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EXISTENCE OF THE CLASSICAL SOLUTION FOR DEGENERATE QUASILINEAR PARABOLIC PROBLEMS WITH SLOW DIFFUSIONS

W. Y. CHAN

Department of Mathematics, Southeast Missouri State University
Cape Girardeau, MO 63701-6700 USA

ABSTRACT. Let $T \leq \infty$, b be a positive number, m be a positive number such that $m > 1$, and q be a nonnegative number. Existence and uniqueness of a classical solution are studied for the following degenerate quasilinear parabolic problem,

$$x^q u_t = (u^m)_{xx} + bf(u) \text{ in } (0,1) \times (0,T),$$

$$u(x,0) = u_0(x) \text{ in } [0,1],\ u(0,t) = 0 = u(1,t) \text{ for } t \in (0,T),$$

where $u_0(x)$ is a positive function for $0 < x < 1$, $u_0^m(x) \in C^{2+\alpha}([0,1])$ for some $\alpha \in (0,1)$, $u_0(0) = u_0(1) = 0$, and $f(u)$ is a given function such that $f(0) \geq 0$ and $f'(u) \geq 0$ for $u \geq 0$. Furthermore, a criterion for u to blow up in a finite time is given.

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1. INTRODUCTION

Let $T \leq \infty$, b be a positive number, m be a positive number such that $m > 1$, q be a nonnegative number, $D = (0,1)$, $\Omega_T = D \times (0,T)$, $\bar{D}$ and $\bar{\Omega}_T$ denote the closures of D and Ω_T respectively, and $\partial\Omega_T$ denote the parabolic boundary $\left(\bar{D} \times \{0\}\right) \cup (\{0,1\} \times (0,T))$. We consider the following degenerate parabolic problem,

$$x^q u_t = (u^m)_{xx} + bf(u) \text{ in } \Omega_T, \tag{1.1}$$

$$u(x,0) = u_0(x) \text{ on } \bar{D},\ u(0,t) = 0 = u(1,t) \text{ for } t \in (0,T), \tag{1.2}$$

where $u_0(x)$ is a positive function in D, $u_0^m(x) \in C^{2+\alpha}\left(\bar{D}\right)$ for some $\alpha \in (0,1)$, $u_0(0) = u_0(1) = 0$, and $f(u)$ is a given function such that $f(0) \geq 0$ and $f'(u) \geq 0$ for $u \geq 0$. We assume that

$$(u_0^m)'' + bf(u_0) \geq 0 \text{ in } D. \tag{1.3}$$

The problem (1.1)–(1.2) arises in plasma physics (cf. Berryman [1], Berryman & Holland [2], and Budd, Galaktionov and Chen [3]) with u denoting the particle density. It describes a particle diffusion across a magnetic field in a toroidal octupole plasma containment device; x^q is a geometrical factor and mu^{m-1} is the diffusion coefficient. Since mu^{m-1} tends to zero as $u \to 0$, (1.1) describes a phenomenon having a "slow diffusion". When $q = 0$, the problem (1.1)–(1.2) can be used to describe population dynamics (cf. Gurtin and MacCamy [10]) with u^m representing individuals migrating away from a region of high density, and with $bf(u)$ being the population supply due to births.

For an n-dimensional version of the problem (1.1)–(1.2) with $q = 0$, $f(u) = u^p/b$ with $p > 1$, and $u_0(x) \geq 0$, Galaktionov [9] obtained results on existence and the blow-up in a finite time of a weak solution. Since the fundamental eigenvalue σ of the problem,

$$\varphi'' + \sigma\varphi = 0,\ \varphi(0) = 0 = \varphi(1),$$

is greater than 1, his results for $n = 1$ showed that existence of a weak solution u in a finite time, and its blow-up can occur for the case $1 < m < p < 3m + 2$; for the case $1 < m < p$, he also gave a criterion (in terms of the fundamental eigenvalue) for u to blow up in a finite time. Results on existence and blow-up in a finite time of weak solutions for a more general multi-dimensional version of the problem (1.1)–(1.2) with $q = 0$ were obtained by Levine and Sacks [12]. We note that existence and the blow-up in a finite time of a classical solution for a multi-dimensional version of the problem (1.1)–(1.2) with $q = 0$ and $f(u) = u^m/b$ was discussed by Samarskii, Galaktionov, Kurdyumov and Mikhailov [13, pp. 29-30]. Budd, Galaktionov and Chen [3] studied the blow-up point of a weak solution of the problem (1.1)–(1.2) when $f(u) = u^p/b$. They proved that $x = 0$ is the single blow-up point when $(p-1)/m = q$. Our main purpose here is to use a completely different approach from the above-mentioned references to obtain existence, uniqueness and the blow-up in a finite time of a classical solution for the problem (1.1)–(1.2).

For the problem (1.1)–(1.2) with $m = 1$, $f(u) = u^p/b$ and $u_0(x) \geq 0$ in D, existence and uniqueness of a classical solution were studied by Floater [7], and by Chan and Liu [6]. Furthermore, Floater [7] proved that $x = 0$ is the only blow-up point if $1 < p \leq q+1$, and $(u_0(x)/x)' \leq 0$ for $x \in D$. On the other hand, Chan and Liu [6] proved that if $p > q + 1$, and for some positive constant K, $u_0''(x) + u_0^p(x) \geq Ku_0(x)$ for $x \in D$, then $x = 0$ is not a blow-up point, and the blow-up set is a compact subset of D. When $b = 1$, Chan and Chan [5] also proved existence and uniqueness of a classical solution.

Let $v = u^m$. Then, the problem (1.1)–(1.2) becomes

$$x^q v_t = mv^{(m-1)/m} v_{xx} + bmv^{(m-1)/m} g(v) \text{ in } \Omega_T, \tag{1.4}$$

$$v(x,0) = v_0(x) \text{ on } \bar{D},\ v(0,t) = 0 = v(1,t) \text{ for } t \in (0,T), \tag{1.5}$$

where $v_0(x) = u_0^m(x)$ and $g(v) = f(u)$. It is noted that $v_0(x) \in C^{2+\alpha}(\bar{D})$, and $g(0) \geq 0$ and $g'(v) \geq 0$ for $v \geq 0$. (1.3) becomes

$$v_0'' + bg(v_0) \geq 0 \text{ in } D. \tag{1.6}$$

In Section 2, we shall prove existence and uniqueness of a classical solution u by studying the problem (1.4)–(1.5) first. In Section 3, we shall study the blow-up of the solution u in the following cases: (i) $f(u) \geq u^p$ where p is a positive constant such that $p > m$ for $u \geq 0$; (ii) $f(u) = u^m$.

2. EXISTENCE AND UNIQUENESS OF A CLASSICAL SOLUTION

Let ε be a sufficiently small positive number less than 1. We consider the following problem,

$$x^q v_{\varepsilon_t} = m v_\varepsilon^{(m-1)/m} v_{\varepsilon_{xx}} + bm v_\varepsilon^{(m-1)/m} g(v_\varepsilon) \text{ in } \Omega_T, \tag{2.1}$$

$$v_\varepsilon(x,0) = v_0(x) + \varepsilon \text{ on } \bar{D},\ v_\varepsilon(0,t) = \varepsilon = v_\varepsilon(1,t) \text{ for } t \in (0,T). \tag{2.2}$$

Also, let δ ($< 1/2$) be a positive number, $D_\delta = (\delta, 1)$, $\Omega_{\delta T} = D_\delta \times (0,T)$, $\bar{D}_\delta$ and $\bar{\Omega}_{\delta T}$ denote the closures of D_δ and $\Omega_{\delta T}$ respectively, and $\partial\Omega_{\delta T}$ denote the parabolic boundary $\left(\bar{D}_\delta \times \{0\}\right) \cup (\{\delta, 1\} \times (0,T))$. We consider the following problem,

$$x_t^q v_{\varepsilon\delta_t} = m v_{\varepsilon_\delta}^{(m-1)/m} v_{\varepsilon_{\delta xx}} + bm v_{\varepsilon_\delta}^{(m-1)/m} g(v_{\varepsilon_\delta}) \text{ in } \Omega_{\delta T}, \tag{2.3}$$

$$\begin{cases} v_{\varepsilon_\delta}(x,0) = v_0(x) + \varepsilon \text{ on } \bar{D}_\delta, \\ v_{\varepsilon_\delta}(\delta,t) = v_0(\delta) + \varepsilon \text{ and } v_{\varepsilon_\delta}(1,t) = \varepsilon \text{ for } t \in (0,T). \end{cases} \tag{2.4}$$

We would like to show that the problem (2.3)–(2.4) has a classical solution v_{ε_δ}, converging to a classical solution v_ε of the problem (2.1)–(2.2) as $\delta \to 0$. We then prove that v_ε converges to a classical solution v of the problem (1.4)–(1.5) as $\varepsilon \to 0$. With this, we establish a classical solution u of the problem (1.1)–(1.2) either exists globally or blows up in a finite time.

To establish existence of v_{ε_δ}, we construct a sequence $\{w_i\}$ as follows: $w_0 = v_0 + \varepsilon$, and for $i = 1, 2, 3, \ldots$,

$$x^q w_{i_t} = m w_{i-1}^{(m-1)/m} w_{i_{xx}} + bm w_{i-1}^{(m-1)/m} g(w_{i-1}) \text{ in } \Omega_{\delta T}, \tag{2.5}$$

$$\begin{cases} w_i(x,0) = v_0(x) + \varepsilon \text{ on } \bar{D}_\delta, \\ w_i(\delta,t) = v_0(\delta) + \varepsilon \text{ and } w_i(1,t) = \varepsilon \text{ for } t \in (0,T). \end{cases} \tag{2.6}$$

To prove that w_i converges to v_{ε_δ} as i tends to infinity, we modify Lemma 2.2 of Floater [7] to obtain the following lemma.

Lemma 2.1. *There exists some positive number $t_1 < T$ and an a priori bound $\psi \in C^{2,1}\left(\bar{\Omega}_{t_1}\right)$ such that $\psi \geq w_i \geq v_0 + \varepsilon$ on $\bar{\Omega}_{\delta t_1}$ for any positive integer i.*

Proof. From (2.5) with $i = 1$,

$$x^q w_{1_t} = m w_0^{(m-1)/m} w_{1_{xx}} + bm w_0^{(m-1)/m} g(w_0). \tag{2.7}$$

From (1.6),

$$0 \leq m w_0^{(m-1)/m}\left[(v_0+\varepsilon)'' + bg\,(v_0+\varepsilon)\right]. \tag{2.8}$$

Subtracting (2.8) from (2.7), we obtain

$$x^q w_{1_t} \geq m w_0^{(m-1)/m}\left[w_1 - (v_0+\varepsilon)\right]_{xx}. \tag{2.9}$$

Since $w_1(x,0) - w_0(x) = 0$ on $\bar{D}_\delta$, and $w_1(\delta,t) - w_0(\delta) = 0$ and $w_1(1,t) - w_0(1) = 0$ for $0 < t \leq t_1$, by the weak maximum principle (cf. Friedman [8, pp. 39-40]), $w_1 \geq v_0(x) + \varepsilon$ on $\bar{\Omega}_{\delta t_1}$. Suppose $w_j \geq v_0 + \varepsilon$ for some integer $j > 1$. Similar to (2.9), we have

$$\begin{aligned} x^q (w_{j+1} - w_0)_t &\geq m w_j^{(m-1)/m}(w_{j+1} - w_0)_{xx} + bm w_j^{(m-1)/m}(g(w_j) - g(w_0)) \\ &\geq m w_j^{(m-1)/m}(w_{j+1} - w_0)_{xx}. \end{aligned}$$

Since $w_{j+1}(x,0) - w_0(x) = 0$ on $\bar{D}_\delta$, and $w_{j+1}(\delta,t) - w_0(\delta) = 0$ and $w_{j+1}(1,t) - w_0(1) = 0$ for $0 < t \leq t_1$, by the weak maximum principle, $w_{j+1} \geq v_0 + \varepsilon$ on $\bar{\Omega}_{\delta t_1}$. Using the principle of mathematical induction, we have $w_i \geq v_0 + \varepsilon$ on $\bar{\Omega}_{\delta t_1}$ for any positive integer i.

(i) Let $\theta(x) = x(1-x)/2$, and τ_0 be a positive number greater than or equal to 1 such that
$$\theta(x)\tau_0 \geq v_0(x) \text{ for } x \in \bar{D}, \text{ and } g(1) < \frac{\tau_0}{b}.$$

(ii) Let $\gamma \in (0, 1/2)$ be a constant such that
$$g\left(\frac{\gamma(1-\gamma)}{2}\tau_0 + 1\right) < \frac{\tau_0}{b}.$$

(iii) Let $\tau(t)$ be the solution of the initial value problem,
$$\tau' = \frac{2bm(\tau/8+1)^{(m-1)/m} g(\tau/8+1)}{\gamma^{q+2}},\ \tau(0) = \tau_0.$$

From (iii), $\tau(t)$ is an increasing function and $\tau(t) \geq \tau_0$. We choose a positive number t_1 such that

$$g\left(\frac{\gamma(1-\gamma)}{2}\tau(t_1) + 1\right) = \frac{\tau(t_1)}{b}. \tag{2.10}$$

Let $\psi(x,t) = \theta(x)\tau(t) + \varepsilon$, and
$$J = x^q\psi_t - m\psi^{(m-1)/m}\psi_{xx} - bm\psi^{(m-1)/m} g(\psi).$$

Then,

$$J = x^q\theta\tau' + m\psi^{(m-1)/m}\left[\tau - bg(\psi)\right]. \tag{2.11}$$

If (x,t) is in $(0,\gamma) \times (0,t_1]$ or $(1-\gamma,1) \times (0,t_1]$, then $\theta < \gamma(1-\gamma)/2$ for $\gamma \in (0,1/2)$. From (2.11) and $\tau' \geq 0$,

$$J \geq m\psi^{(m-1)/m}\left[\tau - bg(\theta(x)\tau(t) + \varepsilon)\right]$$

$$\geq m\psi^{(m-1)/m}\left[\tau_0 - bg\left(\frac{\gamma(1-\gamma)}{2}\tau + 1\right)\right].$$

By (2.10), $J \geq 0$. If $(x,t) \in [\gamma, 1-\gamma] \times (0, t_1]$, then by (2.11) and $\tau \geq 0$,

$$J \geq x^q \theta \tau' - bm\psi^{(m-1)/m} g(\psi). \tag{2.12}$$

At $x = 1/2$, $\theta^{(m-1)/m}(x)$ attains its maximum $(1/8)^{(m-1)/m}$. From (2.12) and (iii),

$$J \geq \frac{\gamma^{q+2}}{2}\tau' - bm\left(\frac{\tau}{8}+\varepsilon\right)^{(m-1)/m} g\left(\frac{\tau}{8}+\varepsilon\right) \geq 0.$$

We now have

$$x^q \psi_t \geq m\psi^{(m-1)/m}\psi_{xx} + bm\psi^{(m-1)/m} g(\psi) \text{ in } \Omega_{t_1}.$$

By (i),

$$\psi(x,0) = \theta\tau_0 + \varepsilon \geq v_0(x) + \varepsilon \text{ for } x \in \bar{D}_\delta.$$

Since $\tau(t)$ is an increasing function for $0 < t \leq t_1$,

$$\psi(\delta,t) = \frac{\delta(1-\delta)}{2}\tau(t) + \varepsilon \geq \frac{\delta(1-\delta)}{2}\tau_0 + \varepsilon \geq v_0(\delta) + \varepsilon,$$

$$\psi(1,t) = \varepsilon.$$

It follows from (i) and $\tau(t)$ being increasing that $\psi \geq w_0$ on $\bar{\Omega}_{\delta t_1}$. From (2.5) with $i = 1$, we have

$$\begin{aligned}&(\psi - w_1)_{xx} + bg(\psi) - bg(w_0)\\ &\leq \frac{x^q}{m}\left(\psi^{(1-m)/m} - w_0^{(1-m)/m}\right)\psi_t + \frac{x^q w_0^{(1-m)/m}}{m}(\psi - w_1)_t.\end{aligned}$$

Because $w_0^{(1-m)/m} \geq \psi^{(1-m)/m}$ on $\bar{\Omega}_{\delta t_1}$, $\psi_t \geq 0$, and $g' \geq 0$, we obtain

$$\frac{x^q w_0^{(1-m)/m}}{m}(\psi - w_1)_t \geq (\psi - w_1)_{xx}.$$

Since $\psi(x,0) - w_1(x,0) \geq 0$ on $\bar{D}_\delta$, and $\psi(\delta,t) - w_1(\delta,t) \geq 0$ and $\psi(1,t) - w_1(1,t) = 0$ for $0 < t \leq t_1$, by the weak maximum principle, $\psi \geq w_1$ on $\bar{\Omega}_{\delta t_1}$. Suppose that $\psi \geq w_j$ on $\bar{\Omega}_{\delta t_1}$ for some integer $j \geq 1$. Then,

$$\begin{aligned}&\frac{x^q w_j^{(1-m)/m}}{m}(\psi - w_{j+1})_t\\ &\geq (\psi - w_{j+1})_{xx} + b(g(\psi) - g(w_j)) + \frac{x^q}{m}\left(w_j^{(1-m)/m} - \psi^{(1-m)/m}\right)\psi_t\\ &\geq (\psi - w_{j+1})_{xx}.\end{aligned}$$

Since $\psi(x,0) - w_{j+1}(x,0) \geq 0$ on $\bar{D}_\delta$, and $\psi(\delta,t) - w_{j+1}(\delta,t) \geq 0$ and $\psi(1,t) - w_{j+1}(1,t) = 0$ for $0 < t \leq t_1$, by the weak maximum principle, $\psi \geq w_{j+1}$ on $\bar{\Omega}_{\delta t_1}$. By the principle of mathematical induction, $\psi \geq w_i$ on $\bar{\Omega}_{\delta t_1}$ for any nonnegative integer i. □

The following result is useful in estimating the coefficients and the forcing terms of the differential equations.

Lemma 2.2. *Let* a_1, a_2, *and* h *be positive numbers,* $R = [a_1, a_2] \times [0, h]$, *and* $Z(x,t)$ *denote a positive classical solution of one of the following problems:* (1.4)–(1.5), (2.1)–(2.2), (2.3)–(2.4), *and* (2.5)–(2.6). *If there exist positive numbers* k_1, k_2, *and* r *with* $r > 3/(2-\alpha)$, *and positive functions* $\Lambda(x,t)$ *and* $\beta(x,t)$ *on* R *such that* $\beta(x,t) \le Z(x,t) \le \Lambda(x,t) \le k_1$ *on* R *and* $||\Lambda(x,t)||_{L^r(R)} \le k_2$, *then there exist some positive numbers* k_3 *and* k_4 *depending on* β, Λ *and* R *such that*

(i) $\left|\left|mx^{-q}Z^{(m-1)/m}\right|\right|_{H^{\alpha,\alpha/2}(R)} \le k_3$,

(ii) $\left|\left|bmx^{-q}Z^{(m-1)/m}g(Z)\right|\right|_{H^{\alpha,\alpha/2}(R)} \le k_4$.

Proof. Since $Z(x,t) \le \Lambda(x,t) \le k_1$ on R and $g' \ge 0$, we have

$$||Z||_{L^r(R)} \le ||\Lambda||_{L^r(R)} \le k_2,$$

$$\left|\left|bmx^{-q}Z^{(m-1)/m}g(Z)\right|\right|_{L^r(R)} \le bma_1^{-q}\left|\left|\Lambda^{(m-1)/m}g(\Lambda)\right|\right|_{L^r(R)}.$$

By Theorem 4.9.1 of Ladyženskaja, Solonnikov and Ural'ceva [11, pp. 341–342], $Z \in W_r^{2,1}(R)$. Since $r > 3/(2-\alpha)$, it follows from Lemma 2.3.3 there [11, p. 80] that $W_r^{2,1}(R) \hookrightarrow H^{\alpha,\alpha/2}(R)$. Thus, $||Z||_{H^{\alpha,\alpha/2}(R)} \le k_5$ for some positive constant k_5.

(i)

$$\begin{aligned}
&\left|\left|mx^{-q}Z^{(m-1)/m}\right|\right|_{H^{\alpha,\alpha/2}(R)} \\
&\quad\le ma_1^{-q}\left|\left|\Lambda^{(m-1)/m}\right|\right|_\infty + m\left|\left|\Lambda^{(m-1)/m}\right|\right|_\infty \sup_{\substack{(x,t)\in R\\(\tilde{x},t)\in R}} \frac{|x^{-q}-\tilde{x}^{-q}|}{|x-\tilde{x}|^\alpha} \\
&\qquad + ma_1^{-q}\left[\sup_{\substack{(x,t)\in R\\(\tilde{x},t)\in R}} \frac{\left|Z^{(m-1)/m}(x,t) - Z^{(m-1)/m}(\tilde{x},t)\right|}{|x-\tilde{x}|^\alpha}\right. \\
&\qquad \left. + \sup_{\substack{(x,t)\in R\\(x,\tilde{t})\in R}} \frac{\left|Z^{(m-1)/m}(x,t) - Z^{(m-1)/m}(x,\tilde{t})\right|}{\left|t-\tilde{t}\right|^{\alpha/2}}\right].
\end{aligned}$$

By the mean value theorem,

$$\begin{aligned}
&\left|\left|mx^{-q}Z^{(m-1)/m}\right|\right|_{H^{\alpha,\alpha/2}(R)} \\
&\quad\le ma_1^{-q}\left|\left|\Lambda^{(m-1)/m}\right|\right|_\infty + m\left|\left|\Lambda^{(m-1)/m}\right|\right|_\infty \left|\left|x^{-q}\right|\right|_{H^{\alpha,\alpha/2}(R)} \\
&\qquad + (m-1)\,a_1^{-q}\left[\sup_{\substack{(x,t)\in R\\(\tilde{x},t)\in R}} \frac{\left|\xi_1^{-1/m}\right| |Z(x,t)-Z(\tilde{x},t)|}{|x-\tilde{x}|^\alpha}\right. \qquad (2.13)
\end{aligned}$$

$$+ \sup_{\substack{(x,t)\in R \\ (x,\tilde{t})\in R}} \frac{\left|\xi_2^{-1/m}\right| \left|Z(x,t) - Z(x,\tilde{t})\right|}{\left|t-\tilde{t}\right|^{\alpha/2}}\Bigg]$$

for some ξ_1 between $Z(x,t)$ and $Z(\tilde{x},t)$, and some ξ_2 between $Z(x,t)$ and $Z\left(x,\tilde{t}\right)$. Since $\xi_1^{-1/m}$ and $\xi_2^{-1/m}$ are bounded by $\beta^{-1/m}$, inequality (2.13) becomes

$$\left|\left|mx^{-q}Z^{(m-1)/m}\right|\right|_{H^{\alpha,\alpha/2}(R)} \leq k_3,$$

which is a constant for fixed β, Λ and R.

(ii) Replacing $Z^{(m-1)/m}$ with $bZ^{(m-1)/m}g(Z)$ in (i), and by $g' \geq 0$, we have

$$\begin{aligned}
&\left|\left|bmx^{-q}Z^{(m-1)/m}g(Z)\right|\right|_{H^{\alpha,\alpha/2}(R)} \\
&\leq bma_1^{-q}\left|\left|\Lambda^{(m-1)/m}\right|\right|_\infty \left|\left|g(\Lambda)\right|\right|_\infty \\
&+ bm\left|\left|\Lambda^{(m-1)/m}\right|\right|_\infty \left|\left|g(\Lambda)\right|\right|_\infty \left|\left|x^{-q}\right|\right|_{H^{\alpha,\alpha/2}(R)} \\
&+ bma_1^{-q}\left|\left|g(\Lambda)\right|\right|_\infty \sup_{\substack{(x,t)\in R \\ (\tilde{x},t)\in R}} \frac{\left|Z^{(m-1)/m}(x,t) - Z^{(m-1)/m}(\tilde{x},t)\right|}{\left|x-\tilde{x}\right|^{\alpha}} \\
&+ bma_1^{-q}\left|\left|g(\Lambda)\right|\right|_\infty \sup_{\substack{(x,t)\in R \\ (x,\tilde{t})\in R}} \frac{\left|Z^{(m-1)/m}(x,t) - Z^{(m-1)/m}\left(x,\tilde{t}\right)\right|}{\left|t-\tilde{t}\right|^{\alpha/2}} \\
&+ bma_1^{-q}\left|\left|\Lambda^{(m-1)/m}\right|\right|_\infty \sup_{\substack{(x,t)\in R \\ (\tilde{x},t)\in R}} \frac{\left|g(Z(x,t)) - g(Z(\tilde{x},t))\right|}{\left|x-\tilde{x}\right|^{\alpha}} \\
&+ bma_1^{-q}\left|\left|\Lambda^{(m-1)/m}\right|\right|_\infty \sup_{\substack{(x,t)\in R \\ (x,\tilde{t})\in R}} \frac{\left|g(Z(x,t)) - g\left(Z\left(x,\tilde{t}\right)\right)\right|}{\left|t-\tilde{t}\right|^{\alpha/2}}.
\end{aligned}$$

By the mean value theorem, the above inequality is equivalent to

$$\begin{aligned}
&\left|\left|bmx^{-q}Z^{(m-1)/m}g(Z)\right|\right|_{H^{\alpha,\alpha/2}(R)} \\
&\leq bma_1^{-q}\left|\left|\Lambda^{(m-1)/m}\right|\right|_\infty \left|\left|g(\Lambda)\right|\right|_\infty \\
&+ bm\left|\left|\Lambda^{(m-1)/m}\right|\right|_\infty \left|\left|g(\Lambda)\right|\right|_\infty \left|\left|x^{-q}\right|\right|_{H^{\alpha,\alpha/2}(R)} \\
&+ bma_1^{-q}\left|\left|g(\Lambda)\right|\right|_\infty \left|\left|Z^{(m-1)/m}\right|\right|_{H^{\alpha,\alpha/2}(R)} \\
&+ bma_1^{-q}\left|\left|\Lambda^{(m-1)/m}\right|\right|_\infty \left|\left|g'(Z)\right|\right|_\infty \left|\left|Z\right|\right|_{H^{\alpha,\alpha/2}(R)} \\
&\leq k_4,
\end{aligned}$$

which is a constant for fixed β, Λ and R. □

The following result deals with the linear problems.

Lemma 2.3. *For any arbitrarily fixed δ and any positive integer i,*

(i) $w_i \in C^{2+\alpha,1+\alpha/2}\left(\bar{\Omega}_{\delta t_1}\right)$ *and is unique,*

(ii) $w_{i_t} \geq 0$ *on* $\bar{\Omega}_{\delta t_1}$,

(iii) $\{w_{i-1}\}$ *is a monotone nondecreasing sequence on* $\bar{\Omega}_{\delta t_1}$.

Proof. (i) By Lemma 2.2 with $R = \bar{\Omega}_{\delta t_1}$, we have for some positive numbers k_6 and k_7 depending on ε, ψ and $\bar{\Omega}_{\delta t_1}$,

$$\left\| mx^{-q} w_{i-1}^{(m-1)/m} \right\|_{H^{\alpha,\alpha/2}\left(\bar{\Omega}_{\delta t_1}\right)} \leq k_6,$$

$$\left\| bmx^{-q} w_{i-1}^{(m-1)/m} g\left(w_{i-1}\right) \right\|_{H^{\alpha,\alpha/2}\left(\bar{\Omega}_{\delta t_1}\right)} \leq k_7.$$

By Theorem 4.5.2 of Ladyženskaja, Solonnikov and Ural'ceva [11, p. 320], the result follows.

(ii) Since w_i is bounded below and above by positive functions, and $w_i \in C^{2+\alpha,1+\alpha/2}(\bar{\Omega}_{\delta t_1})$, a direct computation and using (3.3.2) of Friedman [8, p. 66] show that $\partial \left(x^{-q} w_{i-1}^{(m-1)/m} \right) / \partial t$, $\partial \left(x^{-q} w_{i-1}^{(m-1)/m} g\left(w_{i-1}\right) \right) / \partial t$, $\partial^2 \left(x^{-q} w_{i-1}^{(m-1)/m} \right) / \partial x^2$, and $\partial^2 \left(x^{-q} w_{i-1}^{(m-1)/m} f\left(w_{i-1}\right) \right) / \partial x^2$ are Hölder continuous of exponent α in $\Omega_{\delta t_1}$. By Theorem 3.11 of Friedman [8, p. 74], $w_{i_{xxt}}$ and $w_{i_{tt}}$ exist and they are Hölder continuous of exponent α in $\Omega_{\delta t_1}$.

When $i = 1$, we differentiate (2.5) with respect to t, then

$$\frac{x^q w_0^{(1-m)/m}}{m} w_{1_{tt}} = w_{1_{xxt}}.$$

Also, for $x \in \bar{D}_\delta$

$$w_{1_t}(x,0) = \lim_{h \to 0^+} \frac{w_1(x,h) - w_0(x)}{h} \geq 0,$$

and $w_{1_t}(\delta, t) = 0 = w_{1_t}(1,t)$ for $0 < t \leq t_1$. By the weak maximum principle, $w_{1_t} \geq 0$ on $\bar{\Omega}_{\delta t_1}$. Suppose that it is true for $i = j$ for some positive integer j. When $i = j+1$, we have

$$\frac{x^q w_j^{(1-m)/m}}{m} w_{j+1_t} = w_{j+1_{xx}} + bg\left(w_j\right).$$

By differentiating this expression with respect to t,

$$\begin{aligned} &\frac{x^q}{m} w_j^{(1-m)/m} w_{j+1_{tt}} + \frac{x^q}{m} \left(\frac{1}{m} - 1 \right) w_j^{(1-2m)/m} w_{j_t} w_{j+1_t} \\ &= w_{j+1_{xxt}} + bg'\left(w_j\right) w_{j_t} \geq w_{j+1_{xxt}}. \end{aligned}$$

Furthermore,

$$w_{j+1_t}(x,0) = \lim_{h \to 0^+} \frac{w_{j+1}(x,h) - w_0(x)}{h} \geq 0 \text{ for } x \in \bar{D}_\delta,$$

and $w_{j+1_t}(\delta, t) = 0 = w_{j+1_t}(1,t)$ for $0 < t \leq t_1$. Since $w_j \in C^{2+\alpha,1+\alpha/2}\left(\bar{\Omega}_{\delta t_1}\right)$, $w_j^{(1-2m)/m} w_{j_t}$ is bounded on $\bar{\Omega}_{\delta t_1}$. By the weak maximum principle, $w_{j+1_t} \geq 0$ on $\bar{\Omega}_{\delta t_1}$. The result follows from the principle of mathematical induction.

(iii) If we let $i = 1$ and 2, then from (2.5) we have

$$\frac{x^q}{m} w_0^{(1-m)/m} w_{1_t} = w_{1_{xx}} + bg\left(w_0\right),$$
$$\frac{x^q}{m} w_1^{(1-m)/m} w_{2_t} = w_{2_{xx}} + bg\left(w_1\right).$$

Therefore,

$$w_{2_{xx}} - w_{1_{xx}} + b\left(g\left(w_1\right) - g\left(w_0\right)\right) = \frac{x^q}{m} w_1^{(1-m)/m}\left(w_{2_t} - w_{1_t}\right) + \frac{x^q}{m} w_1^{(1-m)/m} w_{1_t} - \frac{x^q}{m} w_0^{(1-m)/m} w_{1_t}.$$

According to Lemma 2.1 and (ii), we have $w_1 \geq w_0$ and $w_{1_t} \geq 0$. Thus,

$$\frac{x^q}{m} w_1^{(1-m)/m}\left(w_2 - w_1\right)_t \geq w_{2_{xx}} - w_{1_{xx}} + \frac{x^q}{m}\left[w_0^{(1-m)/m} - w_1^{(1-m)/m}\right] w_{1_t} \geq w_{2_{xx}} - w_{1_{xx}},$$

$w_2\left(x,0\right) - w_1\left(x,0\right) = 0$ on $\bar{D}_\delta$, and $w_2\left(\delta,t\right) - w_1\left(\delta,t\right) = 0$ and $w_2\left(1,t\right) - w_1\left(1,t\right) = 0$ for $0 < t \leq t_1$. By the weak maximum principle, $w_2 \geq w_1$ on $\bar{\Omega}_{\delta t_1}$. Suppose that it is true for $i = j$ for some positive integer j. When $i = j+1$,

$$\frac{x^q}{m} w_j^{(1-m)/m}\left(w_{j+1} - w_j\right)_t \geq w_{j+1_{xx}} - w_{j_{xx}} + \frac{x^q}{m}\left[w_{j-1}^{(1-m)/m} - w_j^{(1-m)/m}\right] w_{j_t} \geq w_{j+1_{xx}} - w_{j_{xx}},$$

$w_{j+1}\left(x,0\right) - w_j\left(x,0\right) = 0$ on $\bar{D}_\delta$, and $w_{j+1}\left(\delta,t\right) - w_j\left(\delta,t\right) = 0$ and $w_{j+1}\left(1,t\right) - w_j\left(1,t\right) = 0$ for $0 < t \leq t_1$. By the weak maximum principle, $w_{j+1} \geq w_j$ on $\bar{\Omega}_{\delta t_1}$. Hence, by the principle of mathematical induction $\{w_{i-1}\}$ is a monotone nondecreasing sequence on $\bar{\Omega}_{\delta t_1}$. □

Let $v_{\varepsilon_{i_\delta}}$ be the solution to the problem (2.3)–(2.4) when $\varepsilon = \varepsilon_i$, and $v_{\varepsilon_{\delta_i}}$ be the solution to the problem (2.3)–(2.4) when $\delta = \delta_i$.

Lemma 2.4. (i) *For any arbitrarily fixed δ and ε, there exists a solution $v_{\varepsilon_\delta} \in C^{2+\alpha,1+\alpha/2}\left(\bar{\Omega}_{\delta t_1}\right)$ of the problem* (2.3)–(2.4), *and $\psi \geq v_{\varepsilon_\delta} \geq v_0 + \varepsilon$ on $\bar{\Omega}_{\delta t_1}$.*

(ii) *$v_{\varepsilon_{\delta_1}} \geq v_{\varepsilon_{\delta_2}}$ on $\bar{\Omega}_{\delta_2 t_1}$ for any arbitrarily fixed ε and any positive δ_1 and δ_2 such that $\delta_1 \leq \delta_2$.*

(iii) *$v_{\varepsilon_{1_\delta}} \leq v_{\varepsilon_{2_\delta}}$ on $\bar{\Omega}_{\delta t_1}$ for any arbitrarily fixed δ and any positive ε_1 and ε_2 such that $\varepsilon_1 \leq \varepsilon_2$.*

Proof. (i) Since $\psi \geq w_i \geq v_0 + \varepsilon$ on $\bar{\Omega}_{\delta t_1}$and $\{w_i\}$ is a monotone nondecreasing sequence, we let $v_{\varepsilon_\delta} = \lim_{i\to\infty} w_i$ on $\bar{\Omega}_{\delta t_1}$. For any point $(x_1, t_2) \in \bar{\Omega}_{\delta t_1}$, let $\tilde{Q}_1 = [\tilde{c}_1, \tilde{c}_2] \times \left[0, \tilde{t}_1\right]$ such that $(x_1, t_2) \in \tilde{Q}_1 \subset \bar{\Omega}_{\delta t_1}$ with $\delta \leq \tilde{c}_1$, $\tilde{c}_2 \leq 1$ and $\tilde{t}_1 \leq t_1$. By Lemma 2.2,

$$\left|\left|m x^{-q} w_{i-1}^{(m-1)/m}\right|\right|_{H^{\alpha,\alpha/2}\left(\tilde{Q}_1\right)} \leq k_8,$$

$$\left|\left|bmx^{-q}w_{i-1}^{(m-1)/m}g\left(w_{i-1}\right)\right|\right|_{H^{\alpha,\alpha/2}\left(\tilde{Q}_1\right)}\leq k_9,$$

for some positive numbers k_8 and k_9 (depending on $v_0+\varepsilon$, ψ and $\tilde{Q}_1$, but independent of i). By Theorem 4.10.1 of Ladyženskaja, Solonnikov and Ural'ceva [11, pp. 351–352], there exists some positive number k_{10} independent of i such that

$$||w_i||_{H^{2+\alpha,1+\alpha/2}\left(\tilde{Q}_1\right)}\leq k_{10}.$$

This implies that w_i, w_{i_t}, w_{i_x} and $w_{i_{xx}}$ are equicontinuous on $\tilde{Q}_1$. By the Ascoli-Arzela theorem,

$$||v_{\varepsilon_\delta}||_{H^{2+\alpha,1+\alpha/2}\left(\tilde{Q}_1\right)}\leq k_{10},$$

and the partial derivatives of v_{ε_δ} are the limits of the corresponding w_i. Therefore, $\psi\geq v_{\varepsilon_\delta}\geq v_0+\varepsilon$ on $\bar{\Omega}_{\delta t_1}$. Since $w_i\left(\delta,t\right)=v_0\left(\delta\right)+\varepsilon$ and $w_i\left(1,t\right)=\varepsilon$, we have $v_{\varepsilon_\delta}\left(\delta,t\right)=v_0\left(\delta\right)+\varepsilon$ and $v_{\varepsilon_\delta}\left(1,t\right)=\varepsilon$, and hence, $v_{\varepsilon_\delta}\in C^{2+\alpha,1+\alpha/2}\left(\bar{\Omega}_{\delta t_1}\right)$ is a solution to the problem (2.3)–(2.4).

(ii) Let $w_{\delta_{1_i}}$ and $w_{\delta_{2_i}}$ be the solutions to the problem (2.5)–(2.6) in $\Omega_{\delta_1 t_1}$ and $\Omega_{\delta_2 t_1}$ respectively. When $i=0$, we have $w_{\delta_{1_0}}\left(x\right)=v_0\left(x\right)+\varepsilon$ for $x\in\bar{D}_{\delta_1}$ and $w_{\delta_{2_0}}\left(x\right)=v_0\left(x\right)+\varepsilon$ for $x\in\bar{D}_{\delta_2}$. From (2.5),

$$\frac{x^q\left(w_{\delta_{1_0}}\right)^{(1-m)/m}}{m}\left(w_{\delta_{1_1}}\right)_t=\left(w_{\delta_{1_1}}\right)_{xx}+bg\left(w_{\delta_{1_0}}\right)\ \text{ in } \Omega_{\delta_1 t_1},$$
$$\frac{x^q\left(w_{\delta_{2_0}}\right)^{(1-m)/m}}{m}\left(w_{\delta_{2_1}}\right)_t=\left(w_{\delta_{2_1}}\right)_{xx}+bg\left(w_{\delta_{2_0}}\right)\ \text{ in } \Omega_{\delta_2 t_1}.$$

Thus,

$$\frac{x^q\left(w_{\delta_{1_0}}\right)^{(1-m)/m}}{m}\left(w_{\delta_{1_1}}-w_{\delta_{2_1}}\right)_t=\left(w_{\delta_{1_1}}-w_{\delta_{2_1}}\right)_{xx}\ \text{ in } \Omega_{\delta_2 t_1}.$$

Also, $w_{\delta_{1_1}}\left(x,0\right)-w_{\delta_{2_1}}\left(x,0\right)=0$ for $x\in\bar{D}_{\delta_2}$ and $w_{\delta_{1_1}}\left(1,t\right)-w_{\delta_{2_1}}\left(1,t\right)=0$ for $0<t\leq t_1$. By Lemma 2.3(ii), $\left(w_{\delta_{1_1}}\right)_t\geq 0$ on $\bar{\Omega}_{\delta_1 t_1}$. Thus, $w_{\delta_{1_1}}\left(\delta_2,t\right)-w_{\delta_{2_1}}\left(\delta_2,t\right)\geq 0$ for $0<t\leq t_1$. Hence, by the weak maximum principle $w_{\delta_{1_1}}-w_{\delta_{2_1}}\geq 0$ on $\bar{\Omega}_{\delta_2 t_1}$. Suppose that it is true for $i=j$ for some positive integer j. Then for $i=j+1$, we have

$$\begin{aligned}
&\frac{x^q\left(w_{\delta_{1_j}}\right)^{(1-m)/m}}{m}\left(w_{\delta_{1_{j+1}}}\right)_t-\frac{x^q\left(w_{\delta_{2_j}}\right)^{(1-m)/m}}{m}\left(w_{\delta_{2_{j+1}}}\right)_t\\
&=\left(w_{\delta_{1_{j+1}}}-w_{\delta_{2_{j+1}}}\right)_{xx}+b\left(g\left(w_{\delta_{1_j}}\right)-g\left(w_{\delta_{2_j}}\right)\right)\\
&\geq\left(w_{\delta_{1_{j+1}}}-w_{\delta_{2_{j+1}}}\right)_{xx}.
\end{aligned}$$

By $\left(w_{\delta_{2_{j+1}}}\right)_t\geq 0$ on $\bar{\Omega}_{\delta_2 t_1}$, we have

$$\left(w_{\delta_{1_{j+1}}}-w_{\delta_{2_{j+1}}}\right)_{xx}\leq\frac{x^q\left(w_{\delta_{1_j}}\right)^{(1-m)/m}}{m}\left(w_{\delta_{1_{j+1}}}-w_{\delta_{2_{j+1}}}\right)_t.$$

Also, $w_{\delta_{1_{j+1}}}(x,0) = w_{\delta_{2_{j+1}}}(x,0)$ for $x \in \bar{D}_{\delta_2}$ and $w_{\delta_{1_{j+1}}}(1,t) = w_{\delta_{2_{j+1}}}(1,t)$ for $0 < t \leq t_1$. By Lemma 2.3(ii), $\left(w_{\delta_{1_{j+1}}}\right)_t \geq 0$ on $\bar{\Omega}_{\delta_1 t_1}$. Thus, $w_{\delta_{1_{j+1}}}(\delta_2,t) - w_{\delta_{2_{j+1}}}(\delta_2,t) \geq 0$ for $0 < t \leq t_1$. Hence, by the weak maximum principle $w_{\delta_{1_{j+1}}} \geq w_{\delta_{2_{j+1}}}$ on $\bar{\Omega}_{\delta_2 t_1}$. By the principle of mathematical induction, $w_{\delta_{1_i}} \geq w_{\delta_{2_i}}$ on $\bar{\Omega}_{\delta_2 t_1}$ for nonnegative integer i. Therefore, $v_{\varepsilon_{\delta_1}} \geq v_{\varepsilon_{\delta_2}}$ on $\bar{\Omega}_{\delta_2 t_1}$.

(iii) Let $\tilde{w}_i$ be the solution to the problem (2.5)–(2.6) in $\Omega_{\delta t_1}$ with $\tilde{w}_0 = v_0 + \varepsilon_1$, $\tilde{w}_i(x,0) = v_0(x) + \varepsilon_1$ on $\bar{D}_\delta$, $\tilde{w}_i(\delta,t) = v_0(\delta) + \varepsilon_1$ and $\tilde{w}_i(1,t) = \varepsilon_1$ for $0 < t \leq t_1$, and let $\hat{w}_i$ be the solution to the problem (2.5)–(2.6) in $\Omega_{\delta t_1}$ with $\hat{w}_0 = v_0 + \varepsilon_2$, $\hat{w}_i(x,0) = v_0(x) + \varepsilon_2$ on $\bar{D}_\delta$, and $\hat{w}_i(\delta,t) = v_0(\delta) + \varepsilon_2$ and $\hat{w}_i(1,t) = \varepsilon_2$ for $0 < t \leq t_1$. From (2.5),

$$\frac{x^q \tilde{w}_0^{(1-m)/m}}{m} \tilde{w}_{1_t} = \tilde{w}_{1_{xx}} + bg(\tilde{w}_0) \text{ in } \Omega_{\delta t_1},$$
$$\frac{x^q \hat{w}_0^{(1-m)/m}}{m} \hat{w}_{1_t} = \hat{w}_{1_{xx}} + bg(\hat{w}_0) \text{ in } \Omega_{\delta t_1}.$$

Thus,

$$\begin{aligned} \frac{x^q}{m}\left[\hat{w}_0^{(1-m)/m}\hat{w}_{1_t} - \tilde{w}_0^{(1-m)/m}\tilde{w}_{1_t}\right] &= (\hat{w}_1 - \tilde{w}_1)_{xx} + b\left(g(\hat{w}_0) - g(\tilde{w}_0)\right) \\ &\geq (\hat{w}_1 - \tilde{w}_1)_{xx}. \end{aligned}$$

By Lemma 2.3(ii), $\tilde{w}_{1_t} \geq 0$ on $\bar{\Omega}_{\delta t_1}$, so we obtain

$$\begin{aligned} (\hat{w}_1 - \tilde{w}_1)_{xx} &\leq \frac{x^q}{m}\hat{w}_0^{(1-m)/m}(\hat{w}_1 - \tilde{w}_1)_t + \frac{x^q}{m}\left[\hat{w}_0^{(1-m)/m} - \tilde{w}_0^{(1-m)/m}\right]\tilde{w}_{1_t} \\ &\leq \frac{x^q}{m}\hat{w}_0^{(1-m)/m}(\hat{w}_1 - \tilde{w}_1)_t. \end{aligned}$$

Also, $\hat{w}_1(x,0) - \tilde{w}_1(x,0) = \varepsilon_2 - \varepsilon_1 \geq 0$ for $x \in \bar{D}_\delta$ and $\hat{w}_1(x,t) - \tilde{w}_1(x,t) = \varepsilon_2 - \varepsilon_1$ at $x = \delta$ and $x = 1$ for $0 < t \leq t_1$. Hence, by the weak maximum principle $\hat{w}_1 \geq \tilde{w}_1$ on $\bar{\Omega}_{\delta t_1}$. Suppose that $\hat{w}_j \geq \tilde{w}_j$ on $\bar{\Omega}_{\delta t_1}$ for some integer $j \geq 1$. Then by $\tilde{w}_{j_t} \geq 0$ on $\bar{\Omega}_{\delta t_1}$ and the above calculation, we have

$$(\hat{w}_{j+1} - \tilde{w}_{j+1})_{xx} \leq \frac{x^q}{m}\hat{w}_j^{(1-m)/m}(\hat{w}_{j+1} - \tilde{w}_{j+1})_t.$$

Also, $\hat{w}_{j+1}(x,0) - \tilde{w}_{j+1}(x,0) = \varepsilon_2 - \varepsilon_1 \geq 0$ for $x \in \bar{D}_\delta$, and $\hat{w}_{j+1}(x,t) - \tilde{w}_{j+1}(x,t) = \varepsilon_2 - \varepsilon_1$ at $x = \delta$ and $x = 1$ for $0 < t \leq t_1$. Hence, by the weak maximum principle $\hat{w}_{j+1} \geq \tilde{w}_{j+1}$ on $\bar{\Omega}_{\delta t_1}$. By the principle of mathematical induction, $\hat{w}_i \geq \tilde{w}_i$ on $\bar{\Omega}_{\delta t_1}$ for any nonnegative integer i. Therefore, $v_{\varepsilon_{2_\delta}} \geq v_{\varepsilon_{1_\delta}}$ on $\bar{\Omega}_{\delta t_1}$. $\square$

Let $\omega_{t_1} = (0,1] \times [0,t_1]$. We now let δ tend to 0.

Lemma 2.5. (i) *There exists a solution* $v_\varepsilon \in C\left(\bar{\Omega}_{t_1}\right) \cap C^{2+\alpha,1+\alpha/2}(\omega_{t_1})$ *of the problem* (2.1)–(2.2), *and* $\psi \geq v_\varepsilon \geq v_0 + \varepsilon$ *on* $\bar{\Omega}_{t_1}$.

(ii) $v_{\varepsilon_1} \leq v_{\varepsilon_2}$ *on* $\bar{\Omega}_{t_1}$ *for any positive* ε_1 *and* ε_2 *such that* $\varepsilon_1 \leq \varepsilon_2$.

Proof. (i) Since $\psi \geq v_{\varepsilon_\delta} \geq v_0 + \varepsilon$ on $\bar{\Omega}_{\delta t_1}$ and $\{v_{\varepsilon_\delta}\}$ is a monotone nonincreasing sequence in δ, we let $v_\varepsilon = \lim_{\delta \to 0} v_{\varepsilon_\delta}$. Then, $\psi \geq v_\varepsilon \geq v_0 + \varepsilon$ on $\bar{\Omega}_{t_1}$. For any point

$(x_2, t_3) \in \omega_{t_1}$, let $\tilde{Q}_2 = [\tilde{c}_3, \tilde{c}_4] \times [0, \tilde{t}_2]$ such that $(x_2, t_3) \in \tilde{Q}_2 \subset \omega_{t_1}$ with $0 < \tilde{c}_3$, $\tilde{c}_4 \leq 1$ and $\tilde{t}_2 \leq t_1$. Since $\psi \geq v_{\varepsilon\delta} \geq v_0 + \varepsilon$ on $\bar{\Omega}_{\delta t_1}$, it follows from Lemma 2.2 that there exist some positive numbers k_{11} and k_{12} (depending on $v_0 + \varepsilon$, ψ and $\tilde{Q}_2$, but independent of δ) such that

$$\left|\left| m x^{-q} v_{\varepsilon_\delta}^{(m-1)/m} \right|\right|_{H^{\alpha,\alpha/2}\left(\tilde{Q}_2\right)} \leq k_{11},$$

$$\left|\left| b m x^{-q} v_{\varepsilon_\delta}^{(m-1)/m} g\left(v_{\varepsilon\delta}\right) \right|\right|_{H^{\alpha,\alpha/2}\left(\tilde{Q}_2\right)} \leq k_{12}.$$

By Theorem 4.10.1 of Ladyženskaja, Solonnikov and Ural'ceva,

$$||v_{\varepsilon\delta}||_{H^{2+\alpha,1+\alpha/2}\left(\tilde{Q}_2\right)} \leq k_{13}$$

for some positive number k_{13} independent of δ. This implies that $v_{\varepsilon\delta}$, $\left(v_{\varepsilon\delta}\right)_t$, $\left(v_{\varepsilon\delta}\right)_x$ and $\left(v_{\varepsilon\delta}\right)_{xx}$ are equicontinuous in $\tilde{Q}_2$. By the Ascoli-Arzela theorem,

$$||v_{\varepsilon}||_{H^{2+\alpha,1+\alpha/2}\left(\tilde{Q}_2\right)} \leq k_{13},$$

and the partial derivatives of v_ε are the limits of the corresponding derivatives of v_{ε_δ}. By $\psi \geq v_\varepsilon \geq v_0 + \varepsilon$ on $\bar{\Omega}_{t_1}$ and the sandwich theorem, $\lim_{x \to 0} v_\varepsilon(x,t) = \varepsilon$. Hence, $v_\varepsilon \in C\left(\bar{\Omega}_{t_1}\right) \cap C^{2+\alpha,1+\alpha/2}\left(\omega_{t_1}\right)$ is a solution to the problem (2.1)–(2.2).

(ii) This follows from Lemma 2.4(iii) by letting $\delta \to 0$. □

Let $P = D \times [0, t_1]$. We now let ε tend to 0 to give a local existence result.

Theorem 2.6. *There exists a solution* $v \in C\left(\bar{\Omega}_{t_1}\right) \cap C^{2+\alpha,1+\alpha/2}(P)$ *of the problem* (1.4)–(1.5).

Proof. By Lemma 2.5(ii), we have $v_{\varepsilon_1} \leq v_{\varepsilon_2}$ for $0 < \varepsilon_1 \leq \varepsilon_2$. Let $v = \lim_{\varepsilon \to 0} v_\varepsilon$. Since $\psi \in C\left(\bar{\Omega}_{t_1}\right)$, there exists a positive constant k_{14} (independent of ε) such that $\psi < k_{14}$ on $\bar{\Omega}_{t_1}$. An argument similar to that in Lemma 2.5(i) shows that $\{v_\varepsilon\}$ converges to $v \in C\left(\bar{\Omega}_{t_1}\right) \cap C^{2+\alpha,1+\alpha/2}(P)$. □

The following result gives local existence of a solution u.

Theorem 2.7. *There exists a solution* $u \in C\left(\bar{\Omega}_{t_1}\right) \cap C^{2+\alpha,1+\alpha/2}(P)$ *of the problem* (1.1)–(1.2).

Proof. Using $u = v^{1/(m+1)}$, it follows from $v \geq v_0 > 0$ in Ω_{t_1} and $v \in C\left(\bar{\Omega}_{t_1}\right) \cap C^{2+\alpha,1+\alpha/2}(P)$ that $u \in C\left(\bar{\Omega}_{t_1}\right)$, and that u_t, u_x and u_{xx} exist in P. The Hölder norm (cf. Friedman [8, p. 61]) of a function G with exponent α is given by

$$||G||_{C^{\alpha,\alpha/2}(P)} = ||G||_\infty + \sup_{\substack{(x,t)\in P \\ (\tilde{x},\tilde{t})\in P}} \frac{\left|G(x,t) - G\left(\tilde{x},\tilde{t}\right)\right|}{\left(\sqrt{\left|x - \tilde{x}\right|^2 + \left|t - \tilde{t}\right|}\right)^\alpha}.$$

Since $v^{-m/(m+1)}$ and $v^{-(2m+1)/(m+1)}$ are differentiable, it follows from the above equation that $\left|\left|v^{-m/(m+1)}\right|\right|_{C^{\alpha,\alpha/2}(P)}$ and $\left|\left|v^{-(2m+1)/(m+1)}\right|\right|_{C^{\alpha,\alpha/2}(P)}$ are bounded. For two given functions F and H, we have the inequalities (cf. Friedman [8, p. 66]):

$$||F + H||_{C^{\alpha,\alpha/2}(P)} \leq ||F||_{C^{\alpha,\alpha/2}(P)} + ||H||_{C^{\alpha,\alpha/2}(P)},$$

$$||FH||_{C^{\alpha,\alpha/2}(P)} \leq ||F||_{C^{\alpha,\alpha/2}(P)} ||H||_{C^{\alpha,\alpha/2}(P)} .$$

Then, by these two inequalities, we obtain

$$||u_t||_{C^{\alpha,\alpha/2}(P)} \leq \frac{1}{m+1} \left|\left|v^{-m/(m+1)}\right|\right|_{C^{\alpha,\alpha/2}(P)} ||v_t||_{C^{\alpha,\alpha/2}(P)} ,$$

$$||u_{xx}||_{C^{\alpha,\alpha/2}(P)} \leq \frac{m}{(m+1)^2} \left|\left|v^{-(2m+1)/(m+1)}\right|\right|_{C^{\alpha,\alpha/2}(P)} ||v_x||^2_{C^{\alpha,\alpha/2}(P)}$$
$$+ \frac{1}{m+1} \left|\left|v^{-m/(m+1)}\right|\right|_{C^{\alpha,\alpha/2}(P)} ||v_{xx}||_{C^{\alpha,\alpha/2}(P)} .$$

Hence, $u \in C^{2+\alpha,1+\alpha/2}(P)$. □

Let $t_s = \sup\{t_1\colon$ the problem (1.1)–(1.2) has a solution $u \in C\left(\bar{\Omega}_{t_1}\right) \cap C^{2+\alpha,1+\alpha/2}(P)\}$. We modify the proof of Theorem 8 of Chan and Chan [5] to obtain the following result.

Theorem 2.8. *The problem* (1.1)–(1.2) *has a solution*

$$u \in C\left(\bar{D} \times [0,t_s)\right) \cap C^{2+\alpha,1+\alpha/2}\left(D \times [0,t_s)\right).$$

If $t_s < \infty$, *then* u *is unbounded in* $D \times (0,t_s)$.

Proof. In order to prove global existence of u, it follows from Theorems 2.6 and 2.7 that it is sufficient to prove global existence of v. Let us suppose that v is bounded above by some positive constant M in $D \times (0,t_s)$. To arrive at a contradiction, we need to show that v can be continued into a larger time interval $[0, t_s + t_5]$ for some positive t_5. This can be achieved by extending the *a priori* bound of ψ: Let

$$\Gamma > \max\left\{bg(M), \max_{x\in\bar{D}} \frac{2v_0(x)}{x(1-x)}\right\},$$

$E(x) = \Gamma x(1-x)/2$ and $\tilde{E}(x) = E(x) + \delta$. Then,

$$-\tilde{E}_{xx} - \left(\frac{x^q v^{-(m-1)/m}}{m} v_t - v_{xx}\right) = \Gamma - bg(v) > 0 \text{ in } D \times (0,t_s), \tag{2.14}$$

$\tilde{E}(x) > v_0(x)$ on $\bar{D}$, and $\tilde{E}(x) - v(x,t) > 0$ at $x = 0$ and $x = 1$. To prove $\tilde{E}(x) > v(x,t)$ in $D \times (0,t_s)$, let $Y(x,t) = \tilde{E}(x) - v(x,t)$. Let us suppose that there is a

$$t_6 = \inf\{t : Y(x_4,t) = 0 \text{ for some } x_4 \in D\}.$$

Then, $\tilde{E}(x_4) = v(x_4,t_6)$, $\tilde{E}_t(x_4) = 0 \leq v_t(x_4,t_6)$, and $\tilde{E}_{xx}(x_4) \geq v_{xx}(x_4,t_6)$. Thus,

$$0 \geq -\frac{x_4^q v^{-(m-1)/m}(x_4,t_6)}{m} v_t(x_4,t_6).$$

On the other hand, it follows from (2.14) that

$$-\frac{x_4^q v^{-(m-1)/m}(x_4,t_6)}{m} v_t(x_4,t_6) > \tilde{E}_{xx}(x_4) - v_{xx}(x_4,t_6) \geq 0.$$

This contradiction shows that $\tilde{E}(x) > v(x,t)$ in $D \times (0,t_s)$. When $\delta \to 0$, $E(x) \geq v(x,t)$ in $D \times (0,t_s)$. In particular, $v(x,t)$ is bounded by $E(x)$ at $t = t_s$.

Let us choose a constant $\tilde{\gamma} \in (0, 1/2)$ such that $bg\left(\Gamma\tilde{\gamma}\left(1-\tilde{\gamma}\right)/2+1\right) < \Gamma$. Then we consider

$$\tilde{\tau}' = \frac{2bm\left(\Gamma\tilde{\tau}/8+1\right)^{(m-1)/m} g\left(\Gamma\tilde{\tau}/8+1\right)}{\Gamma\tilde{\delta}^{q+2}},\ \tilde{\tau}\left(t_s\right) = 1.$$

Let t_5 be some positive constant determined by

$$bg\left(\frac{\Gamma\tilde{\gamma}\left(1-\tilde{\gamma}\right)}{2}\tilde{\tau}\left(t_5\right)+1\right) = \Gamma.$$

Following the previous procedures of constructing $\psi\left(x,t\right)$ in Lemma 2.1, we can construct an upper bound $\Psi\left(x,t\right) = E\left(x\right)\tilde{\tau}\left(t\right)+\varepsilon$ of $v\left(x,t\right)$ on $\bar{D}\times\left[t_s, t_s+t_5\right]$. Following the proof of Lemmas 2.4 and 2.5 and Theorem 2.6 with v_0, $\psi\left(x,t\right)$, 0 and t_1 replaced, respectively, by $v\left(x,t_s\right)$, $\Psi\left(x,t\right)$, t_s and t_s+t_5, we obtain a solution on $\bar{D}\times\left[t_s,t_s+t_5\right]$. Thus, $v \in C\left(\bar{\Omega}_{t_s+t_5}\right)\cap C^{2+\alpha,1+\alpha/2}\left(D\times\left[0,t_s+t_5\right]\right)$. This contradicts the definition of t_s. □

The proof of the following theorem is a modification of that by Wiegner [15].

Theorem 2.9. *The problem* (1.1)–(1.2) *has at most one solution.*

Proof. Suppose that the problem (1.1)–(1.2) has two different solutions $u\left(x,t\right)$ and $z\left(x,t\right)$. Without loss of generality, let us assume that $z > u$ somewhere, say, $\left(\bar{x},\bar{t}\right)$ in Ω_T. Since $z\left(x,0\right)-u\left(x,0\right) = 0$ on $\bar{D}$, and $z\left(0,t\right)-u\left(0,t\right) = 0$, and $z\left(1,t\right)-u\left(1,t\right) = 0$, there exists some nonnegative constants a_3, a_4, a_5, and a_6 such that $\bar{x} \in \left(a_5,a_6\right) \subset \left(a_3,a_4\right) \subset \bar{D}$, and $z\left(a_3,t\right) = u\left(a_3,t\right)$ and $z\left(a_4,t\right) = u\left(a_4,t\right)$ for $0 \le t \le \bar{t}$. Also, $z\left(x,\bar{t}\right) > u\left(x,\bar{t}\right)$ for $x \in \left(a_5,a_6\right)$ and $z \ge u$ on $\left[a_3,a_4\right]\times\left[0,\bar{t}\right]$. Let φ and σ denote the fundamental eigenfunction and eigenvalue of the problem,

$$\varphi''+\sigma\varphi = 0 \text{ for } a_3 < x < a_4,\ \varphi\left(a_3\right) = 0 = \varphi\left(a_4\right).$$

Then, $\varphi = \sin\left[\pi\left(x-a_3\right)/\left(a_4-a_3\right)\right]$, and $\sigma = \left[\pi/\left(a_4-a_3\right)\right]^2$. We have

$$\begin{aligned}0 &\le \int_0^{\bar{t}}\int_{a_3}^{a_4}\left(z^m-u^m\right)\sigma\varphi dxdt = -\int_0^{\bar{t}}\int_{a_3}^{a_4}\left(z^m-u^m\right)\varphi'' dxdt\\ &= -\int_0^{\bar{t}}\int_{a_3}^{a_4}\left(z^m-u^m\right)_{xx}\varphi dxdt.\end{aligned}$$

From (1.1), and $z\left(x,0\right) = u\left(x,0\right)$ on $\bar{D}$,

$$\begin{aligned}0 &\le -\int_0^{\bar{t}}\int_{a_3}^{a_4}\left[x^q z_t - bf\left(z\right)-\left(x^q u_t - bf\left(u\right)\right)\right]\varphi dxdt\\ &= -\int_{a_3}^{a_4}x^q\varphi\left(z\left(x,\bar{t}\right)-u\left(x,\bar{t}\right)\right)dx \qquad (2.15)\\ &\quad + b\int_0^{\bar{t}}\int_{a_3}^{a_4}\left(f\left(z\right)-f\left(u\right)\right)\varphi dxdt.\end{aligned}$$

Since $z(x,\bar{t}) - u(x,\bar{t}) \geq 0$ for $x \in [a_3, a_4]$, it follows from the mean value theorem for integrals [4, p. 5] that there exists some $\zeta \in (a_3, a_4)$ such that

$$\int_{a_3}^{a_4} x^q \varphi(z(x,\bar{t}) - u(x,\bar{t}))\, dx = \zeta^q \int_{a_3}^{a_4} \varphi(z(x,\bar{t}) - u(x,\bar{t}))\, dx. \tag{2.16}$$

By the mean value theorem, there exists some ς between z and u such that

$$f(z) - f(u) = f'(\varsigma)(z-u).$$

Since f' exists, $|f'(\varsigma)| \leq k_{19}$ for some positive constant k_{19}. Then,

$$f(z) - f(u) \leq k_{19}(z-u). \tag{2.17}$$

According to (2.16) and (2.17), (2.15) is transformed to

$$\int_{a_3}^{a_4} \varphi(z(x,\bar{t}) - u(x,\bar{t}))\, dx \leq \frac{bk_{19}}{\zeta^q} \int_0^{\bar{t}} \int_{a_3}^{a_4} \varphi(z-u)\, dxdt.$$

By the Gronwall inequality [14, pp. 14-15],

$$\int_{a_3}^{a_4} \varphi(z(x,\bar{t}) - u(x,\bar{t}))\, dx \leq 0.$$

On the other hand, $\varphi(z(x,\bar{t}) - u(x,\bar{t})) > 0$ for $x \in (a_5, a_6)$ implies

$$\int_{a_3}^{a_4} \varphi(z(x,\bar{t}) - u(x,\bar{t}))\, dx > 0.$$

This contradiction shows that the problem (1.1)–(1.2) has at most one solution. □

3. BLOW-UP OF THE SOLUTION

In this section, we study the blow-up of the solution u in the following cases: (i) $f(u) \geq u^p$ where p is a positive constant such that $p > m$ for $u \geq 0$; (ii) $f(u) = u^m$. Let $\phi(x)$ be the fundamental eigenfunction of the problem,

$$\phi'' + \lambda x^q \phi = 0 \text{ in } D,\ \phi(0) = 0 = \phi(1), \tag{3.1}$$

where λ is its corresponding eigenvalue. From the result of Chan and Chan [5], $\lambda > 0$ and

$$\phi(x) = k_{20}(q+2)^{1/2} x^{1/2} J_{1/(q+2)}\left(\frac{2\lambda^{1/2}}{q+2} x^{(q+2)/2}\right) \Big/ \left| J_{[1/(q+2)]+1}\left(\frac{2\lambda^{1/2}}{q+2}\right)\right|,$$

where $J_{1/(q+2)}$ and $J_{[1/(q+2)]+1}$ are Bessel functions of the first kind of orders $1/(q+2)$ and $[1/(q+2)]+1$ respectively, and $\phi(x) > 0$ in D for some positive constant k_{20}. Let us choose k_{20} such that $\int_0^1 x^q \phi(x)\, dx = 1$. If $p > m$, let $R(s) = bs^{p/m}/2 - \lambda s$. The largest positive root of $R(s) = 0$ is given by

$$s = \left(\frac{2\lambda}{b}\right)^{m/(p-m)}.$$

Let $\mu(t) = \int_0^1 x^q \phi(x) u(x,t)\, dx$. When $f(u) \geq u^p$ with $p > m$, we prove that u blows up in a finite time if the initial condition is sufficiently large for any positive b. When $f(u) = u^m$, we show that u blows up if $b > \lambda$.

Theorem 3.1. (i) *If $f(u) \geq u^p$ with $p > m$ and*

$$\int_0^1 x^q \phi u_0^m dx > \left(\frac{2\lambda}{b}\right)^{m/(p-m)},$$

then u blows up in a finite time.

(ii) *If $f(u) = u^m$ and $b > \lambda$, then u blows up in a finite time.*

Proof. (i) According to (1.1) and (3.1), and $f(u) \geq u^p$, we have

$$\begin{aligned} \int_0^1 x^q \phi u_t dx &= \int_0^1 \phi (u^m)_{xx}\, dx + \int_0^1 \phi b f(u)\, dx \\ &\geq -\lambda \int_0^1 x^q \phi u^m dx + b \int_0^1 x^q \phi (u^m)^{p/m}\, dx. \end{aligned} \tag{3.2}$$

It follows by $p > m$ and the Jensen inequality

$$\left(\int_0^1 x^q \phi u dx\right)_t \geq -\lambda \int_0^1 x^q \phi u^m dx + b\left(\int_0^1 x^q \phi u^m dx\right)^{p/m}.$$

By assumption, $\int_0^1 x^q \phi u_0^m dx > (2\lambda/b)^{m/(p-m)}$. Since u is an increasing function of t, we have $R\left(\int_0^1 x^q \phi u^m dx\right) > 0$ for $\int_0^1 x^q \phi u^m dx > (2\lambda/b)^{m/(p-m)}$. This implies that

$$\left(\int_0^1 x^q \phi u dx\right)_t > \frac{b}{2}\left(\int_0^1 x^q \phi u^m dx\right)^{p/m}.$$

By the Jensen inequality,

$$\left(\int_0^1 x^q \phi u dx\right)_t > \frac{b}{2}\left[\left(\int_0^1 x^q \phi u dx\right)^m\right]^{p/m} = \frac{b}{2}\left(\int_0^1 x^q \phi u dx\right)^p.$$

This is equivalent to

$$\mu'(t) > \frac{b}{2}\mu^p(t).$$

Solving this differential inequality, it yields

$$\mu^{-p+1}(t) < \mu^{-p+1}(0) - \frac{b}{2}(p-1)\, t.$$

Thus, $\mu(t)$ tends to infinity in a finite time. Hence, u blows up in a finite time.

(ii) If $f(u) = u^m$, it follows from (3.2) that

$$\left(\int_0^1 x^q \phi u dx\right)_t \geq (b-\lambda) \int_0^1 x^q \phi u^m dx.$$

By using $b > \lambda$ and the Jensen inequality,

$$\mu'(t) \geq (b-\lambda)\, \mu^m(t).$$

Solving this differential inequality, we have

$$\mu^{-m+1}(t) \leq \mu^{-m+1}(0) - (b-\lambda)(m-1)\,t.$$

Thus, $\mu(t)$ tends to infinity in a finite time. Hence, u blows up in a finite time. $\square$

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