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BLOW UP IN A CLASS OF NON-AUTONOMOUS DYNAMIC SYSTEMS

R. RAUTMANN

Institute of Mathematics, University of Paderborn, 33095 Paderborn, Germany
(rautmann@uni-paderborn.de)

ABSTRACT. For a class of non-autonomous dynamic systems in the positive cone $\mathbb{R}^n_+$ of $\mathbb{R}^n$ we prove the blow up of all solutions having sufficiently large initial values. To more specialized non-autonomous systems we present explicit lower and upper bounds for solutions as well as for the time of their blowing up.

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1. INTRODUCTION

1.1. Problems and results.

In this note we will consider dynamic systems

$$\dot{x} = f(t,x) \qquad \left(\dot{x} = \frac{d}{dt}x\right) \tag{1.1}$$

in the time-space cylinder $[0,\infty) \times \mathbb{R}^n_+$ over the open positive cone $R^n_+ = \{x = (x_i) \in \mathbb{R}^n \big| 0 < x_i \text{ for all } i = 1,\dots,n\}$, $n \geq 2$. The right-hand sides $f = (f_i)$ taken into account are

$$f_i(t,x) = \psi_i\left(t, \left[b_i(t) \cdot \prod_{j\neq i} |x_j|^{\alpha_{ij}} - |x_i|^{\gamma_i}\right]\right) \cdot g_i(t,x). \tag{1.2}$$

Under the assumptions of section 2 below concerning the functions f_i we will show the blow up of all solutions having sufficiently large initial values $x(0) \in \mathbb{R}^n_+$ (Theorem 2.1).

Note that the requirements of Theorem 2.1 below do not imply that system (1.1) is quasimonotone (or cooperative) in the classical sense of [13, 3, 12].

In the case of more special right hand sides

$$f_i(t,x) = \left(b_i(t) \cdot \prod_{j\neq i} |x_j|^{\alpha_{ij}} - |x_i|^{\gamma_i}\right) \cdot |x_i|^{\zeta_i}, \tag{1.3}$$

with the assumptions of section 3 (which ensure the quasimonotonicity of the vector function $f = (f_i(t,x))$ from (1.3) with respect to x) we will construct lower and upper bounds for solutions $x(t)$ to (1.1) as well as for the time of their blowing up, (Theorem 3.1, Corollary 3.1). Some global results will be presented in Section 4, (Proposition 4.1 and its Corollaries).

Related autonomous systems

$$\dot{x} = f(x) \tag{1.4}$$

in $\mathbb{R}^n_+$ with right hand sides $f = (f_i)$,

$$f_i(x) = \psi_i \left(\prod_{j \neq i} |x_j|^{\alpha_{ij}} - |x_i|^{\gamma_i} \right) \cdot g_i(x) \tag{1.5}$$

have been studied in [7-10]. There we have assumed:

(1) $\psi_i : \mathbb{R} \to \mathbb{R}$ being continuous, odd, strictly increasing, $\psi_i(0) = 0$,
(2) $g_i : \mathbb{R}^n_+ \to \mathbb{R}^1_+$ being continuous, $g_i(x) \geq c_0$ if $x \geq E$ with constant $c_0 > 0$,
(3) $\alpha = (\delta_{ij}\gamma_j - \alpha_{ij})$, $\det \alpha \neq 0$,
(4) $\alpha_{ii} = 0 < \alpha_{ij}$, $i \neq j$, $0 < \gamma_i$.

Then the global behavior of the solutions $x(t)$ to (1.4), (1.5) is governed by the geometric properties of the matrix $\alpha = (\delta_{ij}\gamma_j - \alpha_{ij})$:

(a) In case α being M-matrix, a monotone family of attracting n-dimensional rectangles exists which are contracting to the single point set $\{E\}$, $E = (1, \ldots, 1)^T$ denoting the unit point in $\mathbb{R}^n_+$. Thus all solutions to (1.4) starting at $t = 0$ at some point $x_0 \in \mathbb{R}^n_+$ exist globally for all $t \geq 0$ and have the common limit set $\{E\}$, [10].
(b) Otherwise if there exist points $A \in \mathbb{R}^n_+$, $\delta \in \mathbb{R}^n_+$ fulfilling $\alpha \cdot A = -\delta$ (thus α being not an M-matrix), then the unique stationary point E of (1.5) is unstable, and a monotone family of attracting cones above E exists which are contracting to ∞. Therefore all solutions of (1.4), (1.5) starting about E in $\mathbb{R}^n_+$ blow up in finite or infinite time, [9].

1.2. **Applications.** By the well known comparison theorems [5, 13], our results below concerning the system (1.1), (1.2) with $f(t, x)$ being quasi-monotone increasing in $x \in R^n_+$ apply to suitable solutions $u(t, \cdot) : \Omega \to \mathbb{R}^n_+$ of weakly coupled quasi-monotone parabolic systems in a smoothly bounded domain $\Omega \subset \mathbb{R}^m$, $m \geq 1$, $z = (z_1, \ldots, z_m)$ denoting the spatial variable in $u = u(t, z)$:

$$\frac{\partial}{\partial t} u_i = F_i(t, u, u_{iz}, u_{izz}), \qquad (t, z) \in (0, T) \times \Omega, \tag{1.6}$$

where

$$F_i(t, x, 0, 0) \equiv f_i(t, x), \quad f_i \text{ from } (1.2) \tag{1.7}$$

for all $i = 1, \ldots, n$, with Dirichlet- or Neumann boundary conditions on $(0, T) \times \partial\Omega$, [5, 13].

The system (1.1), (1.2) is modelling a society of n members which are cooperative in a generalized sense: The i^{th} member has the prosperity function $x_i(t) > 0$. The exponent α_{ij} measures the support given from member i to member j, the exponent γ_i expresses the self-restriction of member i, while the functions b_i, ψ_i, g_i specify the increase of $x_i(t)$.

Similarly the parabolic system (1.6), (1.7) with Neumann boundary condition would describe some cooperative society with spatial prosperity diffusion in a domain Ω, where the flux of the prosperity functions $u_i(t,z)$ across the boundary $(0,T)\times\partial\Omega$ is given. On the other side, with Dirichlet boundary condition to (1.6), we would model a cooperative society with spatial prosperity diffusion in a domain Ω having a closed boundary on which the values of the prosperity functions are prescribed, c.p. [10] for the autonomous case.

Note: Aside from the short remarks above, here we will restrict us to ordinary differential equations of the general shape (1.1), (1.2). For a guidance to the rapidly evolving theory of nonlinear parabolic problems, the reader should consult the recent book [15] and the many citations there.

2. BLOW UP IN CASE OF LARGE INITIAL VALUES

By any point $a = (a_i) \in \mathbb{R}^n_+$ we define the closed cone

$$Q_a = \{x \in \mathbb{R}^n \mid a \le x\},$$

a being its lowest point. We denote the $n-1$-dimensional faces of Q_a by

$$Q_{a,i} = \{x \in Q_a \mid x_i = a_i\}, \quad i = 1,\dots,n.$$

Here and below we use the partial order of $\mathbb{R}^n$ explained by $x \le y \Leftrightarrow x_i \le y_i$, $x < y \Leftrightarrow x_i < y_i$ for all $i = 1,\dots,n$, and we also will write $x \le c$ or $x \in [c, C]$ if $x_i \le c \in \mathbb{R}$ or $c \le x_i \le C$, respectively, holds for all $i = 1,\dots,n$. For any two points $a, b \in \mathbb{R}^n$, $a < b$, by $[a,b]$ we will denote the n-dimensional closed interval $[a,b] = \{x \in \mathbb{R}^n \mid a \le x \le b\}$.

Proposition 2.1. *Assume the continuous map $f = (f_i) : [0,T) \times \mathbb{R}^n_+ \longrightarrow \mathbb{R}^n$ obeys the following direction condition for its restriction $f_i(t,\cdot)|_{Q_{a,i}}$ to the face $Q_{a,i}$:*

$$f_i(t,\cdot)|_{Q_{a,i}} > 0 \quad \text{for all } i = 1,\dots,n \text{ and all } t \in [0,T). \tag{2.1}$$

Then the closed cone Q_a is flow invariant for the differential equation

$$\dot{x} = f(t,x) \qquad \left(\dot{x} = \frac{d}{dt}x\right), \tag{2.2}$$

i.e. each solution $x(t)$ of (2.2) starting at any point $x(0) \in Q_a$ will remain in Q_a for all t of its right maximal interval of existence in $[0,T)$.

Note: Since (2.1) excludes $f_i(t,x) = 0$ for any $t \in [0,T)$, $x \in Q_{a,i}$, flow invariance of Q_a does not require any uniqueness condition for (2.2).

Proof. By contradiction: Let $x(t)$ for $t \in [0,t_1]$ denote a solution of (2.2) starting at $x(0) \in Q_a$, but reaching $x(t_1) \notin Q_a$. Consequently there exists

$$t^* = \sup\{t \in [0,t_1] \mid x(t) \in Q_a\} < t_1,$$

and we conclude $x(t^*) \in \partial Q_a = \bigcup_{i=1}^{n} Q_{a,i}$ because of the continuity of $x(t)$. Consider the set of indices $J = \{i \mid x_i(t^*) = a_i\} \neq \emptyset$ and its complementary set $J' = \{1,2,\ldots,n\} \setminus J$. From (2.1) we see $0 < \epsilon_1 = \min_{i \in J}\{f_i(t^*, x(t^*))\}$, and in addition $0 < \epsilon_2 = \min_{j \in J'}\{x_j(t^*) - a_j\}$ because of $x(t^*) \in \partial Q_a \subset Q_a$. The solution $x(t)$ and $f(t,x(t))$ being continuous we see that there exists $\delta \in (0, t_1 - t^*)$ such that

(a) $\min_{i \in J}\{f_i(t,x(t))\} \geq \frac{\epsilon_1}{2}$, and
(b) $\min_{j \in J'}\{x_j(t) - a_j\} \geq \frac{\epsilon_2}{2}$

hold for all $t \in [t^*, t^* + \delta]$. But then from (b) and the consequence of (a), namely

$$x_i(t) - a_i = \int_{t^*}^{t} f_i(\tau, x(\tau))d\tau > 0$$

for all $t \in (t^*, t^* + \delta]$ and all $i \in J$ we find $x(t) \in Q_a$ for $t \in [t^*, t^* + \delta]$ in contradiction to the definition of t^*.

We consider the initial value problem

$$\dot{x} = f(t,x)\,, \quad x(0) \in \mathbb{R}^n_+ \tag{2.3}$$

with the vector function $f(t,x) = (f_i)$,

$$f_i(t,x) = \psi_i\left(t, \left[b_i(t) \cdot \prod_{j \neq i} |x_j|^{\alpha_{ij}} - |x_i|^{\gamma_i}\right]\right) \cdot g_i(t,x). \tag{2.4}$$

For all $t \geq 0$ and $i = 1, \ldots, n$ we require:

$$\psi_i : [0,\infty) \times \mathbb{R} \to \mathbb{R} \text{ being continuous in } (t,r) \in [0,\infty) \times \mathbb{R}, \tag{2.5}$$

odd, strictly monotone increasing and continuously differentiable with respect to its second argument r, $\psi_i(t,0) \equiv 0$, $c_\psi \leq \frac{\partial}{\partial r}\psi_i(t,r) \leq C_\psi$ for all $t \geq 0$, $r \in \mathbb{R}$ with constants c_ψ, C_ψ, $0 < c_\psi < C_\psi$.

$$b_i : [0,\infty) \to \mathbb{R}^1_+ \quad \text{continuously differentiable,} \tag{2.6}$$

$c_b \leq b_i \leq C_b$, $c_{\dot{b}} \leq \dot{b}_i(t) \leq C_{\dot{b}}$ with constants c_b, C_b, $c_{\dot{b}} \in \mathbb{R}$, $C_{\dot{b}}$ and $0 < c_b < C_b$, $c_{\dot{b}} < C_{\dot{b}}$.

$$g_i : [0,\infty) \times \mathbb{R}^n_+ \to \mathbb{R}^1_+ \quad \text{continuous, fulfilling} \tag{2.7}$$

$g_i(t,x) \geq c_g$ for all $x \geq \epsilon^* > 0$ with some constants ϵ^*, $c_g > 0$.

$$\text{constants } \alpha_{ii} = 0 < \alpha_{ij}, \quad i \neq j, \quad 0 < \gamma_i, \tag{2.8}$$

the matrix $\alpha = (\delta_{ij}\gamma_j - \alpha_{ij})$ having the determinant $\det\alpha \neq 0$ (δ_{ij} denoting Kronecker's index $\delta_{ii} = 1$, $\delta_{ij} = 0$ for $i \neq j$).

(2.9) There exist some vectors $A = (A_i) \in \mathbb{R}^n_+$, $\quad \delta = (\delta_i) \in \mathbb{R}^n_+$ fulfilling $\alpha \cdot A = -\delta$.

Note: Because of (2.9), α is not $M-$matrix, i.e. its inverse matrix $\alpha^{-1} = (\beta_{ij})$ being not positive [1].

Proposition 2.2. *Assume* (2.3)–(2.9). *Then* (1) *in* $\mathbb{R}^n_+$ *for all* $t \geq 0$, *the vector function* $f(t,x) = (f_i)$ *has the unique critical point* $\xi = (\xi_i(t)) \in \mathbb{R}^n_+$,

$$\xi_i(t) = \prod_{j=1}^{n} b_j^{\beta_{ij}}(t), \qquad \textit{where } (\beta_{ij}) = \alpha^{-1}, \tag{2.10}$$

$c_\xi \leq \xi_i(t) \leq C_\xi$, $c_{\dot\xi} \leq \dot\xi_i \leq C_{\dot\xi}$ *with constants* $c_\xi \in (0,1]$ *and* $C_\xi > c_\xi$, $c_{\dot\xi} \in \mathbb{R}$, $C_{\dot\xi} > c_{\dot\xi}$.

(2) *Written in terms of the variables* $y_i = x_i \cdot \xi_i^{-1}(t)$, *the initial value problem* (2.3), (2.4) *reads*

$$\dot y = F(t,y)\,, \qquad y(0) = (x_i(0)\cdot \xi_i^{-1}(0)) \tag{2.11}$$

with the vector function $F(t,y) = (F_i)$,

$$\begin{aligned} F_i(t,y) &= F_i^*(t,y)\cdot G_i(t,y) - \dot\xi_i(t)\cdot \xi_i^{-1}(t)\cdot y_i\,, \ \textit{ where} \\ F_i^*(t,y) &= \psi_i\left(t, \xi_i^{\gamma_i}(t)\left[\prod_{j\neq i}|y_j|^{\alpha_{ij}} - |y_i|^{\gamma_i}\right]\right), \\ G_i(t,y) &= g_i(t,(y_i\cdot\xi_i(t)))\cdot\xi_i^{-1}(t) \quad \textit{for all } i = 1,\dots,n. \end{aligned} \tag{2.12}$$

Proof: (1) Because of (2.5), (2.6) the equation $f(t,x) = 0$ holds with $x \in \mathbb{R}^n_+$ if and only if we have $x_i^{\gamma_i} = b_i(t)\cdot\prod_{j\neq i} x_j^{\alpha_{ij}}$ or, equivalently, $x_i = \xi_i(t)$ from (2.10), where the bounds have to be fixed in dependence on the bounds for $b_i(t)$, $\dot b_i(t)$ in (2.6).

(2) Recalling our assumption (2.6) on $b_i(t)$ and writing $x_i = y_i\cdot\xi_i(t)$ in (2.3), (2.4) we get the system (2.11), (2.12).

Remark 2.1. From (2.5), (2.6), (2.10) we see that the functions $F_i^*(t,y)$ in (2.12) have continuous gradients with respect to $y \in \mathbb{R}^n_+$ which are uniformly bounded on $[0,\infty)\times K$ for each compact subset $K \subset \mathbb{R}^n_+$, i.e. for each such K there exists some constant $C_{F^*,K}$ with

$$|\nabla_y F_i^*(t,y)| \leq C_{F^*,K} \tag{2.13}$$

for all $y \in K$, $t \geq 0$, $i = 1,\dots,n$.

Remark 2.2. Because of our requirement (2.8), in $\mathbb{R}^n_+$ the system (2.11), (2.12) has the unique critical point $y = E = (1,\dots,1)^T$ for all $t \geq 0$. Evidently the inequalities $y > E$ and $(x_i) \equiv (y_i\cdot\xi_i(t)) > (\xi_i)$ imply each other for any $y \in \mathbb{R}^n_+$.

Remark 2.3. Writing (2.3), (2.4) on $[0,\infty)\times\mathbb{R}^n_+$ in terms of variables $z_i = x_i^{\frac{1}{\mu}}$ with any $\mu > 0$, we get an equivalent system now having the new exponents $\mu\cdot\alpha_{ij}$, $\mu\cdot\gamma_i$ in place of α_{ij}, γ_i (and, of course, with analogous new constants in the estimates for b_i, ξ_i). The requirements (2.8), (2.9) remain valid for the new exponents, (2.9) with $\mu\cdot\delta_i$ instead of δ_i. Thus without any restriction of the generality, below in order to simplify our estimates, we always will suppose

$$1 < \gamma_i, \quad 1 \le A_i, \quad 1 \le \delta_i \text{ for all } i = 1,\dots,n, \text{ and } \epsilon^* \le c_\xi. \tag{2.14}$$

Proposition 2.3. *Assume* (2.3)–(2.9), (2.14), *and* $s \gg 1$, $a_s = (a_{s,i}) = (s^{A_i}) \in \mathbb{R}^n_+$, $Q_s = \{y \in \mathbb{R}^n \big| a_s \le y\}$. *Then*

(1) *there exists* $s^* > 1$ *and some constant* $c_F > 0$ *such that for all* $s \ge s^*$ *we have*

$$c_F \le F_i(t,y) \quad \text{for all } y \in Q_{s,i}, \quad t \ge 0, \quad i = 1,\dots,n, \tag{2.15}$$

(2) *the cone* Q_s *being flow invariant with respect to* (2.11), (2.12) *for all* $s \ge s^*$.

Proof. (1) In (2.12) the vector function

$$F^* = (F_i^*(t,y)), \quad F_i^*(t,y) = \psi_i\left(t, \xi_i^{\gamma_i}(t)\cdot\left[\prod_{j\ne i}|y_j|^{\alpha_{ij}} - |y_i|^{\gamma_i}\right]\right)$$

(with prescribed function $\xi_i(t) \ge c_\xi > 0$) being quasimonotone increasing in y because of (2.5) and $\alpha_{ij} \ge 0$, from (2.5), (2.8)–(2.10) we conclude

$$F_i^*(t,y) \ge F_i^*(t,a_s) \ge c_\psi\cdot c_\xi^{\gamma_i}\cdot s^{\gamma_i\cdot A_i} = c_s \tag{2.16}$$

for all $y \in Q_{s,i}$, $t \ge 0$ if $s \ge 2$.

Recalling (2.7), (2.10), from (2.12), (2.16) we see

$$F_i(t,y) \ge \{c_g\cdot c_\psi\cdot c_\xi^{\gamma_i}\cdot s^{(\gamma_i-1)A_i} - C_{\dot\xi}\}\cdot s^{A_i}\cdot C_\xi^{-1} \tag{2.17}$$

for all $y \in Q_{s,i}$, $t \ge 0$, $i = 1,\dots,n$, if $s \ge 2$. Thus an obvious calculation shows that (2.15) certainly holds true for all $s \ge s^*$, if we take

$$s^* = \max\left\{\left(\left[C_{\dot\xi}+1\right]\cdot c^{-1}\right)^{\frac{1}{(c_\gamma-1)\cdot c_A}}, (c_F C_\xi)^{\frac{1}{c_A}}, 2\right\}, \tag{2.18}$$

where

$$1 < c_\gamma = \min_i\{\gamma_i\}, C_\gamma = \max_i\{\gamma_i\}, \quad 1 \le c_A = \min_i\{A_i\}, \tag{2.19}$$

$$c = c_g\cdot c_\psi\cdot c_\xi^{C_\gamma}. \tag{2.20}$$

The inequalities hold because of our assumption (2.14).

(2) From (2.15) the flow invariance of Q_s with respect to (2.11), (2.12) results by Proposition 2.1 for all $s \ge s^*$.

The following subsets of $\mathbb{R}^n_+$ have evident geometric relations to the cones

$$Q_s = \{y \in \mathbb{R}^n \mid a_s \le y\} \quad \text{where } a_s = (s^{A_i}), \quad s > 1.$$

For $\epsilon = (\epsilon_i) \in \mathbb{R}^n_+$, $\epsilon_i < s^{A_i} - 1$, we will consider the ϵ-neighbourhood of Q_s:

$$Q_s^\epsilon = \{y \in \mathbb{R}^n_+ \mid s^{A_i} - \epsilon_i \le y_i \quad \text{for all } i = 1, \dots, n\},$$

ϵ-retract of Q_s:

$$Q_s^{-\epsilon} = \{y \in \mathbb{R}^n_+ \mid s^{A_i} + \epsilon_i \le y_i \quad \text{for all } i = 1, \dots, n\},$$

η-cone near Q_s with any $\eta = (\eta_i) \in \mathbb{R}^n$, $|\eta_i| < s^{A_i} - 1$:

$$Q_s^\eta = \{y \in \mathbb{R}^n_+ \mid s^{A_i} - \eta_i \le y_i \quad \text{for all } i = 1, \dots, n\},$$

and the ϵ-neighbourhood of the i^{th} $n-1$-dimensional face $Q_{s,i}$ of Q_s:

$$Q_{s,i}^\epsilon = \{y \in Q_s^\epsilon \mid y_i \le s^{A_i} + \epsilon_i\}.$$

Remark 2.4. For all $s > 1$, $\epsilon \in \mathbb{R}^n_+$ with $\epsilon_i < s^{A_i} - 1$ there exists $\sigma > s$ and $\tau \in (1, s)$ such that

(a) $Q_s^{-\epsilon} \subset Q_\sigma \subset Q_s$ and
(b) $Q_\sigma \subset Q_s^\epsilon \subset Q_\tau$ hold.

Corollary 2.1. *Let us assume* (2.4)–(2.9), (2.14).

(1) *Then there exist* $s^{**} \ge s^*$, $\delta^* > 0$ *such that*

$$o < \frac{c_F}{2} \le F_i(t, y) \tag{2.21}$$

holds true for all $y \in Q_{s,i}^\epsilon$, $0 < \epsilon = (\epsilon_i) \le \delta^*$, $s \ge s^{**}$, $t \ge 0$, $i = 1, \dots, n$ *with* c_F *from* (2.15).

(2) *Thus the cone* Q_s^η *is flow invariant with respect to* (2.11), (2.12) *for all* $\eta = (\eta_i) \in \mathbb{R}^n, |\eta_j| \le \delta^*$, $t \ge 0$.

Proof: (1) We take $0 < \epsilon = (\epsilon_i) \le \delta^*$ with some $\delta^* > 0$. In order to find a positive lower bound for the restriction $F_i(t, \cdot)|_{Q_{s,i}^\epsilon}$ of $F_i(t, \cdot)$, recalling (2.16) we project each point $y \in Q_{s,i}^\epsilon$ on the point $y(i) \le y$, $y(i)$ having the coordinates.

$$y(i)_i = y_i \, , \; y(i)_j = a_{s,j} - \epsilon_j \quad \text{for } j \ne i,$$

thus $y(i)$ being the lowest point of the $n-1$-dimensional cone of all points $z \in Q_{s,i}^\epsilon$ which belong to the hyperplane $z_i = y_i$. Since $y(i)$ differs from $a_s - \epsilon$ in the i^{th} coordinate only, we get $|y(i) - (a_s - \epsilon)| \le 2\epsilon_i$.

The quasimonotonicity of the first factor $F^*(t, \cdot) = (F_i^*(t, \cdot))$ of $F(t, \cdot)$ from (2.12) gives $F_i^*(t, y(i)) \le F_i^*(t, y)$ for all i and all $t \ge 0$. Recalling Remark 2.1 we take $\delta^* \in (0, 1)$ so small that

$$|F_i^*(t, a_s + u) - F_i^*(t, a_s + v)| \le \frac{c_s}{4} \quad \text{with } c_s \text{ from (2.16)}$$

for all $u, v \in \mathbb{R}^n$, $|u_i|, |v_i| \le 2\delta^*$, all $t \ge 0$, and $i = 1, \dots, n$. Then from (2.16) we see

$$\begin{aligned}(2.22)\qquad \frac{c_s}{2} &\le F_i^*(t, a_s) - |F_i^*(t, a_s) - F_i^*(t, a_s - \epsilon)| \\ &\quad - |F_i^*(t, a_s - \epsilon) - F_i(t, y(i))| \le F_i^*(t, y(i)) \le F_i^*(t, y).\end{aligned}$$

Recalling our bounds in (2.7), (2.10), (2.14), (2.19) and noting $y_i \in [s^{A_i} - \epsilon_i, s^{A_i} + \epsilon_i]$ for $y \in Q^{\epsilon}_{s,i}$, from (2.12), (2.22) we get

$$(2.23)\qquad \frac{1}{2} s^{A_i} \cdot C_\xi^{-1} \cdot \{c_g c_\psi c_\xi^{C_\gamma} \cdot s^{(c_\gamma - 1)c_A} - 2C_{\dot{\xi}}\} \le F_i(t, y).$$

Finally from (2.23), by a short calculation we find the sufficient condition

$$(2.24)\qquad s \ge s^* = \max\left\{([4C_{\dot{\xi}} + 1]c^{-1})^{\frac{1}{(c_\gamma - 1)c_A}}, (c_F C_\xi)^{\frac{1}{c_A}}, 2\right\}$$

for the direction condition (2.21) being valid.

(2) For all $\eta \in \mathbb{R}^n, (|\eta_i| \le \epsilon$, each $n-1$-dimensional face $(Q_s^\eta)_i$ of the cone Q_s^η near Q_s is contained in the ϵ-neighbourhood $(Q_{s,i})^\epsilon$ of $Q_{s,i}$. Therefore with s^{**}, δ^* from (1) and any $\epsilon, 0 < \epsilon \le \delta^*$, the direction condition (2.21) holds true on all faces $(Q_s^\eta)_i$ for all $s \ge s^{**}$, $t \ge 0$. But then Proposition 2.1 shows the flow invariance of the cone Q_s^η with respect to (2.11), (2.12).

Corollary 2.2. *Let $y(t) \in \mathbb{R}^n_+$ for $t \in [0, \infty)$ denote a solution of* (2.11), (2.12).

Then for all $s \ge s^{**}$, with s^{**} from the last Corollary 2.1 *there exist some $\delta(s) \in \mathbb{R}^n_+$ and $\tau > 0$, such that $y(t_0) \in Q_s^{\delta(s)}$ implies $y(t_0 + \tau) \in Q_s^{-\delta(s)}$.*

Proof. (c.p. [9, p. 675): By definition of the η-neighbourhood of Q_s, we have $x \in Q_s^\eta$ if and only if for each $i = 1, \dots, n$ either (a) $a_{s,i} + \eta_i \le y_i(t)$ or (b) $a_{s,i} - \eta_i \le y_i(t) < a_{s,i} + \eta_i$ holds. Taking $\eta \in (0, \delta^*]$, from Corollary 2.1 we find that inequality (a) is flow invariant for (2.11), (2.12): If it holds for any $t_0 \ge 0$ then it holds for all $t \ge t_0$, too. If additionally we require $2\eta \le \tau \cdot \frac{c_F}{2}$, from (b) for $t = t_0$ we conclude:

$$a_{s,i} + \eta_i \le a_{s,i} - \eta_i + \tau \cdot \frac{c_F}{2} \le y_i(t_0 + \tau)$$

for each $i = 1, \dots, n$, thus $y(t_0 + \tau) \in Q_s^{-\eta}$.

Theorem 2.1. *Assume* (2.4)–(2.9), (2.14). *Let $y(t)$ denote any solution of* (2.11), (2.12), *$[0, T)$ being its right maximal interval of existence, $y(0) \in Q_{s^{**}}$ with s^{**} from Corollary* 2.1.

Then either (1)*: with $t \to T \le \infty$, $y(t)$ is entering each $Q_s, s > s^{**}$, or* (2)*: $\tilde{s} = \sup\{s \ge s^{**} \mid \exists\, t = t_s, y(t_s) \in Q_s\} < \infty$ holds, and $|y(t_s)| \to \infty$ with $s \to \tilde{s}$ (thus $t_s \to T$).*

Proof [9, p. 675–676]: The supremum $\tilde{s}$ is well defined since $y(t) \in Q_s$ holds for some $t \ge 0$, $s \ge s^{**}$ because of $y(0) \in Q_{s^{**}}$, and $\tilde{s} > s^{**}$ results from Corollary 2.2. (a) In case $\tilde{s} = \infty$ we have (1). (b) Otherwise in case $\tilde{s} < \infty$, there exist a strictly increasing sequence $(s_k) \uparrow \tilde{s}$ and a sequence $(t_k) \subset [0,T)$ with $y(t_k) \in Q_{s_k}$, where the sequence (Q_{s_k}) is contracting to the closed cone $Q_{\tilde{s}}$.

(b1) If $(y(t_k))$ does not contain a bounded subsequence, we have (2).

(b2) Otherwise there would exist a bounded subsequence $(y(t_{k'})) \subset (y(t_k))$, $|y(t_{k'})| \leq M < \infty$. But then $(y(t_{k'}))$ contains a convergent subsequence $(y(t_{k''})) \to \tilde{y}$, thus $\tilde{y} \in \partial Q_{\tilde{s}} = \bigcup_{i=1}^{n} Q_{\tilde{s},i}$. Without loss of generality we may assume that the related sequences $(t_k), (t_{k''})$ are monotone increasing because each $Q_s, s \geq s^{**}$, is flow invariant for (2.11), (2.12).

(b2.1) In case $(t_{k''})$ being bounded there exists $\tilde{t} = \lim_{k'' \to \infty} t_{k''} \in (0,T]$. Thus by the extension theorem for ordinary differential equations [2, p. 13-14, Lemma 3.1] the solution $y(t)$ can be extended to $[0,\tilde{t}\,]$ with $y(\tilde{t}) = \tilde{y}$, and subsequently to $[0, \tilde{t} + \tau)$ for some $\tau > 0$ by the local existence theorem. However, as shown in Corollary 2.1, the vector $F(\tilde{t}, y(\tilde{t}))$ is pointing strictly inwards to $\overset{\circ}{Q}_{\tilde{s}} = \bigcup_{s > \tilde{s}} Q_s$. From this we get $y(t) \in Q_s$ for some $s > \tilde{s} > s^{**}$ in contradiction to the definition of $\tilde{s}$.

(b2.2) Otherwise the sequence $(t_{k''})$ is unbounded: We have $(t_{k''}) \uparrow \infty = T$. Then because of $Q_{s_k} \downarrow Q_{\tilde{s}}$, for each $\eta \in \mathbb{R}^n_+$ there exists some $k_\eta \in \mathbb{N}$ such that $Q_{s_{k''}} \subset (Q_{\tilde{s}})^\eta$ for all $k'' \geq k_\eta$. Choosing $\eta \in (0, \delta(\tilde{s}))$ and recalling Corollary 2.2, from $y(t_{k''}) \in Q^\eta_{\tilde{s}}$ we find $y(t_{k''} + \tau) \in Q^{-\eta}_{\tilde{s}} \subset \bigcup_{s > \tilde{s}} Q_s$ for some $\tau > 0$, which again contradicts the definition of $\tilde{s}$.

An evident consequence of Theorem 2.1 is the following

Corollary 2.3. *The cone* $Q^*_{s^{**}} = \{x = (x_i) \in \mathbb{R}^n_+ \big| C_\xi \cdot (s^{**})^{A_i} \leq x_i \text{ for all } i = 1, \ldots, n\}$ *belongs to the domain of attraction of* ∞ *with respect to* (2.3), (2.4).

Proof: Recalling $x(t) \equiv \xi(t) \cdot y(t)$, Remark 2.2 and the uniform lower and upper bounds for $\xi(t)$ in (2.10), we find the claim of Corollary 2.3 from Theorem 2.1.

3. CONSTRUCTION OF SUB- AND SUPERSOLUTIONS

In addition to Theorem 2.1, to systems (2.3) with more special right hand sides than in (2.4) we will find explicit subsolutions as well as supersolutions, both types blowing up in finite time, by the well known comparison method [13]. With respect to $x \in \mathbb{R}^n_+$ the systems below are quasimonotone in the sense of [13] and also cooperative in the sense of [3, 12].

We consider the initial value problem

$$\dot{x} = f(t,x), \quad x(0) = x_0 \in \mathbb{R}^n_+\,, \tag{3.1}$$

with the vector function $f = (f_i) : [0,\infty) \times \mathbb{R}^n_+ \to \mathbb{R}^n$,

$$f_i(t,x) = (b_i(t) \cdot \prod_{j \neq i} |x_j|^{\alpha_{ij}} - |x_i|^{\gamma_i}) \cdot |x_i|^{\zeta_i}\,, \quad i = 1, \ldots, n, \tag{3.2}$$

where $\zeta_i \in \mathbb{R}$ are prescribed real constants. Concerning $b_i(t)$ and α_{ij}, γ_i we require again (2.6), (2.8), (2.9), (2.14). Thus the unique critical point $\xi(t) \in \mathbb{R}^n_+$ of $f(t,x)$ is given by (2.10) in Proposition 2.2 for all $t \geq 0$.

Additionally we assume

$$A_i \cdot (1 - [\gamma_i + \zeta_i]) < \delta_i\,, \quad i = 1, \ldots, n, \tag{3.3}$$

and with any $c_0 \in (0,1)$ we define the real constants

$$M = \inf_{i,t \geq 0} \left\{ c_0 \cdot [A_i \cdot \xi_i^{1-[\gamma_i+\zeta_i]}(t)]^{-1} \right\}, \tag{3.4}$$

$$N = \sup_{i,t \geq 0} \left\{ \dot{\xi}_i(t) \cdot [c_0 \cdot \xi_i^{\gamma_i+\zeta_i}(t)]^{-1} \right\}, \tag{3.5}$$

$$\mu = \min_i \left\{ \delta_i - A_i(1 - [\gamma_i + \zeta_i]) \right\}, \tag{3.6}$$

$$\varphi_0 \geq (1 - c_0)^{-1} \text{ and } \varphi_0^{\mu} > N. \tag{3.7}$$

In the following we will only consider the case $N \neq 0$. Otherwise if $N = 0$, lower and upper bounds for solutions $x(t)$ to (3.1), (3.2) and their blow up time can be found in a similar manner.

Theorem 3.1. *Let $x(t) = (x_i)$ denote the solution of* (3.1), (3.2) *on its right maximal interval $[0, T)$ of existence. Then for each sufficiently large initial value*

$$x(0) = (x_i(0)) \geq (\xi_i(0) \cdot \varphi_0^{A_i}) \in \mathbb{R}^n_+ \tag{3.8}$$

we find: The solution $\varphi : [0, T^*) \to \mathbb{R}^1_+$ of the initial value problem

$$\dot{\varphi} = M \cdot \varphi \cdot \{\varphi^{\mu} - N\}\,, \quad \varphi(0) = \varphi_0 \tag{3.9}$$

gives the lower bound

$$v(t) = (v_i) = (\xi_i(t) \cdot \varphi^{A_i}(t)) \leq x(t) \tag{3.10}$$

for all $t \in [0, T)$, the function $\varphi(t)$ being strictly monotone blowing up to ∞ with

$$t \uparrow T^* = \frac{-\ln(1 - N \cdot \varphi_0^{-\mu})}{\mu \cdot M \cdot N} = \frac{ln(1 + (-N)\varphi_0^{-\mu})}{\mu \cdot M \cdot (-N)} > 0, \tag{3.11}$$

thus the real number

$$T^* \geq T \tag{3.12}$$

represents an upper bound for T.

Proof. Recalling the unique critical point $\xi(t) = (\xi_i(t)) \in \mathbb{R}^n_+$ of the vector function $f(t,x) = (f_i)$ from (3.2), ξ given again by (2.10) in Proposition 2.2, and using the transformed variables $y_i = \xi_i^{-1}(t) \cdot x_i$, from (3.1), (3.2) we are led to the equivalent system

$$\dot{y}_i = |y_i|^{\zeta_i} \cdot \left(\prod_{j \neq i} |y_j|^{\alpha_{ij}} - |y_i|^{\gamma_i} \right) \cdot \xi_i^{\gamma_i + \zeta_i - 1} - \dot{\xi}_i \cdot \xi_i^{-1} \cdot y_i \tag{3.13}$$

$$= H_i(t,y), \tag{3.14}$$

$$y_i(0) = \xi_i^{-1}(0) \cdot x_i(0)\,, \quad i = 1, \ldots, n.$$

Since the right hand side $H = (H_i(t,y))$ is quasimonotone increasing in $y \in R^n_+$ and locally Lipschitz continuous, each solution $w(t) = (w_i)$ of

$$\dot{w}_i \leq H_i(t,w)\,, \quad w_i(0) \leq y_i(0)\,, \quad i = 1, \ldots, n, \tag{3.15}$$

defines a subfunction $w(t) \leq y(t)$ to $y(t)$ from (3.13), thus $v(t) = (v_i) = (\xi_i(t) \cdot w_i(t)) \leq x(t)$ on the right maximal interval $[0,T) \subset [0,T^*)$, where these solutions exist, [13, pp. 65, 69, 92–96].

Setting

$$w_i(t) = \varphi^{A_i}(t) \tag{3.16}$$

with some function $\varphi(t) > 0$ which has to be calculated, we get

$$\dot{w}_i = A_i \cdot \varphi^{A_i - 1} \cdot \dot{\varphi} \tag{3.17}$$

and

$$H_i(t,w) = \xi_i^{\gamma_i + \zeta_i - 1}(t) \cdot \varphi^{A_i(\gamma_i + \zeta_i) + \delta_i}(t) \cdot (1 - \varphi^{-\delta_i}) - \dot{\xi}_i \cdot \xi_i^{-1} \cdot \varphi^{A_i} \tag{3.18}$$

because of $-\alpha \cdot A = \delta$ in (2.9).

A short calculation shows that (3.15), (3.17), (3.18) are fulfilled all together for all $i = 1, \ldots, n$, $t \geq 0$, if we define $w_i(t)$ in (3.16) by the solution $\varphi(t)$ of (3.9). Evidently, this solution $\varphi(t)$ from

$$\begin{aligned} \varphi^\mu(t) &= \varphi_0^\mu \cdot N \cdot \{\varphi_0^\mu - e^{\mu M N t} \cdot (\varphi_0^\mu - N)\}^{-1} \\ &= \varphi_0^\mu \cdot (-N) \cdot \{e^{\mu M N t} \cdot (\varphi_0^\mu - N) - \varphi_0^\mu\}^{-1} \end{aligned} \tag{3.19}$$

calculated to the Bernoulli-type equation (3.9), exists on $[0,T^*)$, T^* from (3.11), $\varphi(t)$ being strictly monotone blowing up with $t \uparrow T^*$ because of our second assumption in (3.7), the first assumption in (3.7) implying $0 < c_0 \leq 1 - \varphi_0^{-\delta_i}$ for all i. Similarly, requiring

$$\dot{w}_i^* \geq H_i(t,w^*)\,, \quad w_i^*(0) \geq y_i(0), \quad i = 1, \ldots, n, \tag{3.20}$$

each solution $w^*(t) = (w_i^*)$ defines a superfunction $w^*(t) \geq y(t)$ to $y(t)$ from (3.13), thus $v^*(t) = (v_i^*) = (\xi_i(t) \cdot w_i^*(t)) \geq x(t)$ on the right maximal interval $[0, T_*) \subset [0, T^*)$, where $w^*(t)$ exists.

Now setting

$$M_* = \sup_{i,t\geq 0}\{[A_i \cdot \xi_i^{1-(\gamma_i+\zeta_i)}(t)]^{-1}\}, \tag{3.21}$$

$$N_* = \inf_{i,t\geq 0}\{\dot{\xi}_i(t) \cdot \xi_i^{-(\gamma_i+\zeta_i)}\} \neq 0, \tag{3.22}$$

$$\mu_* = \max_i\{\delta_i - A_i(1 - \cdot[\gamma_i + \zeta_i])\}, \tag{3.23}$$

$$\varphi_{0*} > \varphi_0\,, \quad \varphi_{0*}^{\mu_*} > N_*, \tag{3.24}$$

we solve (3.20) by $w_i^*(t) = \varphi_*^{A_i}(t)$ for all $i = 1, \ldots, n$, getting

$$v^*(t) = (v_i^*) = (\xi_i(t) \cdot \varphi_*^{A_i}(t)), \tag{3.25}$$

now φ_* denoting the solution of the initial value problem

$$\dot{\varphi}_* = M_* \cdot \varphi_* \cdot \{\varphi^{\mu_*} - N_*\}, \varphi_*(0) = \varphi_{0*} \tag{3.26}$$

on its right maximal interval of existence $[0, T_*)$, where

$$T_* = \frac{-\ln(1 - N_*\varphi_{0*}^{-\mu_*})}{\mu_* \cdot M_* \cdot N_*} \leq T \leq T^*, \tag{3.27}$$

quite analogous to (3.11). The latter two inequalities hold true by definition of the superfunction $w^*(t) \geq y(t)$. Thus we have proved

Corollary 3.1. *Each solution $x(t)$ to the initial value problem* (3.1)*,* (3.2)*, $x(t)$ having any initial value $(x_i(0)) \in [(\xi_i(0) \cdot \varphi_0^{A_i}),\ (\xi_i(0) \cdot \varphi_{0*}^{A_i})]$, blows up at some finite time*

$$T \in [T_*, T^*]. \tag{3.28}$$

The bounds φ_0, $\varphi_{0,*}$, T_*, T^* are given in (3.7)*,* (3.24)*,* (3.11)*,* (3.27)*, respectively.*

4. GLOBAL BOUNDS IMPLYING GLOBAL SOLUTIONS

The initial value problem (3.26) with large $\varphi_{0*} > 0$ has the global solution $\varphi_*(t)$ existing for all $t \in [0, \infty)$ certainly, if instead of (3.3) we require $\mu_* \leq 0$. Then our construction in section 3 leads to global sub- and superfunctions $v(t), v_*(t)$ to (3.1), (3.2), and the extension theorem for ordinary differential equations shows that the solution $x(t) \in [v(t), v_*(t)]$ to this initial value problem has the right maximal interval of existence $[0, \infty)$.

Remark 4.1. Because of (2.9) $\alpha \cdot A = -\delta$, the matrix $\alpha = (\delta_{ij}\gamma_j - \alpha_{ij})$ is not M-matrix, and our new assumption $\mu_* = \max_i\{\delta_i - A_i[1 - (\gamma_i + \zeta_i)]\} \leq 0$ together with (2.9) gives the inequality $(\alpha_{ij}) \cdot A \leq (\delta_{ij} \cdot [1 - \zeta_i]) \cdot A$. Therefore we must take each ζ_i small or even negative large enough in order to fulfill the latter inequality and to avoid the estimate $1-\zeta_i<\gamma_i$, which, holding for all $i=1,\ldots,n$, would imply $(\alpha_{ij})\cdot A<(\delta_{ij}\cdot\gamma_j)\cdot A$, thus α being M-matrix [1, p. 137].

Proposition 4.1. *We assume* (2.6), (2.8), (2.9), (2.14). *Recalling the notations* (3.4)–(3.6), (3.21)–(3.23), *we require* (3.7), (3.24) *and*

$$\mu\le\mu_*<0\,,\quad N_*\le N<0. \tag{4.1}$$

(1) *Then the solutions* $\varphi(t)$ *to* (3.9), $\varphi_*(t)$ *to* (3.26) *as well as the sub- and super-functions* $v(t)$ *from* (3.10), $v_*(t)$ *from* (3.25) *exist for all* $t\in[0,\infty)$.

(2) *For all initial values*

$$x_0=(x_i(0))\in[(\xi_i(0)\cdot\varphi_0^{A_i}),(\xi_i(0)\cdot\varphi_{0*}^{A_i})], \tag{4.2}$$

the solution $x(t)$ *to* (3.1), (3.2) *fulfills*

$$x(t)\in[v(t),v^*(t)] \tag{4.3}$$

for all t *of its right maximal interval of existence* $[0,T)$.

Proof: With the requirements of Proposition 4.1, the solution $\varphi(t)$ to (3.9) is presented by (3.19) or, equivalently, by

$$\varphi(t)=\varphi_0\cdot(-N)^{\frac{1}{\mu}}\cdot[e^{(-\mu)\cdot M\cdot(-N)\cdot t}\cdot(\varphi_0^{\mu}+(-N))-\varphi_0^{\mu}]^{-\frac{1}{\mu}}, \tag{4.4}$$

evidently for all $t\in[0,\infty)$. Similarly the solution $\varphi_*(t)$ to (3.26) is given by (4.4) with φ_{0*}, μ_*, M_*, N_* written in place of φ_0, μ, M, N. Consequently for all $t\in[0,\infty)$ we get $v(t)$ from (3.10), $v^*(t)$ from (3.25), and (4.3) holds true for $t\in[0,T)$ because of our assumptions above, due to the comparison principle [13, p. 94, 96].

In case $\mu<0<N$, the equivalence of the inequalities

$$N<\varphi^{\mu}\Longleftrightarrow\varphi<N^{\frac{1}{\mu}}\quad\text{for all values}\quad\varphi>0 \tag{4.5}$$

shows that then the value $\varphi=N^{\frac{1}{\mu}}$ is the unique stationary point in $\mathbb{R}^1_+$ of the differential equation (3.9), $\varphi=N^{\frac{1}{\mu}}$ being stable and having the domain of attraction $\mathbb{R}^1_+$. Consequently the solution $\varphi(t)$ to (3.9) exists for all initial values $\varphi_0>0$ for all $t\in[0,\infty)$, $\varphi(t)$ being monotone increasing (decreasing) to $N^{\frac{1}{\mu}}$ in case $\varphi_0<N^{\frac{1}{\mu}}$ (in case $N^{\frac{1}{\mu}}<\varphi_0$, respectively). Thus in case $\varphi_0^{\mu}<N$ and $\varphi_{0*}^{\mu*}<N_*$ the brackets in (3.9), (3.26) becoming negative, in order to fulfill (3.15), (3.20) we have to interchange the factors M and M_* of these brackets in both equations.

Corollary 4.1. *Assume* (2.6), (2.8), (2.9), (2.14). *In addition we require*

$$\mu\le\mu_*<0<N_*\le N<(1-c_0)^{-\frac{1}{\mu}}, \tag{4.6}$$

$$(1-c_0)^{-1}<\varphi_0\le\varphi_{0*},\quad N_*<(1-c_0)^{-\frac{1}{\mu^*}}.$$

(a) *In case* $\varphi_0^{\mu}\ge N$ *and* $\varphi_{0*}^{\mu_*}\ge N_*$, *we calculate* $\varphi(t)$ *from* (3.9), $\varphi_*(t)$ *from* (3.26).

(b) *In case* $\varphi_0^{\mu} < N$ *and* $\varphi_{0*}^{\mu_*} < N_*$, *first of all we rewrite* (3.9) *with* M_* *instead of* M, (3.26) *with* M *instead of* M_* *and calculate* $\varphi(t), \varphi_*(t)$ *from the modified equations.*

Then the conclusions (1) *and* (2) *of Proposition* 4.1 *hold true, but now, in addition, the functions* $v(t), v^*(t)$ *remain uniformly bounded in* $t \in [0, \infty)$.

Proof: Recalling the assumptions of Corollary 4.1, in case (a) the solution to (3.9) is given by (3.19) or, equivalently, by

$$\varphi(t) = \varphi_0 \cdot N^{\frac{1}{\mu}} \cdot \{\varphi_0^{\mu} - e^{\mu M N t} \cdot (\varphi_0^{\mu} - N)\}^{-\frac{1}{\mu}} \tag{4.7}$$

evidently for all $t \in [0, \infty)$. In case (b) we have to rewrite (4.7) with M_* in place of M. Similarly we find the solution $\varphi_*(t)$ to (3.26) from (4.7) with φ_{0*}, μ_*, M_*, N_* in place of φ_0, μ, M, N in case (a), and with φ_{0*}, μ_*, N_* instead of φ_0, μ, N in case (b). Then, due to the comparison principle [13, pp. 94, 96], from (3.10), (3.25) we get the subfunction $v(t)$ and the superfunction $v^*(t)$, respectively, to (3.1), (3.2) because of our requirements above for case (a) as well as for case (b).

Corollary 4.2. *Under the assumptions of Proposition* 4.1 *or Corollary* 4.1, *each solution* $x(t)$ *to* (3.1), (3.2), $x(t)$ *having any initial value*

$$x_0 = (x_i(0)) \in [(\xi_i(0) \cdot \varphi_0^{A_i}), (\xi_i(0) \cdot \varphi_{0*}^{A_i})], \tag{4.8}$$

exists globally for all $t \in [0, \infty)$, *and* $x(t)$ *obeys* (4.3).

Proof by contradiction: Under the assumptions of Proposition 4.1 or Corollary 4.1, the lower function $v(t)$ from (3.10) and the upper function $v^*(t)$ from (3.25) exist for all $t \in [0, \infty)$, defining the subsets

$$S_t = \{(\tau, z) \in [0, t] \times \mathbb{R}_+^n \big| 0 \le \tau \le t\,, \quad v(\tau) \le z \le v^*(\tau)\}. \tag{4.9}$$

For each solution $x(t)$ to (3.1), (3.2) on its right maximal interval $[0, T)$, $x(t)$ with initial value x_0 fulfilling (4.8) we find: If we had $T < \infty$, the subset S_T being compact, then the extension theorem for ordinary differential equations [2, pp. 12–14] would imply $(t, x(t)) \notin S_T$ for all $t \in [0, T)$, t being sufficiently near to T, in contradiction to the result $(t, x(t)) \in S_T$ for all $t \in [0, T)$ because of (4.3) in Proposition 4.1 or Corollary 4.1.

Obviously, similar results can be proved also in the special cases where some of the constants μ, N, μ_*, N_* are zero or where we have $N_* \le 0 \le N$.

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