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FRACTIONAL DIFFERENTIAL EQUATIONS WITH DEVIATING ARGUMENTS

TADEUSZ JANKOWSKI

Gdansk University of Technology, Department of Differential Equations
11/12 G. Narutowicz Str., 80–952 Gdańsk, Poland

ABSTRACT. This paper deals with initial problems for fractional differential equations with deviating arguments. Sufficient conditions are formulated under which such problems have unique or extremal solutions. Corresponding inequalities for such problems are also discussed.

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1. INTRODUCTION

Recently, the differential equations involving Riemann–Liouville differential operators of fractional order $q \in (0,1)$ are discussed, see for example [1]–[8]. Existence of solutions for initial problems was investigated in papers [1],[3]–[8]. As far as I know only paper [2] deals with the initial problem for delayed fractional differential equations. In this paper we discuss an initial value problem

$$(1.1)\qquad \begin{cases} D^q x(t) & = & f(t, x(t), x(\alpha(t))) \equiv Fx(t), \quad t \in J = [0,T],\ T > 0, \\ [x(t)t^{1-q}]\,|_{t=0} & = & x_0, \end{cases}$$

where

$H_1:\ f \in C(J \times \mathbb{R} \times \mathbb{R}, \mathbb{R}),\ \alpha \in C(J,J),\ \alpha(t) \le t,\ t \in J$ and $0 < q < 1$.

Since f is continuous, problem (1.1) is equivalent to the following Volterra fractional integral

$$(1.2)\qquad x(t) = x_0 t^{q-1} + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s, x(s), x(\alpha(s)))ds, \quad t \in J,$$

where Γ denotes the Gamma function. In this paper, we formulate sufficient conditions under which problem (1.1) has a unique solution or extremal solutions. To obtain extremal solutions we use the monotone iterative method. The problem of inequalities is also discussed. It is important to add that in paper [3] this technique was also used for initial fractional differential equations without delays.

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2. EXISTENCE SOLUTIONS OF PROBLEM (1.1)

By $C_p(J,\mathbb{R}),\ p>0$, we denote the space of all functions $x \in C(J,\mathbb{R})$ such that $t^p x \in C(J,\mathbb{R})$ with the norm $\|x\|_p = \max_{t\in J} t^p|x(t)|$. A function $x \in C_p(J,\mathbb{R})$ is a solution of problem (1.1) when $x_0 \neq 0$ if $D^q x$ exists, is continuous and satisfies (1.1). If $x_0 = 0$, then $C_p(J,\mathbb{R})$ is replaced by $C(J,\mathbb{R})$.

Our first existence result for problem (1.1) is based on the Banach contraction principle.

Theorem 2.1. *Let assumption H_1 hold and $x_0 = 0$. In addition, we assume that*

$H_2:$ *there exist nonnegative constants K, L such that*

(2.1) $|f(t,u_1,u_2)-f(t,v_1,v_2)| \le K|v_1-u_1|+L|v_2-u_2|$ *if* $t\in J,\ u_i,v_i\in\mathbb{R},\ i=1,2,$

$H_3:\ \dfrac{(K+L)T^q}{\Gamma(q+1)} < 1$ *if* $0<q\le\frac{1}{2}$.

Then problem (1.1) *has a unique solution $x \in C(J,\mathbb{R})$.*

Proof. Consider the problem $x = Nx$, where N denotes the operator defined by the right–hand–side of problem (1.2). Now we need to show that operator N has a fixed point. To do it we shall show that $N : C(J,\mathbb{R}) \to C(J,\mathbb{R})$ is a contraction map. Put $\|x\| = \max_{t\in J}|x(t)|$. We consider two cases.

Case 1. Let $0 < q \le \frac{1}{2}$. Then in view of assumption H_2, for $x, y \in C(J,\mathbb{R})$ we have

$$\begin{aligned}\|x-y\| &\le \frac{1}{\Gamma(q)}\max_{t\in J}\int_0^t (t-s)^{q-1}|f(s,x(s),x(\alpha(s)))-f(s,y(s),y(\alpha(s)))|ds\\ &\le \frac{K+L}{\Gamma(q)}\|x-y\|\max_{t\in J}\int_0^t (t-s)^{q-1}ds\\ &= \frac{(K+L)T^q}{\Gamma(q+1)}\|x-y\|\end{aligned}$$

because

$$\int_0^t (t-s)^{q-1}ds = t^q\int_0^1(1-\sigma)^{q-1}d\sigma = \frac{\Gamma(q)}{\Gamma(q+1)}t^q.$$

This and assumption H_3 prove that operator N is a contraction. Therefore, N has a unique fixed point by a Banach fixed point theorem.

Case 2. Assume that $\frac{1}{2} < q < 1$. Let

$$\|x\|_* = \max_{t\in J} e^{-\lambda t}|x(t)| \quad \text{with} \quad \lambda > \left(\frac{K+L)}{\Gamma(q)}\right)^2 T^{2q-1}\frac{\Gamma(2q-1)}{2\Gamma(2q)}.$$

Note that

$$\int_0^t e^{2\lambda t}dt \le \frac{1}{2\lambda}e^{2\lambda t}, \quad \int_0^t (t-s)^{2(q-1)}ds = t^{2q-1}\frac{\Gamma(2q-1)}{\Gamma(2q)}. \tag{2.2}$$

We will use the Schwarz inequality for integrals

$$\int_0^t |a(s)||b(s)|ds \leq \sqrt{\int_0^t a^2(s)ds}\sqrt{\int_0^t b^2(s)ds}.$$

Let $x, y \in C(J, \mathbb{R})$. Using assumption H_2, the Schwarz inequality and (2.2), we have

$$\begin{aligned}
\|x-y\|_* &\leq \frac{1}{\Gamma(q)} \max_{t\in J} e^{-\lambda t} \int_0^t (t-s)^{q-1}|f(s,x(s),x(\alpha(s))) - f(s,y(s),y(\alpha(s)))|ds \\
&\leq \frac{1}{\Gamma(q)} \|x-y\|_* \max_{t\in J} e^{-\lambda t} \int_0^t (t-s)^{q-1}[Ke^{\lambda s} + Le^{\lambda\alpha(s)}]ds \\
&\leq \frac{(K+L)}{\Gamma(q)} \|x-y\|_* \max_{t\in J} e^{-\lambda t} \int_0^t (t-s)^{q-1} e^{\lambda s} ds \\
&\leq \frac{(K+L)}{\Gamma(q)} \|x-y\|_* \max_{t\in J} e^{-\lambda t} \sqrt{\int_0^t (t-s)^{2(q-1)}ds}\sqrt{\int_0^t e^{2\lambda s}ds} \\
&= \frac{(K+L)}{\Gamma(q)} \sqrt{\frac{T^{2q-1}\Gamma(2q-1)}{2\lambda\Gamma(2q)}} \|x-y\|_* \equiv \rho(\lambda)\|x-y\|.
\end{aligned}$$

Note that $\rho = \rho(\lambda) < 1$, so operator N is a contraction, so operator N has a unique fixed point, by the Banach fixed point theorem. It ends the proof. □

Theorem 2.2 (see [6])**.** *Let assumption H_1 hold and $x_0 \neq 0$. In addition, we assume that f does not depend on the third argument and*

H_2' : *there exists a nonnegative constant K such that*

$$|f(t,u_1) - f(t,v_1)| \leq K|v_1 - u_1|, \quad t \in J, \ u_1, v_1 \in \mathbb{R},$$

H_3' : $\dfrac{KT^q\Gamma(q)}{\Gamma(2q)} < 1.$

Then problem (1.1) has a unique solution $x \in C_p(J, \mathbb{R})$ with $p = 1 - q$.

3. EXISTENCE OF EXTREMAL SOLUTIONS OF PROBLEM (1.1)

First we need to investigate some problems connected with inequalities.

Theorem 3.1. *Let assumption H_1 hold. Let $v, w : J \to \mathbb{R}$ be continuous and satisfy*

$$(3.1) \qquad \begin{cases} v(t) \leq v(0) + \dfrac{1}{\Gamma(q)} \displaystyle\int_0^t (t-s)^{q-1} f(s, v(s), v(\alpha(s)))ds, \\ w(t) \geq w(0) + \dfrac{1}{\Gamma(q)} \displaystyle\int_0^t (t-s)^{q-1} f(s, w(s), w(\alpha(s)))ds, \end{cases} \quad t \in J,$$

and one of the inequalities being strict. Let f be nondecreasing with respect to the last two arguments for each the first argument of f. Then $v(0) < w(0)$ implies $v(t) < w(t)$, $t \in J$.

Proof. Suppose that the assertion is not true and let the first inequality of (3.1) be strict. Because of the condition $v(0) < w(0)$ and continuity of v, w, there exists a point $t_1 \in (0, T]$ such that $v(t_1) = w(t_1),\ v(t) < w(t),\ t \in [0, t_1)$. In view of the fact that $0 \le \alpha(t) \le t$, we have $v(\alpha(t)) < w(\alpha(t))$ for $t \in [0, t_1)$.

Using the monotone character of f, we see that

$$\begin{aligned} v(t_1) &< v(0) + \frac{1}{\Gamma(q)} \int_0^{t_1} (t_1 - s)^{q-1} f(s, v(s), v(\alpha(s)))ds, \\ &< w(0) + \frac{1}{\Gamma(q)} \int_0^{t_1} (t_1 - s)^{q-1} f(s, w(s), w(\alpha(s)))ds \le w(t_1). \end{aligned}$$

It is a contradiction. It shows that the assertion of Theorem 3.1 holds and this ends the proof. □

Theorem 3.2. *Assume that assumption H_1 holds. Let $v, w : J \to \mathbb{R}$ be continuous and such that* (3.1) *hold. In addition, we assume that there exist nonnegative constants K and L such that*

$$\Gamma(q+1) > K(1 + T^q) + L(1 + S^q) \quad \textit{with} \quad S = \max_{t \in J} \alpha(t) \tag{3.2}$$

and

$$f(t, x_1, y_1) - f(t, x_2, y_2) \le K(x_1 - x_2) + L(y_1 - y_2) \quad \textit{if} \quad x_1 \ge x_2,\ y_1 \ge y_2. \tag{3.3}$$

Then $v(0) \le w(0)$ implies $v(t) \le w(t)$ on J.

Proof. For $\epsilon > 0$, we put $w_\epsilon(t) = w(t) + \epsilon(1 + t^q),\ t \in J$. Then, in view of (3.3), we have

$$\begin{aligned} Q(t, q, w_\epsilon) &\equiv \frac{1}{\Gamma(q)} \int_0^t (t - s)^{q-1} [f(s, w(s), w(\alpha(s))) - f(s, w_\epsilon(s), w_\epsilon(\alpha(s)))]ds + \epsilon t^q \\ &\ge \frac{-1}{\Gamma(q)} \int_0^t (t - s)^{q-1} \{K[w_\epsilon(s) - w(s)] + L[w_\epsilon(\alpha(s)) - w(\alpha(s))]\}ds + \epsilon t^q \\ &= -\frac{\epsilon}{\Gamma(q)} \int_0^t (t - s)^{q-1} \{K[1 + s^q] + L[1 + \alpha^q(s)]\}ds + \epsilon t^q \\ &\ge \frac{-\epsilon}{\Gamma(q)} [K(1 + T^q) + L(1 + S^q)] \int_0^t (t - s)^{q-1} ds + \epsilon t^q \\ &= \epsilon t^q \left[1 - \frac{K(1 + T^q) + L(1 + S^q)}{\Gamma(q+1)}\right] > 0, \end{aligned}$$

by condition (3.2). This and the definition of w_ϵ yield

$$\begin{aligned} w_\epsilon(t) &= w(t) + \epsilon(1 + t^q) \\ &\ge w_\epsilon(0) + \frac{1}{\Gamma(q)} \int_0^t (t - s)^{q-1} f(s, w_\epsilon(s), w_\epsilon(\alpha(s)))ds + Q(t, w, w_\epsilon) \\ &> w_\epsilon(0) + \frac{1}{\Gamma(q)} \int_0^t (t - s)^{q-1} f(s, w_\epsilon(s), w_\epsilon(\alpha(s)))ds. \end{aligned}$$

Now, in view of Theorem 3.1, we get $v(t) < w_\epsilon(t)$ on J. Hence, if $\epsilon \to 0$, then we have the assertion. This ends the proof. $\square$

Now we shall discuss the problem of the existence of extremal solutions for problems of type (1.1). To do it we apply the monotone iterative technique.

Theorem 3.3. *Assume that assumption H_1 holds and $x_0 = 0$. Let $v_0, w_0 : J \to \mathbb{R}$ be continuous such that*

$$(3.4) \qquad \begin{cases} v_0(t) \;\leq\; \dfrac{1}{\Gamma(q)} \displaystyle\int_0^t (t-s)^{q-1} f(s, v_0(s), v_0(\alpha(s)))ds, \\ w_0(t) \;\geq\; \dfrac{1}{\Gamma(q)} \displaystyle\int_0^t (t-s)^{q-1} f(s, w_0(s), w_0(\alpha(s)))ds, \end{cases} \quad t \in J,$$

and $v_0(t) \leq w_0(t)$ on J. In addition, we assume that there exist nonnegative constants K and L such that both condition (3.2)*, assumption H_3 and the following condition*

$$(3.5) \qquad f(t, x_1, y_1) - f(t, x_2, y_2) \leq K(x_2 - x_1) + L(y_2 - y_1) \;\text{ if }\; x_1 \leq x_2, \; y_1 \leq y_2$$

are satisfied.

Then there exist the extremal solutions v, w of problem (1.1) *in the sector*

$$[v_0, w_0]^* = \{y : v_0(t) \leq w(t) \leq w_0(t), \; t \in J\}.$$

Proof. Let us define two sequences $\{v_n, w_n\}$ by formulas

$$(3.6) \qquad \begin{cases} v_{n+1}(t) \;=\; \dfrac{1}{\Gamma(q)} \displaystyle\int_0^t (t-s)^{q-1} \mathcal{F}(v_{n+1}, v_n)(s)ds, \\ w_{n+1}(t) \;=\; \dfrac{1}{\Gamma(q)} \displaystyle\int_0^t (t-s)^{q-1} \mathcal{F}(w_{n+1}, w_n)(s)ds, \end{cases} \quad t \in J,$$

where

$$\mathcal{F}(x, y)(s) = Fy(s) - K[x(s) - y(s)] - L[x(\alpha(s)) - y(\alpha(s))].$$

We need to show that $v_n \to v, \; w_n \to w$ as $n \to \infty$ uniformly and monotonically on J. First we see that elements v_1, w_1 are well defined by Theorem 2.1.

Put $p = v_0 - v_1$. Then, by conditions (3.4) and (3.6), we have

$$\begin{aligned} p(t) &\leq \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} [Fv_0(s) - \mathcal{F}(v_1, v_0)(s)]ds \\ &= -\frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} [Kp(s) + Lp(\alpha(s))]ds. \end{aligned}$$

Indeed, the problem

$$z(t) = -\frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} [Kz(s) + Lz(\alpha(s))]ds, \; t \in J$$

has a unique solution, by Theorem 2.1. We see that $z(t) = 0, \; t \in J$. This and Theorem 3.1 yield $p(t) \leq z(t) = 0, \; t \in J$ proving that $v_0(t) \leq v_1(t), \; t \in J$. Similarly, we can show that $w_1(t) \leq w_0(t), \; t \in J$.

To show that $v_1 \le w_1$ we put $p = v_1 - w_1$. By (3.6) and (3.5), we obtain

$$\begin{aligned} p(t) &= \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}[\mathcal{F}(v_1,v_0)(s) - \mathcal{F}(w_1,w_0)(s)]ds \\ &\le \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}\{K[w_0(s)-v_0(s)] + L[w_0(\alpha(s)) - v_0(\alpha(s))] \\ &\quad -K[v_1(s)-v_0(s)-w_1(s)+w_0(s)] \\ &\quad -L[v_1(\alpha(s)) - v_0(\alpha(s)) - w_1(\alpha(s)) + w_0(\alpha(s))]\}ds \\ &= -\frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}[Kp(s) + Lp(\alpha(s))]ds. \end{aligned}$$

As before, this and Theorem 3.1 yield $v_1(t) \le w_1(t),\ t \in J$. In this way we proved that $v_0(t) \le v_1(t) \le w_1(t) \le w_0(t),\ t \in J$.

By induction in n, we can prove that

$$v_0(t) \le v_1(t) \le \cdots \le v_n(t) \le v_{n+1}(t) \le w_{n+1}(t) \le w_n(t) \le \cdots \le w_1(t) \le w_0(t) \tag{3.7}$$

for $n = 0, 1, \ldots$ and $t \in J$.

Now we will prove that the sequences $\{v_n, w_n\}$ converge to their limit functions v, w, respectively. First, we need to show that the sequences are bounded and equicontinuous on J. Indeed, v, w are uniformly bounded by M in view of (3.7). Let $0 \le t_1 < t_2 \le T$. Note that

$$\int_{t_1}^{t_2} (t_2-s)^{q-1}ds = \int_0^{t_2-t_1} (t_2-t_1-s)^{q-1}ds = (t_2-t_1)^q \frac{\Gamma(q)}{\Gamma(q+1)}. \tag{3.8}$$

Then, in view of (3.8), we have

$$\begin{aligned} |v_n(t_1) - v_n(t_2)| &= \frac{1}{\Gamma(q)}\left|\int_0^{t_1} [(t_1-s)^{q-1} - (t_2-s)^{q-1}]\mathcal{F}(v_n,v_{n-1})(s)ds \right. \\ &\quad \left. - \int_{t_1}^{t_2} (t_2-s)^{q-1}\mathcal{F}(v_n,v_{n-1})(s)ds\right| \\ &\le \frac{M}{\Gamma(q)}\left(\int_0^{t_1} [(t_1-s)^{q-1} - (t_2-s)^{q-1}]ds + \int_{t_1}^{t_2} (t_2-s)^{q-1}ds\right) \\ &= \frac{M}{\Gamma(q)}\left(\int_0^{t_1} (t_1-s)^{q-1}ds - \int_0^{t_2} (t_2-s)^{q-1}ds + 2\int_{t_1}^{t_2} (t_2-s)^{q-1}ds\right) \\ &= \frac{M}{\Gamma(q+1)}(t_1^q - t_2^q + 2(t_2-t_1)^q) \le \frac{2M}{\Gamma(q+1)}(t_2-t_1)^q < \epsilon \end{aligned}$$

provided that $|t_2 - t_1| < \left[\frac{\epsilon\Gamma(q+1)}{2M}\right]^{\frac{1}{q}}$ with $\epsilon > 0$. The Arzela–Ascoli Theorem guarantees the existence of subsequences $\{v_{n_k}, w_{n_k}\}$ of $\{v_n, w_n\}$, respectively, and continuous functions v, w with v_{n_k}, w_{n_k} converging uniformly on J to v and w, respectively. If $n_k \to \infty$, then we see that v, w are continuous solutions of (1.1).

Now we need to prove that v is the minimal solution of (1.1) and w is a maximal solution of (1.1) in the sector $[v_0, w_0]^*$. Assume that u is any solution of problem (1.1) such that $v_0(t) \le u(t) \le w_0(t),\ t \in J$. Put $p_1 = v_1 - u,\ p_2 = u - w_1$. Then

$$\begin{aligned} p_1(t) &= \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} [\mathcal{F}(v_1, v_0)(s) - Fu(s)] ds \\ &\le -\frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} [Kp_1(s) + Lp_1(\alpha(s))] ds, \\ p_2(t) &= \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} [Fu(s) - \mathcal{F}(w_1, w_0)(s)] ds \\ &\le -\frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} [Kp_2(s) + Lp_2(\alpha(s))] ds, \end{aligned}$$

by condition (3.5). This and Theorem 3.1 yield $v_1(t) \le u(t) \le w_1(t),\ t \in J$. By induction in n, we can show that $v_n(t) \le u(t) \le w_n(t),\ t \in J,\ \ n = 1, 2, \ldots$. Now if $n \to \infty$, then we have the assertion. This ends the proof. $\square$

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