \le x < \infty, \tag{1.2}$$

belong to $L^2(0, \infty)$, that is Weyl limit circle case holds for the differential expression $Ly = -[p(x)y']' + q(x)y$ (see [2],[14]).

(C2) The function $f(x, y, z)$ is real-valued and continuous in $(x, y, z) \in [0, \infty) \times \mathbf{R} \times \mathbf{R}$, and

$$|f(x, y, z)| \le a|y| + b|z| + g(x) \tag{1.3}$$

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for all $(x, y, z) \in [0, \infty) \times \mathbf{R} \times \mathbf{R}$, where $g(x) \geq 0$, $g \in L^2(0, \infty)$ and, a and b are positive constants.

(C3) $\int_0^\infty \int_0^\infty |K(x, s)|^2 \, dxds < \infty$.

Let D be the linear manifold of all elements $y \in L^2(0, \infty)$ such that Ly is defined and $Ly \in L^2(0, \infty)$. Let $y^{[1]}(x) = p(x)y'(x)$ denote the quasi-derivative of $y(x)$. For two given differentiable functions $y = y(x)$ and $z = z(x)$ we define the Wronskian of y and z by

$$W_x(y, z) = y(x)z^{[1]}(x) - y^{[1]}(x)z(x), \; x \in [0, \infty).$$

Using the Green's formula

$$\int_0^b [(Ly)z - y(Lz)](x)dx = W_b(y, z) - W_0(y, z), \tag{1.4}$$

for all $y, z \in D$, we have the limit

$$W_\infty(y, z) = \lim_{b \to \infty} W_b(y, z)$$

exists and is finite.

Let $u = u(x)$ and $v = v(x)$ be solutions of (1.2) satisfying the initial conditions

$$u(0) = 0, \; u^{[1]}(0) = 1; \; v(0) = -1, \; v^{[1]}(0) = 0 \tag{1.5}$$

From these conditions and the constancy of the Wronskian it follows that $W_x(u, v) = 1$. Hence, u and v are linearly independent and they form a fundamental system of solutions of (1.2). It follows from the condition (C1) that $u, v \in L^2(0, \infty)$; moreover $u, v \in D$. Consequently for each $y \in D$, the values $W_\infty(y, u)$ and $W_\infty(y, v)$ exist and are finite.

We deal with the equation (1.1) whose boundary conditions are

$$\alpha y(0) + \beta y^{[1]}(0) = d_1, \quad \gamma W_\infty(y, u) + \delta W_\infty(y, v) = d_2 \tag{1.6}$$

where $\alpha, \beta, \gamma,$ and δ are given real numbers satisfying the condition

(C4) $\omega := \alpha\delta - \beta\gamma \neq 0$,

and d_1, d_2 are arbitrary given real numbers.

We define the set $\mathbb{D} = \{y \in L^2(0, \infty) : y'$ is continuous and py' is differentiable on $[0, \infty)$ and $(py')'$ is continuous on $[0, \infty)$ and $\alpha y(0) + \beta y^{[1]}(0) = d_1$, $\gamma W_\infty(y, u) + \delta W_\infty(y, v) = d_2\}$.

A function y is called a solution of the problem (1.1), (1.6) if $y \in \mathbb{D}$ and the equation $-[p(x)y']' + q(x)y = f(x, y, Ty)$ holds for all $x \in [0, \infty)$.

In the recent paper [8], the existence and uniqueness of solutions of the BVP

$$\begin{aligned} &-[p(x)y']' + q(x)y = f(x, y), \quad 0 \leq x < \infty, \\ &\alpha y(0) + \beta y^{[1]}(0) = d_1, \quad \gamma W_\infty(y, u) + \delta W_\infty(y, v) = d_2, \end{aligned}$$

has been discussed. These boundary conditions at infinity are used in [3−8], and [10]. The problem to find solutions of a second order dynamics with assigned conditions at infinity also arises in other contexts and recent contributions dealing with different situations. We refer, in particular to [1,11-13] and the references there contained.

By using Green's formula (1.4) and the initial conditions (1.5), the formulas

$$\begin{aligned} W_\infty(y,u) &= y(0)+\int_0^\infty u(x)Ly(x)dx, \\ W_\infty(y,v) &= y^{[1]}(0)+\int_0^\infty v(x)Ly(x)dx \end{aligned}$$

are obtained. For the BVP (1.1), (1.6), we have

$$\begin{aligned} W_\infty(y,u) &= y(0)+\int_0^\infty u(x)f(x,y(x),Ty(x))dx, \\ W_\infty(y,v) &= y^{[1]}(0)+\int_0^\infty v(x)f(x,y(x),Ty(x))dx. \end{aligned}$$

Put $\varphi(x)=\alpha u(x)+\beta v(x)$ and $\psi(x)=\gamma u(x)+\delta v(x)$. Since $W_x(\varphi,\psi)=\omega\neq 0$, the functions φ and ψ are linearly independent solutions of (1.2). So we obtain

$$\begin{aligned} &\varphi(0)=W_x(\varphi,u)=-\beta, \quad \varphi^{[1]}(0)=W_x(\varphi,v)=\alpha; \\ &\psi(0)=W_x(\psi,u)=-\delta, \quad \psi^{[1]}(0)=W_x(\psi,v)=\gamma. \end{aligned}$$

For the boundary conditions

$$\alpha y(0)+\beta y^{[1]}(0)=0,\ \gamma W_\infty(y,u)+\delta W_\infty(y,v)=0,$$

φ satisfies the boundary condition at zero, and ψ satisfies the boundary condition at infinity. Let

$$G(x,s)=-\frac{1}{\omega}\begin{cases} \varphi(x)\psi(s) & 0\le x\le s<\infty, \\ \varphi(s)\psi(x) & 0\le s\le x<\infty. \end{cases}$$

Since $\varphi,\ \psi\in L^2(0,\infty)$, we get

$$\int_0^\infty\int_0^\infty |G(x,s)|^2dxds<\infty. \tag{1.7}$$

From [8, Corollary 1], the nonlinear BVP (1.1), (1.6) is equivalent to the nonlinear integral equation

$$y(x)=w(x)+\int_0^\infty G(x,s)f(s,y(s),Ty(s))ds, \quad 0\le x<\infty,$$

where $w(x)=\frac{d_2}{\omega}\varphi(x)-\frac{d_1}{\omega}\psi(x)$. Then investigating the existence of solutions of the nonlinear BVP (1.1), (1.6) is equivalent to investigating fixed points of the operator $A:L^2(0,\infty)\to L^2(0,\infty)$ by the formula

$$Ay(x)=w(x)+\int_0^\infty G(x,s)f(s,y(s),Ty(s))ds, \quad 0\le x<\infty, \tag{1.8}$$

where $y\in L^2(0,\infty)$.

In Section 2, we will investigate the existence of solutions of the BVP (1.1), (1.6) by using the Schauder Fixed Point Theorem.

Finally, in Section 3, we will study BVPs on the whole axis.

2. EXISTENCE OF SOLUTIONS ON THE SEMI-AXIS

In this section to show the existence of solutions of the BVP (1.1), (1.6), we will use the following Schauder Fixed Point Theorem: *Let $\mathcal{B}$ be a Banach space and $\mathcal{S}$ a nonempty bounded, convex, and closed subset of $\mathcal{B}$. Assume $A : \mathcal{B} \to \mathcal{B}$ is a completely continuous operator. If the operator A leaves the set $\mathcal{S}$ invariant then A has at least one fixed point in $\mathcal{S}$.*

Theorem 2.1. *A set $\mathcal{S} \subset L^2(0,\infty)$ is relatively compact iff $\mathcal{S}$ is bounded and for every $\varepsilon > 0$ (i) there exists a $\delta > 0$ such that $\int_0^\infty |y(x+h) - y(x)|^2 dx < \varepsilon$ for all $y \in \mathcal{S}$ and all $h \geq 0$ with $h < \delta$, (ii) there exists a number $N > 0$ such that $\int_N^\infty |y(x)|^2 dx < \varepsilon$ for all $y \in \mathcal{S}$.*

Theorem 2.2. *Assume conditions* (C1), (C2), (C3)*, and* (C4) *are satisfied. In addition, let there exist a number $R > 0$ such that*

$$\{\int_0^\infty |w(x)|^2 dx\}^{1/2} + M^{1/2}\{\sup_{y\in\mathcal{S}} \int_0^\infty |f(s, y(s), Ty(s))|^2 ds\}^{1/2} \leq R \tag{2.1}$$

where $M = \int_0^\infty \int_0^\infty |G(x,s)|^2 dxds$ and $\mathcal{S} = \{y \in L^2(0,\infty) : \|y\| \leq R\}$. Then the BVP (1.1), (1.6) *has at least one solution $y \in L^2(0,\infty)$ with*

$$\int_0^\infty |y(x)|^2 dx \leq R^2.$$

Proof. To show that the operator A defined in (1.8) is completely continuous, we must prove the operator A is continuous, and $A(Y)$ is a relatively compact set in $L^2(0,\infty)$ where $Y \subset L^2(0,\infty)$ is a bounded set. First, we want to show that when $\varepsilon > 0$ and $y_0 \in L^2(0,\infty)$, there exists $\delta > 0$ such that

$$y \in L^2(0,\infty) \text{ and } \|y - y_0\| < \delta \text{ implies } \|Ay - Ay_0\| < \varepsilon. \tag{2.2}$$

It can be easily seen that the inequality

$$|Ay(x) - Ay_0(x)|^2 \leq \int_0^\infty |G(x,s)|^2 ds \int_0^\infty |f(s, y(s), Ty(s)) - f(s, y_0(s), Ty_0(s))|^2 ds$$

holds. Hence we get

$$\begin{aligned} \|Ay - Ay_0\|^2 &\leq M \int_0^\infty |f(s, y(s), Ty(s)) - f(s, y_0(s), Ty_0(s))|^2 ds \\ &= M \int_0^N |f(s, y(s), Ty(s)) - f(s, y_0(s), Ty_0(s))|^2 ds \\ &\quad + M \int_N^\infty |f(s, y(s), Ty(s)) - f(s, y_0(s), Ty_0(s))|^2 ds, \end{aligned}$$

where

$$M = \int_0^\infty \int_0^\infty |G(x,s)|^2 dx ds$$

and N is an arbitrary positive number. By the condition (1.3) and the inequalities

$$(x+y)^2 \le 2(x^2+y^2), \quad (x+y+z+t+u)^2 \le 5(x^2+y^2+z^2+t^2+u^2),$$

we have

$$\begin{aligned}
& \int_N^\infty |f(s,y(s),Ty(s)) - f(s,y_0(s),Ty_0(s))|^2 ds \\
\le & \int_N^\infty [|f(s,y(s),Ty(s))| + |f(s,y_0(s),Ty_0(s))|]^2 ds \\
\le & \int_N^\infty [2g(s) + a|y(s)| + b|Ty(s)| + a|y_0(s)| + b|Ty_0(s)|]^2 ds \\
\le & \int_N^\infty [20g^2(s) + 5a^2|y(s)|^2 + 5b^2|Ty(s)|^2 + 5a^2|y_0(s)|^2 + 5b^2|Ty_0(s)|^2] ds \\
\le & \ 20\int_N^\infty g^2(s)ds + 10a^2\int_0^\infty |y(s)-y_0(s)|^2 ds + 15a^2\int_N^\infty |y_0(s)|^2 ds \\
+ & \ 10b^2\int_0^\infty |Ty(s)-Ty_0(s)|^2 ds + 15b^2 \int_N^\infty |Ty_0(s)|^2 ds.
\end{aligned}$$

Choose N such that

$$\int_N^\infty g^2(s)ds < \frac{\varepsilon^2}{120M}, \quad \int_N^\infty |y_0(s)|^2 ds < \frac{\varepsilon^2}{90a^2M}, \quad \int_N^\infty |Ty_0(s)|^2 ds < \frac{\varepsilon^2}{90b^2M}.$$

Since T is continuous, we can find a $\delta_0 > 0$ such that

$$\|y - y_0\| < \delta_0 \text{ implies } \|Ty - Ty_0\|^2 < \frac{\varepsilon^2}{60b^2M}.$$

Then we get

$$\int_N^\infty |f(s,y(s),Ty(s)) - f(s,y_0(s),Ty_0(s))|^2 ds < \frac{\varepsilon^2}{6M} + 10a^2\delta^2 + \frac{\varepsilon^2}{6M} + \frac{\varepsilon^2}{6M} + \frac{\varepsilon^2}{6M}.$$

It is known (see [9]) that under the condition (C2) the operator F defined by $Fy(x) = f(x,y(x),Ty(x))$ is continuous in $L^2(0,\infty)$. So we can find a $\delta_1 > 0$ such that

$$\|y - y_0\| < \delta_1 \text{ implies } \int_0^N |f(s,y(s),Ty(s)) - f(s,y_0(s),Ty_0(s))|^2 ds < \frac{\varepsilon^2}{6M}.$$

Taking

$$\delta^2 = min\{\frac{\varepsilon^2}{60a^2M}, \delta_0^2, \delta_1^2\},$$

we obtain desired result (2.2). Hence, the operator A is continuous.

Now, we must show that $A(Y)$ is a relatively compact set in $L^2(0,\infty)$ where $\|y\| \le c_1$ for all $y \in Y$. For this purpose, we will use Theorem 2.1.

For all $y \in Y$, we have

$$\begin{aligned} \|Ay\| &= \|w(x) + \int_0^\infty G(x,s) f(s, y(s), Ty(s)) ds\| \\ &\leq \|w\| + \| \int_0^\infty G(x,s) f(s, y(s), Ty(s)) ds\|. \end{aligned}$$

Then, we obtain

$$\|Ay\| \leq \|w\| + \{\int_0^\infty \int_0^\infty |G(x,s)|^2 dx ds\}^{1/2} \{\int_0^\infty |f(s, y(s), Ty(s))|^2 ds\}^{1/2}. \tag{2.3}$$

At the same time, using (1.3) and the inequality $(x+y+z)^2 \leq 3(x^2+y^2+z^2)$, the following inequality holds;

$$\begin{aligned} \int_0^\infty |f(s, y(s), Ty(s))|^2 ds &\leq \int_0^\infty [a|y(s)| + b|Ty(s)| + g(s)]^2 ds \\ &\leq 3\int_0^\infty [a^2|y(s)|^2 + b^2|Ty(s)|^2 + g^2(s)] ds \\ &= 3(a^2\|y\|^2 + b^2\|Ty\|^2 + \|g\|^2). \end{aligned}$$

By virtue of the condition (C3) and $\|y\| \leq c_1$ for all $y \in Y$, we have $\|Ty\| \leq c_2$ for all $y \in Y$. Then

$$\int_0^\infty |f(s, y(s), Ty(s))|^2 ds \leq 3(a^2 c_1^2 + b^2 c_2^2 + \|g\|^2).$$

Hence $\|Ay\| \leq \|w\| + \{3M(a^2c_1^2 + b^2c_2^2 + \|g\|^2)\}^{1/2}$ for all $y \in Y$, that is, $A(Y)$ is a bounded set in $L^2(0,\infty)$.

Besides, for all $y \in Y$, we have

$$\begin{aligned} \int_0^\infty |Ay(x+h) - Ay(x)|^2 dx &= \int_0^\infty | \int_0^\infty [G(x+h,s) - G(x,s)] f(s, y(s), Ty(s)) ds|^2 dx \\ &\leq \int_0^\infty \{\int_0^\infty |G(x+h,s) - G(x,s)|^2 ds \int_0^\infty |f(s, y(s), Ty(s))|^2 ds\} dx \\ &\leq 3(a^2c_1^2 + b^2c_2^2 + \|g\|^2) \int_0^\infty \int_0^\infty |G(x+h,s) - G(x,s)|^2 dx ds. \end{aligned}$$

Thus we get by (1.7) that for a given $\varepsilon > 0$ there exists a $\delta > 0$, depending only on ε, such that

$$\int_0^\infty |Ay(x+h) - Ay(x)|^2 dx < \varepsilon^2$$

for all $y \in Y$ and all $h \geq 0$ with $h < \delta$.

Also for all $y \in Y$, we have

$$\int_N^\infty |Ay(x)|^2 dx \leq 3(a^2c_1^2 + b^2c_2^2 + \|g\|^2) \int_N^\infty \int_0^\infty |G(x,s)|^2 ds dx.$$

Hence we get again by (1.7) that for given $\varepsilon > 0$ there exists a positive number N, depending only on ε, such that $\int_N^\infty |Ay(x)|^2 dx < \varepsilon^2$ for all $y \in Y$.

Thus, $A(Y)$ is a relatively compact set in $L^2(0,\infty)$. Then, the operator A is completely continuous. Further, it is obvious that the set $\mathcal{S}$ is bounded, convex, and closed. By (2.3) and (2.1), A maps the set $\mathcal{S}$ into itself, and thus the proof is completed. □

Example 2.1. We consider the following problem

$$\left\{\begin{array}{c} -(e^{3x}y')' - 2e^{3x}y = \int_0^\infty e^{-x}y(s)ds, \\ y(0) + y^{[1]}(0) = 1, \quad W_\infty(y,u) - W_\infty(y,v) = 0. \end{array}\right. \tag{2.4}$$

When taking $p(x) = e^{3x}$, $q(x) = -2e^{3x}$, $\alpha = \beta = \gamma = 1$, $\delta = -1$, $d_1 = 1$, $d_2 = 0$, $f(x,y,Ty) = Ty$ and $Ty(x) = \int_0^\infty e^{-x}y(s)ds$, the conditions (C1), (C2), (C3), and (C4) hold. It is clear that $u(x) = e^{-x} - e^{-2x}$ and $v(x) = 2e^{-x} + e^{-2x}$. Then, we obtain $\varphi(x) = -e^{-x}$, $\psi(x) = 3e^{-x} - 2e^{-2x}$ and $\omega(x) = \frac{1}{2}(3e^{-x} - 2e^{-2x})$. For the Green's function

$$G(x,s) = -\frac{1}{2}\left\{\begin{array}{ll} -e^{-x}(3e^{-s} - 2e^{-2s}) & 0 \le x \le s < \infty, \\ -e^{-s}(3e^{-x} - 2e^{-2x}) & 0 \le s \le x < \infty, \end{array}\right.$$

$M = \int_0^\infty \int_0^\infty |G(x,s)|^2 dxds = \frac{59}{240}$ and $\int_0^\infty |w(x)|^2 dx = \frac{3}{8}$. If we take $R = 2$, then (2.1) is satisfied. Finally, the BVP (2.4) has at least one solution $y \in L^2(0,\infty)$ with

$$\int_0^\infty |y(x)|^2 dx \le 4.$$

3. EXISTENCE OF SOLUTIONS ON THE WHOLE-AXIS

We consider the equation

$$-[p(x)y']' + q(x)y = f(x,y,Iy), \quad -\infty < x < \infty \tag{3.1}$$

where $y = y(x)$ is a desired solution, and

$$(Iy)(x) = \int_{-\infty}^\infty K(x,s)y(s)ds,$$

$K \in C[\mathbf{R} \times \mathbf{R}, \mathbf{R}^+]$.

Assume that the following conditions are satisfied.

(H1) The coefficients $p(x)$ and $q(x)$ are real-valued measurable functions on $\mathbf{R} = (-\infty,\infty)$ such that

$$\int_a^b \frac{dx}{|p(x)|} < \infty, \qquad \int_a^b |q(x)|dx < \infty$$

for all finite real numbers a and b with $a < b$. Moreover, the functions $p(x)$ and $q(x)$ are such that all solutions of the second order linear differential equation

$$-[p(x)y']' + q(x)y = 0, \quad -\infty < x < \infty, \tag{3.2}$$

belong to $L^2(-\infty,\infty)$.

(H2) The function $f(x,y,z)$ is real-valued and continuous in $(x,y,z)\in \mathbf{R}\times\mathbf{R}\times\mathbf{R}$, and

$$|f(x,y,z)|\le a|y|+b|z|+g(x)$$

for all (x,y,z) in $\mathbf{R}\times\mathbf{R}\times\mathbf{R}$, where $g(x)\ge 0,\quad g\in L^2(-\infty,\infty)$ and, a and b are positive constants.

(H3) $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|K(x,s)|^2dxds<\infty$.

Denote by D the linear manifold of all elements $y\in L^2(-\infty,\infty)$ such that $Ly=-[p(x)y']'+q(x)y$ is defined and $Ly\in L^2(-\infty,\infty)$. It follows from the Green's formula

$$\int_a^b[(Ly)z-y(Lz)](x)dx=W_b(y,z)-W_a(y,z) \tag{3.3}$$

that, for all $y,z\in D$ the limits

$$W_{-\infty}(y,z)=\lim_{a\to-\infty}W_a(y,z),\qquad W_{\infty}(y,z)=\lim_{b\to\infty}W_b(y,z)$$

exist as finite numbers.

Let $u=u(x)$ and $v=v(x)$ be solutions of (3.2) satisfying the initial conditions

$$u(0)=0,\ u^{[1]}(0)=1;\quad v(0)=-1,\ v^{[1]}(0)=0. \tag{3.4}$$

By condition (H1) the solutions u and v belong to $L^2(-\infty,\infty)$. Moreover, they belong to D. Therefore for each $y\in D$ the values $W_{\pm\infty}(y,u)$ and $W_{\pm\infty}(y,v)$ exist and are finite. By using (3.3) and (3.4), we get the formulas

$$\begin{aligned}
&W_{-\infty}(y,u)=y(0)-\textstyle\int_{-\infty}^0u(x)Ly(x)dx, && W_{-\infty}(y,v)=y^{[1]}(0)-\textstyle\int_{-\infty}^0v(x)Ly(x)dx,\\
&W_{\infty}(y,u)=y(0)+\textstyle\int_0^{\infty}u(x)Ly(x)dx, && W_{\infty}(y,v)=y^{[1]}(0)+\textstyle\int_0^{\infty}v(x)Ly(x)dx.
\end{aligned}$$

Now we study the equation (3.1) together with the boundary conditions

$$\alpha W_{-\infty}(y,u)+\beta W_{-\infty}(y,v)=d_1,\qquad \gamma W_{\infty}(y,u)+\delta W_{\infty}(y,v)=d_2, \tag{3.5}$$

where α,β,γ, and δ are given real numbers satisfying the condition

(H4) $\omega:=\alpha\delta-\beta\gamma\ne 0$,

and d_1,d_2 are given arbitrary real numbers.

We define the set $\mathbb{D}=\{y\in L^2(-\infty,\infty): y'$ is continuous and py' is differentiable on $\mathbf{R}$ and $(py')'$ is continuous on $\mathbf{R}$ and $\alpha W_{-\infty}(y,u)+\beta W_{-\infty}(y,v)=d_1$, $\gamma W_{\infty}(y,u)+\delta W_{\infty}(y,v)=d_2\}$. If $y\in\mathbb{D}$ and the equation $-[p(x)y']'+q(x)y=f(x,y,Iy)$ holds for all $x\in\mathbf{R}$, then y is called a solution of the problem (3.1), (3.5).

Let us set

$$\varphi(x)=\alpha u(x)+\beta v(x),\quad \psi(x)=\gamma u(x)+\delta v(x)$$

and define the function

$$G(x,s) = -\frac{1}{\omega}\begin{cases} \varphi(x)\psi(s) & -\infty < x \le s < \infty, \\ \varphi(s)\psi(x) & -\infty < s \le x < \infty, \end{cases}$$

and

$$w(x) = \frac{d_2}{\omega}\varphi(x) - \frac{d_1}{\omega}\psi(x).$$

Thus we have

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |G(x,s)|^2 dxds < \infty, \qquad \int_{-\infty}^{\infty} |w(x)|^2 dx < \infty.$$

The BVP (3.1), (3.5) is equivalent to the integral equation

$$y(x) = w(x) + \int_{-\infty}^{\infty} G(x,s) f(s, y(s), Iy(s)) ds, \quad -\infty < x < \infty.$$

Reasoning as in the previous section we can prove the following theorem.

Theorem 3.1. *Assume conditions* (H1), (H2), (H3), *and* (H4) *are satisfied. In addition, let there exist a number* $R > 0$ *such that*

$$\{\int_{-\infty}^{\infty} |w(x)|^2 dx\}^{1/2} + M^{1/2}\{\sup_{y\in\mathcal{S}} \int_{-\infty}^{\infty} |f(s, y(s), Iy(s))|^2 ds\}^{1/2} \le R, \tag{3.6}$$

where $M = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |G(x,s)|^2 dxds$ *and* $\mathcal{S} = \{y \in L^2(-\infty,\infty) : \|y\| \le R\}$. *Then the BVP* (3.1), (3.5) *has at least one solution* $y \in L^2(-\infty,\infty)$ *with* $\|y\| \le R$.

Example 3.1. We consider the following problem

$$\begin{cases} -(e^{2x^2}y')' + (4x^2+2)e^{2x^2}y = \int_{-\infty}^{\infty} e^{-x^2} y(s) ds, \\ W_{-\infty}(y,u) - W_{-\infty}(y,v) = 0, \quad W_{\infty}(y,u) + W_{\infty}(y,v) = 1. \end{cases} \tag{3.7}$$

When taking $p(x) = e^{2x^2}$, $q(x) = (4x^2+2)e^{2x^2}$, $\alpha = \gamma = \delta = 1$, $\beta = -1$, $d_1 = 0$, $d_2 = 1$, $f(x,y,Iy) = Iy$ and $Iy(x) = \int_{-\infty}^{\infty} e^{-x^2} y(s) ds$, the conditions (H1), (H2), (H3), and (H4) are satisfied. It is clear that $u(x) = xe^{-x^2}$ and $v(x) = -e^{-x^2}$. Then, we have $\varphi(x) = (x+1)e^{-x^2}$, $\psi(x) = (x-1)e^{-x^2}$ and $\omega(x) = \frac{1}{2}(x+1)e^{-x^2}$. For the Green's function

$$G(x,s) = -\frac{1}{2}\begin{cases} (x+1)(s-1)e^{-(x^2+s^2)} & -\infty < x \le s < \infty, \\ (s+1)(x-1)e^{-(x^2+s^2)} & -\infty < s \le x < \infty, \end{cases}$$

$M = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |G(x,s)|^2 dxds = \frac{25\pi}{32} - \frac{9\sqrt{\pi}}{8}$ and $\int_{-\infty}^{\infty} |w(x)|^2 dx = \frac{5\sqrt{2\pi}}{32}$. If we get $R = 3$, then (3.6) holds. Finally, the BVP (3.7) has at least one solution $y \in L^2(-\infty,\infty)$ with

$$\int_{-\infty}^{\infty} |y(x)|^2 dx \le 9.$$

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