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EFFECTS OF A CONCENTRATED NONLINEAR SOURCE ON QUENCHING IN $\mathbb{R}^N$

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ABSTRACT. Let T and α be positive real numbers, β be a real number, B be a N-dimensional ball $\{x \in \mathbb{R}^N : |x| < R\}$ centered at the origin with a radius R, and ∂B be its boundary. Also, let $\nu(x)$ denote the unit inward normal at $x \in \partial B$, and $\chi_B(x)$ be the characteristic function, which is 1 for $x \in B$, and 0 for $x \in \mathbb{R}^N \setminus B$. This article studies the following parabolic Cauchy problem with a concentrated nonlinear source on ∂B:

$$u_t - \triangle u = \alpha(1+|x|)^\beta \frac{\partial \chi_B(x)}{\partial \nu} f(u) \text{ in } \mathbb{R}^N \times (0,T],$$
$$u(x,0) = 0 \text{ for } x \in \mathbb{R}^N,\ u(x,t) \to 0 \text{ as } |x| \to \infty \text{ for } 0 < t \leq T,$$

where f is a given function such that $\lim_{u \to c^-} f(u) = \infty$ for some positive constant c, and $f(u)$ and its derivatives $f'(u)$ and $f''(u)$ are positive for $0 \leq u < c$. It is shown that the solution u always quenches for $N \leq 2$, and quenching can be prevented for any β for $N \geq 3$. For given R and β, the effects of α on quenching are discussed. Similarly for a given α, the effects of R and β on quenching are investigated.

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1. INTRODUCTION

Let $H = \partial/\partial t - \triangle$, T and α be positive real numbers, β be a real number, $x = (x_1, x_2, \ldots, x_N)$ be a point in the N-dimensional Euclidian space $\mathbb{R}^N$, and $\Omega = \mathbb{R}^N \times (0,T]$. Dai and Zeng [2] studied quenching phenomena of the following Cauchy problem:

$$Hu = \alpha(1+|x|)^\beta \frac{1}{1-u} \text{ in } \Omega,$$
$$u(x,0) = 0 \text{ for } x \in \mathbb{R}^N,\ u(x,t) \to 0 \text{ as } |x| \to \infty \text{ for } 0 < t \leq T.$$

They proved that there exists a number $t_q \in (0,\infty)$ such that

$$\lim_{t \to t_q^-} \sup_{x \in \mathbb{R}^N} u(x,t) = 1 \tag{1.1}$$

for $N \leq 2$ while for $N \geq 3$ and $\beta \geq -2$, (1.1) holds for any α. They also showed that for $N \geq 3$ and $\beta < -2$, there exists a critical number α^* such that u exists globally for $\alpha < \alpha^*$ and (1.1) holds for $\alpha > \alpha^*$.

Here, we would like to study the effects of a concentrated nonlinear source. Let B be a N-dimensional ball $\{x \in \mathbb{R}^N : |x - \bar{b}| < R\}$ centered at a given point $\bar{b}$ with a radius R, ∂B be the boundary of B, $\nu(x)$ denote the unit inward normal at $x \in \partial B$, and

$$\chi_B(x) = \begin{cases} 1 \text{ for } x \in B, \\ 0 \text{ for } x \in \mathbb{R}^N \setminus B, \end{cases}$$

be the characteristic function. Without loss of generality, let $\bar{b}$ be the origin. We consider the parabolic Cauchy problem with a concentrated nonlinear source on the surface of the ball:

$$\text{(1.2)} \qquad \begin{cases} Hu = \alpha(1+|x|)^{\beta} \dfrac{\partial \chi_B(x)}{\partial \nu} f(u) & \text{in } \Omega, \\ u(x,0) = 0 \text{ for } x \in \mathbb{R}^N,\ u(x,t) \to 0 \text{ as } |x| \to \infty & \text{for } 0 < t \leq T. \end{cases}$$

This model is motivated by a N-dimensional ball B having a radius R and situated in $\mathbb{R}^N$; on the surface ∂B of the ball, there is a concentrated nonlinear source of strength $\alpha(1+|x|)^{\beta} f(u)$, where $u(x,t)$ in Ω is the unknown to be determined. We assume that $\lim_{u \to c^-} f(u) = \infty$ for some positive constant c, and $f(u)$ and its derivatives $f'(u)$ and $f''(u)$ are positive for $0 \leq u < c$.

A solution u of the problem (1.2) is said to quench if there exists an extended real number $t_q \in (0, \infty]$ such that

$$\sup\{u(x,t) : x \in \mathbb{R}^N\} \to c^- \text{ as } t \to t_q.$$

If $t_q < \infty$, then u is said to quench in a finite time. If $t_q = \infty$, then u quenches in infinite time.

We note that a quenching problem in $\mathbb{R}^N$ with a concentrated nonlinear source $\alpha f(u)$ was studied by Chan and Tragoonsirisak [1]. Since for given α, R and β, the term $\alpha(1+R)^{\beta}$ is a constant, it follows from Theorem 3.1 of Chan and Tragoonsirisak [1] that we have the following result.

Theorem 1.1. *For $N \leq 2$, u always quenches, in a finite time, everywhere on ∂B only.*

For $N \geq 3$, u behaves differently. In Section 2, we study the effects of the coefficient α when the radius R and β are fixed. We also derive a formula for the critical value α^* such that u exists globally for $\alpha \leq \alpha^*$ and quenches in a finite time for $\alpha > \alpha^*$. In Section 3, the effects of R and β on quenching are investigated for any given α.

2. EFFECTS OF α ON QUENCHING FOR $N \geq 3$

The integral equation corresponding to the problem (1.2) is given by

$$u(x,t) = \alpha \int_0^t \int_{\partial B} g(x,t;\xi,\tau)(1+|\xi|)^\beta f(u(\xi,\tau))dS_\xi d\tau$$
$$= \alpha(1+R)^\beta \int_0^t \int_{\partial B} g(x,t;\xi,\tau) f(u(\xi,\tau))dS_\xi d\tau \tag{2.1}$$

(cf. Chan and Tragoonsirisak [1]), where

$$g(x,t;\xi,\tau) = \frac{1}{[4\pi(t-\tau)]^{N/2}} \exp\left(-\frac{|x-\xi|^2}{4(t-\tau)}\right).$$

Let $M(t)$ denote $\sup_{x\in\mathbb{R}^N} u(x,t)$, and t_q denote the supremum of all t_1 such that the integral equation (2.1) has a unique continuous nonnegative solution for $0 \leq t \leq t_1$. Since for given α, R and β, the term $\alpha(1+R)^\beta$ is a constant, it follows from Theorems 2.1, 2.2 and 2.3 of Chan and Tragoonsirisak [1] that we have the following results.

Theorem 2.1. *There exists some t_q such that for $0 \leq t < t_q$, the integral equation* (2.1) *has a unique continuous nonnegative solution u. Furthermore, u is the solution of the problem* (1.2), *and is a strictly increasing function of t. For any $t > 0$,*

$$u(x,t) = M(t) \text{ for } x \in \partial B,\ M(t) > u(y,t) \text{ for any } y \notin \partial B.$$

If t_q is finite, then at t_q, u quenches everywhere on ∂B only.

The fundamental solution (cf. Evans [3, pp. 22 and 615]) of the Laplace equation for $N \geq 3$ is given by

$$G(x) = \frac{\Gamma\left(\frac{N}{2}+1\right)}{N(N-2)\pi^{N/2}} \frac{1}{|x|^{N-2}}.$$

The following result follows from Theorem 4.2 of Chan and Tragoonsirisak [1].

Theorem 2.2. *If $u(x,t) \leq C$ for some constant $C \in (0,c)$, then $u(x,t)$ converges from below to a solution $U(x) = \lim_{t\to\infty} u(x,t)$ of the nonlinear integral equation,*

$$U(x) = \alpha (1+R)^\beta \int_{\partial B} G(x-\xi) f(U(\xi)) dS_\xi.$$

The next result follows from Theorems 4.3 to 4.5 of Chan and Tragoonsirisak [1].

Theorem 2.3. *For $N \geq 3$, there exists a unique*

$$\alpha^* = \frac{(N-2)\pi^{(N-3)/2}}{(1+R)^\beta R\Gamma\left(\frac{N-1}{2}\right)\left(\prod_{i=1}^{N-3}\int_0^\pi \sin^i\varphi d\varphi\right)} \max_{0\leq s\leq c}\left(\frac{s}{f(s)}\right),$$

where for $N = 3$, $\prod_{i=1}^{N-3}\int_0^\pi \sin^i\varphi d\varphi = 1$, such that u exists globally for $\alpha \leq \alpha^*$, and u quenches in a finite time for $\alpha > \alpha^*$.

From the above theorem, $U(b)$ exists when $\alpha = \alpha^*$. This rules out the possibility of quenching in infinite time. In contrast to that with a source which is not concentrated, we note that the presence of the concentrated source can prevent the occurence of quenching for any given β.

For illustration, let $f(u) = 1/(1-u)$. A direct computation shows that $s\,(1-s)$ attains its maximum when $s = 0.5$. Hence,

$$\alpha^* = \frac{0.25(N-2)\pi^{(N-3)/2}}{(1+R)^{\beta} R\Gamma\left(\frac{N-1}{2}\right)\left(\prod_{i=1}^{N-3}\int_0^{\pi}\sin^i\varphi d\varphi\right)}.$$

3. EFFECTS OF R AND β ON QUENCHING FOR $N \geq 3$

In this section, we study the effects of R and β on quenching for a given α.

Lemma 3.1. *For $N \geq 3$,*

(i) *if*

$$(1+R)^{\beta} R \leq \frac{(N-2)\,\pi^{(N-3)/2}}{\alpha\Gamma\left(\frac{N-1}{2}\right)\left(\prod_{i=1}^{N-3}\int_0^{\pi}\sin^i\varphi d\varphi\right)} \max_{0\leq s\leq c}\left(\frac{s}{f(s)}\right), \tag{3.1}$$

then u exists globally.

(ii) *if*

$$(1+R)^{\beta} R > \frac{(N-2)\pi^{(N-3)/2}}{\alpha\Gamma\left(\frac{N-1}{2}\right)\left(\prod_{i=1}^{N-3}\int_0^{\pi}\sin^i\varphi d\varphi\right)} \max_{0\leq s\leq c}\left(\frac{s}{f(s)}\right), \tag{3.2}$$

then u quenches in a finite time.

Proof. (i) (3.1) is equivalent to $\alpha \leq \alpha^*$. By Theorem 2.3, u exists globally.
(ii) Since (3.2) is equivalent to $\alpha > \alpha^*$, it follows from Theorem 2.3 that u quenches in a finite time. □

Let $\varphi(R) = (1+R)^{\beta} R$. We have

$$\varphi'(R) = (1+R)^{\beta-1}\left[1+(\beta+1)R\right]. \tag{3.3}$$

Theorem 3.2. *For $N \geq 3$, and a given α, if $\beta > -1$, then there exists a unique R^* such that u exists globally for $R \leq R^*$ and quenches in a finite time for $R > R^*$.*

Proof. Using (3.3), we have for $\beta > -1$,

$$\begin{cases} \varphi'(R) > 0 \text{ for } R \geq 0, \\ \varphi(0) = 0, \\ \lim_{R\to\infty}\varphi(R) = \infty. \end{cases} \tag{3.4}$$

By solving

$$(1+R)^{\beta} R = \frac{(N-2)\pi^{(N-3)/2}}{\alpha\Gamma\left(\frac{N-1}{2}\right)\left(\prod_{i=1}^{N-3}\int_0^{\pi}\sin^i\varphi d\varphi\right)}\max_{0\le s\le c}\left(\frac{s}{f(s)}\right)$$

for R, it follows from (3.4) that there exists exactly one solution, denoted by R^*. The theorem then follows from Lemma 3.1 and (3.4). □

Theorem 3.3. *For $N \ge 3$ and $\beta = -1$,*

(i) *if*

$$\alpha \le \frac{(N-2)\pi^{(N-3)/2}}{\Gamma\left(\frac{N-1}{2}\right)\left(\prod_{i=1}^{N-3}\int_0^{\pi}\sin^i\varphi d\varphi\right)}\max_{0\le s\le c}\left(\frac{s}{f(s)}\right), \tag{3.5}$$

then u exists globally for any R.

(ii) *if*

$$\alpha > \frac{(N-2)\pi^{(N-3)/2}}{\Gamma\left(\frac{N-1}{2}\right)\left(\prod_{i=1}^{N-3}\int_0^{\pi}\sin^i\varphi d\varphi\right)}\max_{0\le s\le c}\left(\frac{s}{f(s)}\right),$$

then there exists a unique R^* such that u exists globally for $R \le R^*$ and quenches in a finite time for $R > R^*$.

Proof. (i) It follows from (3.3) that for $\beta = -1$,

$$\begin{cases} \varphi'(R) > 0 \text{ for } R \ge 0, \\ \varphi(0) = 0, \\ \lim_{R\to\infty}\varphi(R) = 1. \end{cases} \tag{3.6}$$

(3.5) is equivalent to $\alpha \le \alpha^*$. By Theorem 2.3, u exists globally.

(ii) We note from the assumption that

$$\frac{(N-2)\pi^{(N-3)/2}}{\alpha\Gamma\left(\frac{N-1}{2}\right)\left(\prod_{i=1}^{N-3}\int_0^{\pi}\sin^i\varphi d\varphi\right)}\max_{0\le s\le c}\left(\frac{s}{f(s)}\right) < 1.$$

By solving

$$\frac{R}{1+R} = \frac{(N-2)\pi^{(N-3)/2}}{\alpha\Gamma\left(\frac{N-1}{2}\right)\left(\prod_{i=1}^{N-3}\int_0^{\pi}\sin^i\varphi d\varphi\right)}\max_{0\le s\le c}\left(\frac{s}{f(s)}\right)$$

for R, it follows from (3.6) that there exists only one solution, denoted by R^*. The theorem then follows from Lemma 3.1 and (3.6). □

Theorem 3.4. *For $N \ge 3$ and $\beta < -1$,*

(i) *if*

$$\alpha \leq \frac{(N-2)\pi^{(N-3)/2}(-\beta-1)}{\Gamma\left(\frac{N-1}{2}\right)\left(\prod_{i=1}^{N-3}\int_0^\pi \sin^i\varphi d\varphi\right)\left(\frac{\beta}{\beta+1}\right)^\beta}\max_{0\leq s\leq c}\left(\frac{s}{f(s)}\right), \tag{3.7}$$

then u exists globally for any R.

(ii) *if*

$$\alpha > \frac{(N-2)\pi^{(N-3)/2}(-\beta-1)}{\Gamma\left(\frac{N-1}{2}\right)\left(\prod_{i=1}^{N-3}\int_0^\pi \sin^i\varphi d\varphi\right)\left(\frac{\beta}{\beta+1}\right)^\beta}\max_{0\leq s\leq c}\left(\frac{s}{f(s)}\right),$$

then there exist R^{**} and R^{***} such that u exists globally for $R \leq R^{**}$ or $R \geq R^{***}$, and quenches in a finite time for $R^{**} < R < R^{***}$.

Proof. (i) From (3.3), we have for $\beta < -1$,

$$\begin{cases} \varphi(R) \text{ attains its maximum at } R = -\dfrac{1}{\beta+1}, \\ \varphi'(R) > 0 \text{ for } R < -\dfrac{1}{\beta+1}, \\ \varphi'(R) < 0 \text{ for } R > -\dfrac{1}{\beta+1}. \end{cases} \tag{3.8}$$

(3.7) is equivalent to $\alpha \leq \alpha^*$. By Theorem 2.3, u exists globally.

(ii) We note from the assumption that

$$\frac{(N-2)\pi^{(N-3)/2}}{\alpha\Gamma\left(\frac{N-1}{2}\right)\left(\prod_{i=1}^{N-3}\int_0^\pi \sin^i\varphi d\varphi\right)}\max_{0\leq s\leq c}\left(\frac{s}{f(s)}\right) < \varphi\left(-\frac{1}{\beta+1}\right).$$

By solving

$$(1+R)^\beta R = \frac{(N-2)\pi^{(N-3)/2}}{\alpha\Gamma\left(\frac{N-1}{2}\right)\left(\prod_{i=1}^{N-3}\int_0^\pi \sin^i\varphi d\varphi\right)}\max_{0\leq s\leq c}\left(\frac{s}{f(s)}\right)$$

for R, it follows from (3.8) that there exist two solutions. Let us denote the solution less than $(-\beta-1)^{-1}$ by R^{**}, and the one larger than $(-\beta-1)^{-1}$ by R^{***}. The theorem then follows from Lemma 3.1 and (3.8). □

For illustration, let $f(u) = 1/(1-u)$, $\alpha = 1$, and $N = 3$. A direct computation gives $\max_{0\leq s\leq c}(s/f(s)) = 0.25$. We give below examples for the three cases: $\beta > -1$, $\beta = -1$, and $\beta < -1$.

Example 3.1. Let $\beta = 3$. Since $\beta > -1$, it follows from Theorem 3.2 that $(1 + R^*)^3 R^* = 0.25$. By using Mathematica version 6.0, we have $R^* \approx 0.160116$.

Example 3.2. Let $\beta = -1$. It follows from Theorem 3.3(ii) that $R^* / (1 + R^*) = 0.25$, which gives $R^* = 1/3$.

Example 3.3. Let $\beta = -1.5$. It follows from Theorem 3.4(ii) that

$$\frac{R}{(1 + R)^{1.5}} = 0.25.$$

By using Mathematica version 6.0, we have $R^{**} \approx 0.425485$ and $R^{***} \approx 12.7587$.

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