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MULTI-NEW PRODUCT COMPETITION IN DUOPOLY: A DIFFERENTIAL GAME ANALYSIS

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ABSTRACT. We use a differential game approach to study some aspects of the dynamics and competition in marketing. A market is a big complex competition environment, and some of the factors that should be considered are the number of competitors, different dynamics of different products, different strategies in the different phases of product life cycles, different marketing goals, etc. In this paper a differential game model is presented to carefully analyze optimal advertising policies over finite planning horizon for two companies. Each company has different brands but similar products. The optimal competitive strategies are given by Nash equilibrium based on their different marketing goals. Numerical computations will be used to get optimal policies and dynamic curves. The numerical algorithms can be use to deal with realistic problems.

Keywords. differential games, marketing, competitive strategy

AMS (MOS) Subject Classification. 91A23, 91A80, 91B50, 91B60

1. INTRODUCTION

Advertising is one of the most important tools in marketing. Advertising is naturally dynamic. A company cannot survive without a delicate and timely consideration of competition. Managers should adjust and/or change their advertising policies based on what is happening in the market. Game theory provides us with a framework to study the interactive and dynamic nature of competitive advertising.

There are different periods one has to consider in the general life cycle of a product. In this paper we concentrate on competition in the first stage of product life cycle. There are few references in the literature dealing with competition in this stage.

We also subdivide the same kind of product into different 'quality' levels. In real markets, a company always sells the same kind of product in different brands. Usually different brands will differ a little in their quality, price, etc. This means competition strategy for different brand products should be different. In this paper we investigate the different competition strategies for different brands of the same product.

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We give more specific practical guidelines to competition strategy based on the subdivision of the market, and different preferences of marketing managers. Such a study is rarely seen in the literature.

From mathematical view, the model in this paper is mathematically accurate. It is appropriate to explain the driving forces (natural growth and competition) of the dynamics, and we give an appropriate objective function based on a specific product life cycle and rationality of people.

In most differential game models in marketing, the models are relatively simple, and can be solved or discussed analytically. Thus, more realistic models are needed. This paper gives a numerical algorithm, which can be used to solve general differential game models. The algorithm is based on full discretization of the continuous space, and is an iterative method. The results from the algorithm are reasonable and explainable.

In the case of two players, the classical differential game has the following set-up: player 1 chooses his control/strategy $\mathbf{u}_1$ to minimize(or maximize) a payoff

$$J_1(u_1,u_2)=\int_{t_0}^{t_f} f_1(t,x_1(t),x_2(t),u_1(t),u_2(t))dt+h_1(t_f,x_1(t_f),x_2(t_f))$$

and player 2 chooses his control/strategy $\mathbf{u_2}$ to minimize(or maximize) a payoff

$$J_2(u_1,u_2)=\int_{t_0}^{t_f} f_2(t,x_1(t),x_2(t),u_1(t),u_2(t))dt+h_2(t_f,x_1(t_f),x_2(t_f))$$

where both of them are subject to the same dynamics:

$$\begin{aligned}\dot{x}_i(t) =&\ \ g_i(t,x_1(t),x_2(t),u_1(t),u_2(t))\\ x_i(t_0) =&\ \ x_{i0}, i=1,2\end{aligned}$$

In this classical game, the information about the game is open to each player. This means that each player knows how many players there are. In addition each player knows how his control and others' affect the dynamics. The players choose their controls at the same time. *Nash equilibrium* is reached when there is no incentive for each player to change his control any more. The Nash strategy is optimal in the sense that if one of the player deviates from the Nash equilibrium, his cost will increase. The following is the formal definition.

Definition 1.1: Suppose $\mathbf{J_1(u_1,u_2)},\mathbf{J_2(u_1,u_2)}$ are performance indices for player 1 and player 2 respectively. Then, the strategy pair $\{\mathbf{u_1^*},\mathbf{u_2^*}\}$ is Nash equilibrium if

$$\begin{aligned}J_1(u_1,u_2^*) &\ \geq J_2(u_1^*,u_2^*)\\ J_2(u_1^*,u_2) &\ \geq J_2(u_1^*,u_2^*)\end{aligned}$$

In the application of differential games, there are two types of controls that are commonly used: *Open-loop* and *Closed-loop*.

Definition 1.2: Control $\mathbf{u}$ is called *open-loop* control if $\mathbf{u} = \mathbf{u}(\mathbf{t})$.
Definition 1.3: Control $\mathbf{u}$ is called *closed-loop* if $\mathbf{u} = \mathbf{u}(\mathbf{t}, \mathbf{x_1}(\mathbf{t}), \mathbf{x_2}(\mathbf{t}))$.

From the above definitions, we can see that an open-loop control is just a function of time. A closed-loop is a function of time and the state. Open-loop controls are easier to compute than closed loop controls, and the disadvantage is that the players choose their controls at the beginning of the game and comply with their strategies in the game, which means that they neglect valuable information about the game. The advantage of a closed-loop control is that the players adjust their controls according to the state. Since the players use the most recent information to make decisions, closed-loop controls will bring bigger benefit to them. The disadvantage of closed-loop controls is that they are more difficult to compute.

The following assumptions are in force.

1. There are two companies/players.
2. Players make their advertising decisions simultaneously and with complete information, which means each competitor has full knowledge of the nature of the competitive interaction and the motivations and profit structure of the other competitors, so that each competitor can infer with certainty the strategies of his rivals.
3. There is no cooperation between competitors, which means that the competitors cannot collude. It is a non-cooperative game.
4. The solution of the game is interpreted as an open-loop Nash equilibrium.

In the past 30 years, differential game models have been constructed based on three basic dynamics: Vidale-Wolfe, Lanchester, and Diffusion models. Vidale-Wolfe advertising model(1957)([40]) is the first mathematical model to describe the dynamics in markets, and which is derived from actual market phenomena and consistent with experimental observations. Because of its capacity to describe the relationship between advertising and sales in a reasonable manner, Vidale-Wolfe type dynamics are used in many differential game models in marketing. Vidale-Wolfe model is based on the equation

$$x'(t) = \rho\mu(1 - \tfrac{x(t)}{M}) - kx(t), \quad x(0) = x_0$$

where $\mathbf{x}$ is the sales rate, $\mathbf{M}$ is the maximum sales potential, and the parameters ρ and $\mathbf{k}$ are the response constant of advertising and sales decay constant, respectively. μ is a control variable, representing the rate of advertising expenditure.

From this model, we can see that the product sales change rate depends on two factors: one is positive response to advertising that acts on the unsold portion of the market, and the other is decay caused by forgetting, which is linearly proportional to the sold portion of the market. The general Vidale-Wolfe in duopoly is:

$$x_i' = \rho_i\mu_i(1 - f(x_i, x_1 + x_2)) - k_i x_i, \quad i = 1, 2$$

where ρ_i is the effect factor of control/advertisment, and $\mathbf{k}_i$ is the natural decrease factor of sales. The function $\mathbf{f}(\mathbf{x}, \mathbf{y})$ is smooth and $\mathbf{f}(\mathbf{x}, \mathbf{y}) > \mathbf{0}$, $\frac{\partial \mathbf{f}}{\partial \mathbf{x}} > \mathbf{0}$, $\frac{\partial \mathbf{f}}{\partial \mathbf{y}} < \mathbf{0}$.

Kenneth R. Deal(1979)([5]) first used deterministic differential game model involving Vidale-Wolfe dynamics to optimize advertising expenditures in a dynamic duopoly

$$\begin{aligned} \max_{u_1} J_1 = & \int_{t_0}^{t_f} (c_1 x_1(t) - u_1^2(t))dt + \omega_1 \frac{x_1(t_f)}{x_1(t_f)+x_2(t_f)} \\ \max_{u_2} J_2 = & \int_{t_0}^{t_f} (c_2 x_2(t) - u_2^2(t))dt + \omega_2 \frac{x_2(t_f)}{x_1(t_f)+x_2(t_f)} \end{aligned}$$

and system dynamics:

$$\begin{aligned} x_1'(t) &= -a_1 x_1(t) + b_1 u_1(t) \frac{M - x_1(t) - x_2(t)}{M} \\ x_2'(t) &= -a_2 x_2(t) + b_2 u_2(t) \frac{M - x_1(t) - x_2(t)}{M} \end{aligned}$$

where $\mathbf{c_i}$ is the price of the $\mathbf{i}-th$ product. The quantity $\mathbf{w}_i$ is weight, which can explain how much the players emphasize final market share. From the above dynamics we can see that it deals with competition at the final stage of product life cycle and the assumption dynamics curve can been seen in *Figure 1*. A major drawback of

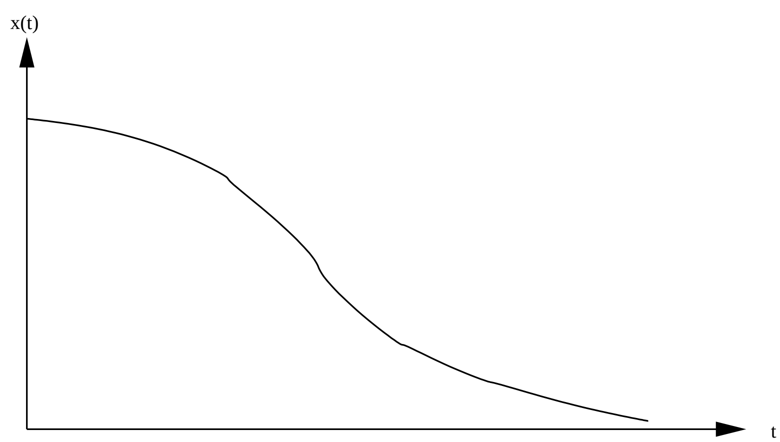

FIGURE 1. sales vs time

the dynamics is both controls do not appear in each of the state dynamics, so players have no direct influence on competitors' sales. We can see that when player $\mathbf{i}$ does not employ any advertising, then $\mathbf{x}_i$ develops as $\mathbf{x_i'} = -\mathbf{a_i x_i}$, and the player $\mathbf{j}, \mathbf{j} \neq \mathbf{i}$ cannot influence the sale of player $\mathbf{i}$. The only indirect influence $\mathbf{j}$ can employ on $\mathbf{i}$ is player $\mathbf{i}$'s potential market. Another drawback is that it just deals with the competition at final stage of product life cycle. In Deal's work, the open-loop controls are solved by maximization of Hamiltonians, and substituted back into the state and adjoint equations yielding a two-point boundary value problem(TPBVP) in the state and costate variables, which are solved numerically.

Lanchester combat model ([32]) is another dynamic model in the advertising competitive environment. Kimball (1957) (George E. Kimball, 1957) ([19]) recognized the application of this model in advertising. The dynamics are given by:

$$x_1'(t) = bu_1x_2(t) - au_2x_1(t)$$
$$x_2'(t) = au_2x_1(t) - bu_1x_2(t)$$

where $\mathbf{x_1(t) + x_2(t) = M}$. By substituting $\mathbf{x_2(t) = M - x_1(t)}$ we can see that the dynamics for the $\mathbf{i}th$ player are a Vidale-Wolfe model with time-varying decay parameter. And the general Lanchester model is:

$$x_i' = g(u_i)x_j - h(u_j)x_i, \quad i \neq j$$

where $\mathbf{g(\cdot), h(\cdot)}$ are continuous and satisfy $\mathbf{g(\cdot), h(\cdot) > 0, g'(x) > 0, h'(x) > 0}$.

The typical 'Lanchester' type differential game is from Case (Case J.H., 1979) ([10]):

$$\begin{aligned} \max_{u_1} J_1 = & \int_{t_0}^{+\infty} e^{-rt}(q_1x(t) - \tfrac{c_1}{2}u_1^2(t))dt \\ \max_{u_2} J_2 = & \int_{t_0}^{+\infty} e^{-rt}(q_2(1 - x(t)) - \tfrac{c_2}{2}u_2^2(t))dt \end{aligned}$$

where system dynamics is given by

$$x'(t) = \quad u_1(t)(1 - x(t)) - u_2(t)x(t), \quad x(0) = x_0$$

Diffusion model emanated from Bass(Frank M. Bass,1969)([6]):

$$\begin{aligned} x'(t) \quad &= (a(u(t)) + b(u(t))x(t))(M - x(t)) \\ &= a(u(t))(M - x(t)) + b(u(t))x(t)(M - x(t)) \end{aligned}$$

This model is an aggregate model and does not model explicitly the adoption patterns of individual customers. There are two effects in this model: one is external influence factor, $\mathbf{a(u)(M - x)}$, the other is internal influence, $\mathbf{b(u)x(M - x)}$, where $\mathbf{a, b}$ may or may not depend on the advertising $\mathbf{u}$. The external effect is interpreted as the effect of advertising on the unsold market. The internal effect is interpreted as a *word-of-mouth* effect. We can see from $\mathbf{b(u)x(M - x)}$ that the untapped market $\mathbf{(M - x)}$ is influenced by customers who have already purchased $(\mathbf{x})$.

Teng and Thompson (Jinn-Tsair Teng, 1983) ([26]) first set up diffusion type deterministic differential game model in n-player oligopoly:

$$J_i = \int_0^T e^{-\rho_i t}\{[p_i - C_{i0}(\frac{S_{i0}}{S_i})^{e_i}]\dot{S}_i - (\alpha_i A_i^2 + \beta_i A_i + \delta_i)\}dt + W_i e^{-\rho_i T} S_i(T), \quad i = 1, \ldots, n$$

Its dynamics is:

$$\dot{S}_i = (\gamma_{i1} + \gamma_{i2}A_i)(1 - S) + (\gamma_{i3} + \gamma_{i4}A_iS_i(1 - S)), \quad i = 1, \ldots, n,$$

where $S = \sum_{i=1}^n S_i$.

2. FORMATION OF SYSTEM DYNAMICS

From the above three basic models, we can see that regardless of the kind of differential game models, the key issue is how the advertising strategies of competition drive or affect the dynamics which describe the changes in the sales or market shares state variables. In this paper, we will set up differential game model based on typical dynamics at the beginning of product life cycle. The typical product life cycle can be approximated by the following graph (Suresh P. Sethi, 1977) ([37]): Our problem is

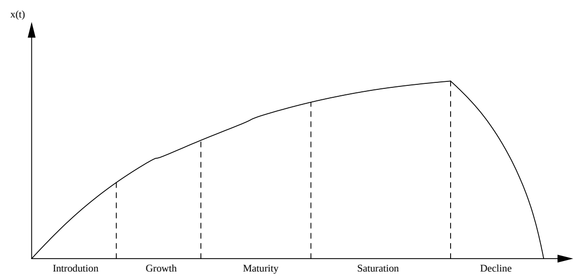

FIGURE 2. sales vs time

how should marketing managers choose advertising strategy to maximize their own-defined objective function when they put a new product into the market. When new products come into the market, the typical dynamics of sales can been seen in *Figure 3*. Because the managers face the dynamics in *Figure 3*, which is the first

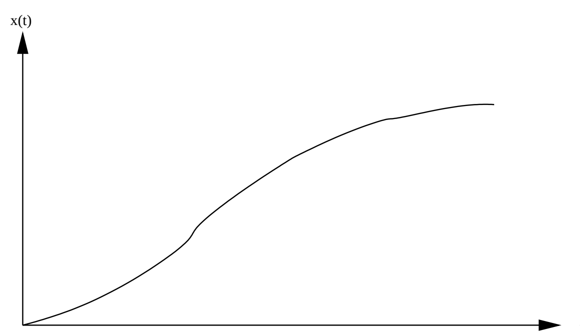

FIGURE 3. sales vs time

stage of the product life cycle, the basic dynamics can not be drawn from Vidale-Wofle. From the curve in *Figure 3*, we can see an approximate logistics growth, so in our basic dynamics we will adopt the logistic equation to describe the growth of sale when a new product comes into the market. Further Olsder (Olsder, G.J. 1976) ([34]) gave the excess advertising model, which says the sales flow in the direction of the company that advertises in the excess:

$$\begin{cases} x_i' &= u_i - u_{3-i}, \quad i = 1, 2 \\ x_1 + x_2 &= 1 \end{cases}$$

So the greater the excess in advertising, the greater the sales flow from one company to the other. This dynamics can be used in the situation where the products are not one-time buying and the customer lack of loyalty and just change to other brand by advertising.

The following is the problem setup. In the system of our model, there are two companies. At initial time $\mathbf{t}_0$, there are $\mathbf{n}$ classes of new products to be put into the market. These two companies will produce these $\mathbf{n}$ kinds of products and sell respectively. The state variables are the sales rate (per unit/per unit time) for two companies, which are $\mathbf{x_i}, \mathbf{i} = \mathbf{1}, \ldots, \mathbf{n}; \mathbf{y_i}, \mathbf{i} = \mathbf{1}, \ldots, \mathbf{n}$ respectively. Here $\mathbf{x_i}, \mathbf{y_i}$ are the sales rate of the **ith** kind product. Customers will switch between the same kind of product. The functions $\mathbf{u_i}(\mathbf{t}), \mathbf{i} = \mathbf{1}, \mathbf{2}$ are the controls used by the marketing managers to compete for customers, which is interpreted as advertising rate. The systems dynamics is as follows:

$$\begin{cases} \frac{1}{x_i}\frac{dx_i}{dt} & = \alpha_{1i}(1 - \frac{x_i}{k_{1i}}) + (\beta_{i1}\frac{u_1}{\sum_{i=1}^n x_i} - \beta_{i2}\frac{u_2}{\sum_{i=1}^n y_i})\frac{y_i}{x_i+y_i} \\ \frac{1}{y_i}\frac{dy_i}{dt} & = \alpha_{2i}(1 - \frac{y_i}{k_{2i}}) + (\gamma_{i1}\frac{u_2}{\sum_{i=1}^n y_i} - \gamma_{i2}\frac{u_1}{\sum_{i=1}^n x_i})\frac{x_i}{x_i+y_i} \\ & i = 1, \ldots, n \\ & x_i(t_0), y_i(t_0) \;\; is \;\; known, \end{cases}$$

In this model, the system is driven by two factors:

1) The natural growth, which can be interpreted as logistic growth model (*Figure 3*). In above dynamics, $a_{ki}(1 - \frac{x_i}{k_{ki}})x_i$ explains the natural growth of sales rate. It is logistic equation because at the beginning of product life cycle, the sale rate will increase, but the potential market of one product is limited, so the sale rate cannot increase indefinitely;

2) The excess advertising effect: In above dynamics, this effect is not just difference of advertising rate, which is the original idea from Olsder (Olsder G.J., 1976) ([34]). By $\frac{u_1}{\sum_{i=1}^n x_i}$, we accept the assumption that the effect of advertising will decrease with the increase of sales. We multiply excess advertising effect by the counterpart's sale rate, because the larger the counterpart's sale rate, the larger the excess advertising effect.

Above all, the change of sales rate will be driven by the addition effect from logistic growth and the excess advertising.

3. THE PERFORMANCE INDICES

The construction of the dynamics of a system is a first step in the description of that system, and a detailed set of objective function is necessary to use the model for planning. Different companies have different goals, even as to the same company, it has different goals when it is in its different development phase. Generally speaking, in the competition for sales in the marketing, the marketing managers have two goals:

one is minimizing cost to attract customers, the other is maximizing the market share in the specific system. Based on these analyses, we set up the performance index by combining these two goals: one is minimizing the total cost (= cost in advertising − income of sales), the other is capture the final market share. It is natural that different companies will emphasize one goal more than the other, which can be done by choosing different weight in the performance index function. So the performance index is mathematically represented as follows:

$$\min_{u_1} J_1 = \int_{t_0}^{t_f} \{(u_1(t))^2 - \delta \sum_{i=1}^{3} p_i x_i\} dt - \omega_1 \frac{\sum_{i=1}^{3} p_i x_i(t_f)}{\sum_{j=1}^{3}(p_j(x_i(t_f) + y_i(t_f)))}$$

$$\min_{u_2} J_2 = \int_{t_0}^{t_f} \{(u_2(t))^2 - \delta \sum_{i=1}^{3} p_i y_i\} dt - \omega_2 \frac{\sum_{i=1}^{3} p_i y_i(t_f)}{\sum_{j=1}^{3}(p_j(x_i(t_f) + y_i(t_f)))}$$

where $\mathbf{p}_i$ refers to the price of product $\mathbf{i}$, and δ is a modification of scales. The quantity $\omega_{(i)}$ is weighting factor, represents different emphasis of goals by the marketing manager. In the following analysis of the model, ω_i is allowed to vary from total profit orientation to where the primary goal is the terminal capital share.

Till now we have set up the differential game model, but in order to get a complete model, some assumptions have to be made. First, because the time interval is not long, we assume that there is no diminishing returns to the expenditures. Second, since the system dynamics and performance index are continuous over time, it is assumed that the managers expend money in relatively continuous ways. Third, cooperation is not allowed. Fourth, we will use open-loop control.

4. NECESSARY CONDITIONS FOR NASH EQUILIBRIUM SOLUTIONS

The necessary conditions for differential game model come from Pontryagin's Minimum Principle. The typical optimal control problem is:

$$\begin{aligned} MinJ(u) = &\ h(x(t_f), t_f) + \textstyle\int_{t_0}^{t_f} g(x(t), u(t), t)dt \\ s.t.\ \ \dot{x} = &\ a(x(t), u(t), t) \end{aligned}$$

Its Hamiltonian is give by:

$$\mathcal{H}(x(t), u(t), p(t), t) \triangleq g(x(t), u(t), t) + p^T[a(x(t), u(t), t)]$$

The necessary conditions for $\mathbf{u}^*$ to be an optimal control are:

$$\begin{cases} \dot{x}^*(t) = & \frac{\partial \mathcal{H}}{\partial p}(x^*(t), u^*(t), p^*(t), t) \\ \dot{p}^*(t) = & -\frac{\partial \mathcal{H}}{\partial x}(x^*(t), u^*(t), p^*(t), t) \\ \mathcal{H}(x^*(t), u^*(t), p^*(t), t) \leq & \mathcal{H}(x^*(t), u(t), p^*(t), t), \quad for\ \ all\ \ admissible\ \ u(t) \end{cases}$$

and boundary condition:

$$[\frac{\partial h}{\partial x}(x^*(t_f), t_f) - p^*(t_f)]^T \delta x_f + [\mathcal{H}(x^*(t), u^*(t), p^*(t), t) + \frac{\partial h}{\partial t}]\delta t_f = 0$$

Theorem 1.1: Solving the $\mathbf{n}$ players' open-loop Nash differential game is equivalent to solving $\mathbf{n^2+2n}$ differential-algebraic equation.

Proof : We assume that players $\mathbf{2,3,\ldots,n}$ give their optimal open-loop controls/advertisment strategies $\mathbf{u_2^*,u_3^*,\ldots,u_n^*}$, at the beginning of game, so player $\mathbf{1}$ has the following problem:

$$\min_{u_1} J_1 = h_1(x(t_f),t_f) + \int_{t_0}^{t_f} f_1(x(t),u_1(t),u_2^*(t),\ldots,u_n^*(t))$$
$$s.t.\ \ \dot{x} = a(x(t),u_1(t),u_2^*(t),\ldots,u_n^*(t),t)$$

Player $\mathbf{1}$'s Hamiltonian is:

$$\mathcal{H}_1 = g_1(x(t),u_1(t),u_2^*(t),\ldots,u_n^*(t)) + \lambda_1^T\cdot a(x(t),u_1(t),u_2^*(t),\ldots,u_n^*(t),t)$$

According to the definition of Nash equilibrium, the Hamiltonian should satisfy:

$$\mathcal{H}_1(x^*(t),u_1^*(t),u_2^*(t),\ldots,u_n^*(t),p^*(t),t) \le \mathcal{H}_1(x^*(t),u_1(t),u_2^*(t),\ldots,u_n^*(t),p^*(t),t)$$

In addition we have:

$$\begin{cases} \dot{x}^*(t) = & \frac{\partial\mathcal{H}_1}{\partial p}(x^*(t),u_1^*(t),u_2^*(t),\ldots,u_n^*(t),p^*(t),t) \\ \dot{p}^*(t) = & -\frac{\partial\mathcal{H}_1}{\partial x}(x^*(t),u_1^*(t),u_2^*(t),\ldots,u_n^*(t),p^*(t),t) \\ 0 = & \frac{\partial\mathcal{H}_1}{\partial u_1}(x^*(t),u_1(t),u_2^*(t),\ldots,u_n^*(t),p^*(t),t) \\ & x^*(0)\ \ given,\ \ p^*(t_f) = \frac{\partial h_1}{\partial x}(x^*(t_f),t_f) \end{cases}$$

The status of each player is symmetric, so each player faces the same kind of optimal control problem, so we can draw $\mathbf{n}$ necessary conditions for all players, and in each necessary condition, there are $\mathbf{n}$ state, $\mathbf{n}$ co-state equations and $\mathbf{n}$ algebraic equation for controls. By combing all necessary conditions we get $\mathbf{n}$ state equations, $\mathbf{n^2}$ co-state equations and $\mathbf{n}$ algebraic equations, so we get $\mathbf{n^2+2n}$. Further, if we can solve for the controls explicitly in terms of state and costate variables, then we can get $\mathbf{n^2+n}$ two point-boundary value problems (TPBVP). ■

5. NUMERICAL ALGORITHM

In order to solve our differential game model numerically, we set up the following concrete system:

$$\begin{cases} \frac{1}{x_1}\frac{dx_1}{dt} & = \alpha_{11}(1-\frac{x_1}{k_{11}}) + (\beta_{11}\frac{u_1}{\sum_{i=1}^n x_i} - \beta_{12}\frac{u_2}{\sum_{i=1}^n y_i})\frac{y_1}{x_1+y_1} \\ \frac{1}{x_2}\frac{dx_2}{dt} & = \alpha_{12}(1-\frac{x_2}{k_{12}}) + (\beta_{21}\frac{u_1}{\sum_{i=1}^n x_i} - \beta_{22}\frac{u_2}{\sum_{i=1}^n y_i})\frac{y_2}{x_2+y_2} \\ \frac{1}{x_3}\frac{dx_3}{dt} & = \alpha_{13}(1-\frac{x_3}{k_{13}}) + (\beta_{31}\frac{u_1}{\sum_{i=1}^n x_i} - \beta_{32}\frac{u_2}{\sum_{i=1}^n y_i})\frac{y_3}{x_3+y_3} \\ \frac{1}{y_1}\frac{dy_1}{dt} & = \alpha_{21}(1-\frac{y_1}{k_{21}}) + (\gamma_{11}\frac{u_2}{\sum_{i=1}^n y_i} - \gamma_{12}\frac{u_1}{\sum_{i=1}^n x_i})\frac{x_1}{x_1+y_1} \\ \frac{1}{y_2}\frac{dy_2}{dt} & = \alpha_{22}(1-\frac{y_2}{k_{22}}) + (\gamma_{21}\frac{u_2}{\sum_{i=1}^n y_i} - \gamma_{22}\frac{u_1}{\sum_{i=1}^n x_i})\frac{x_2}{x_2+y_2} \\ \frac{1}{y_3}\frac{dy_3}{dt} & = \alpha_{23}(1-\frac{y_3}{k_{23}}) + (\gamma_{31}\frac{u_2}{\sum_{i=1}^n y_i} - \gamma_{32}\frac{u_1}{\sum_{i=1}^n x_i})\frac{x_3}{x_3+y_3} \\ & i = 1,\ldots,n \\ & x_i(t_0), y_i(t_0)\ \ is\ \ given, \end{cases}$$

The necessary condition for optimality of the differential game is:

$$\begin{cases} \frac{dx^*}{dt} &= f_1(t, x^*, y^*, u_1^*, u_2^*), \quad x^*(0) = x_0 \\ \frac{dy^*}{dt} &= f_2(t, x^*, y^*, u_1^*, u_2^*), \quad y^*(0) = y_0 \\ u_1^* &= argmin \mathcal{H}_1(t, \lambda, x^*, y^*, u_1, u_2^*) \\ u_2^* &= argmin \mathcal{H}_2(t, r, x^*, y^*, u_1^*, u_2) \\ \frac{d\lambda}{dt} &= -\frac{\partial}{\partial x}\mathcal{H}_1(t, \lambda, n^*, u^{(1)*}, u^{(2)*}) \\ \frac{dr}{dt} &= -\frac{\partial}{\partial y}\mathcal{H}_2(t, \gamma, n^*, u^{(1)*}, u^{(2)*}) \\ \lambda(t_f) &= \frac{\partial}{\partial x} h_1(x^*(t_f)) \\ r(t_f) &= \frac{\partial}{\partial y} h_2(y^*(t_f)) \end{cases}$$

where

$$\begin{cases} \mathcal{H}_1(t, \lambda, x, y, u_1, u_2) &= g_1(t, x, y, u_1, u_2) + \lambda^T \cdot f(t, x, y, u_1, u_2) \\ \mathcal{H}_2(t, r, x, y, u_1, u_2) &= g_2(t, x, y, u_1, u_2) + r^T \cdot f(t, x, y, u_1, u_2) \end{cases}$$

Because $\mathcal{H}_1, \mathcal{H}_2$ are convex function of u_1, u_2 respectively, we obtain u_i using $\frac{\partial \mathcal{H}_i}{\partial u_i} = 0$. In above system there are 6 equations with 6 given initial conditions and 12 costate equations with 12 terminal conditions and 2 equations for the controls.

We use iterative algorithm to solve above system, which is based on the following process:

1. The two players give their initial controls randomly;
2. We use these controls to solve the state equations forward by Euler or Runge-Kutta methods;
3. Using $x(t_f), y(t_f)$ solved in second step we proceed to get $\lambda(t_f), r(t_f)$. Then, we find the costate equations by solving the state and costate equations backwards by Euler or Runge-Kutta methods.
4. Using the values of state and costate variables we check if $\frac{\partial \mathcal{H}_1}{\partial u_1} = 0$, $\frac{\partial \mathcal{H}_2}{\partial u_2} = 0$ are satisfied. If yes, we have the optimal control strategies; if not, using steepest descent algorithm to get new control trajectories, repeat the above steps starting with the first step.

Based on the above process, we get the following steepest descent algorithm to solve open loop differential game:

Algorithm 1.1:

1. Generate randomly a discrete approximation to the controls $u_1(t), u_2(t), t \in [t_0, t_f]$, that is:

$$\begin{aligned} u_1(t) = &\quad u_1(t_k), \quad t \in [t_k, t_{k+1}), \quad k = 1, 2, \ldots, N \\ u_2(t) = &\quad u_2(t_k), \quad t \in [t_k, t_{k+1}), \quad k = 1, 2, \ldots, N \end{aligned}$$

2. Use $u_1(t), u_2(t)$ to integrate the state equation from t_0 to t_f with initial condition $x(t_0) = x_0, y(t_0) = y_0$. The resulting state trajectory x, y is stored as piecewise-constant vector.

3. Calculate $\lambda(t_f), r(t_f)$ using $x(t_f), y(t_f)$ from $\lambda(t_f) = \frac{\partial}{\partial x}h_1(x(t_f)), r(t_f) = \frac{\partial}{\partial y}h_2(y^*(t_f))$ and integrate the costate equations backward.
4. Use the discrete value of state and costate variables x, y to evaluate $\frac{\partial H_1}{\partial u_1}, \frac{\partial H_2}{\partial u_2}$.
5. If $\| \frac{\partial H_1}{\partial u_1} \| \leq \epsilon, \| \frac{\partial H_2}{\partial u_2} \| \leq \epsilon$,where $\| \frac{\partial H_i}{\partial u} \|^2 = \int_{t_0}^{t_f} [\frac{H_i}{\partial u}(t)]^T [\frac{H_i}{\partial u}(t)]dt$, then terminate the iterative procedure and output the extremal state and control.

   If the stopping criterion is not satisfied, generate a new pair of piecewise constant controls given by

$$\begin{cases} u_1(t_{k+1}) = & u_1(t_k) - \tau_1 \frac{\partial H_1}{\partial u_1}(t_k), \quad k = 1, 2, \dots, N \\ u_2(t_{k+1}) = & u_2(t_k) - \tau_2 \frac{\partial H_2}{\partial u_2}(t_k), \quad k = 1, 2, \dots, N \end{cases}$$

   where step length τ_1, τ_2 will be chosen to decrease H_1, H_2. Then go back to step 2.

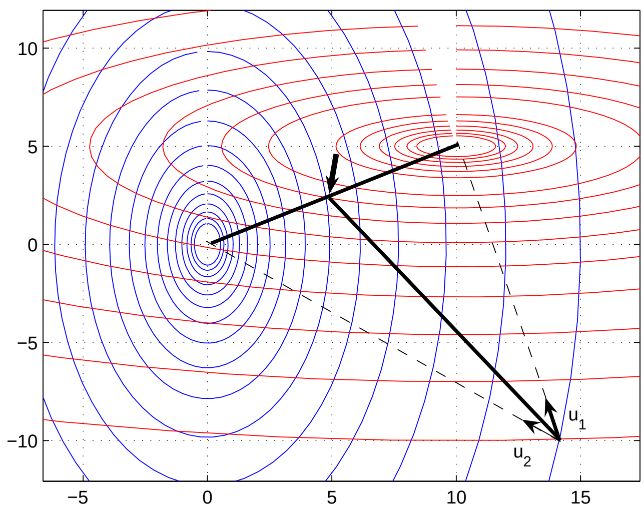

FIGURE 4. Trajectory in Steepest Descent

From *Figure 4*, we can feel the possible trajectory in the above algorithm provided that the Hamiltonian is a convex function of the control.

6. ANALYSIS OF MODEL

In order to test the model, we set up the value of parameters based on some practical situations, so there are some assumptions put on the parameters.

Assumption 1: Company 1 is relatively 'bigger' than company 2, which means that the effectiveness of company 1 is larger than company 2;

Assumption 2: The prices of three kinds of products under consideration are such that: $p_1 < p_2 < p_3$;

Assumption 3: $\beta_{11} > \beta_{12}, \gamma_{11} < \gamma_{12}, \beta_{21} \simeq \beta_{22}, \gamma_{21} \simeq \gamma_{22}, \beta_{31} < \beta_{32}, \gamma_{31} > \gamma_{32}$, which means that the same-product-buyers are more easily driven by the bigger company's advertising.

Assumption 4: $\beta_{11} > \beta_{21} > \beta_{31}, \gamma_{11} > \gamma_{21} > \gamma_{31}$, which means that for the customers of same company, relatively-expensive-product-buyers are less driven by advertising.

Assumption 5: $\alpha_{11} > \alpha_{12} > \alpha_{13}, \alpha_{21} > \alpha_{22} > \alpha_{23}$, which means that the natural sale growth of cheaper products is larger.

Assumption 6: $\alpha_{i1} > \alpha_{i2}, i = 1, 2, 3$, which means that for the same kind of product, the natural sale growth of bigger company is larger;

Assumption 7: $k_{11} > k_{12} > k_{13}$, which means that for one company, cheaper product has larger market potential.

Assumption 8: $k_{1j} > k_{2j}, j = 1, 2, 3$, which means that for the same product, bigger company has larger market potential.

In the following we will discuss the different results that correspond to different choices of parameters ω_i. The quantities x_1, x_2, x_3 represent the sales rate of three products of company 1, and y_1, y_2, y_3 the sales rate of three products of company 2. The time interval is six months. Other parameters values are as follows:

Scenario 1: $\omega_1 = \omega_2 = 1000$. Both marketing managers prefer bigger final market share to profit. The outcome can be seen in *Figure* 5. The control/advertising strategy for manager 1 is at first bigger control than manager 2 and decrease control/advertising as time goes on. And manager 2's strategy is increasing control/advertising gradually and then decrease. Most of the time, manager 2's control/advertising is bigger than that of manager 1. This makes sense because the company of manager 1 is bigger and customers are sensitive to its control/advertising. However for manager 2, in order to get more customers, he should use bigger control/advertising. Another observation is the migration tendency of expensive-product-buyer can easily reflect the change of strength of advertising. First, manager 1's control is larger, so the expensive product sale increases sharply. When manager 2's control is much larger than manager 1's , the expensive-product-buyers migrate to company 2. The change of control reflects that, in order to get larger final market share, they put their eyes on the expensive products.

Scenario 2: $\omega_1 = 1, \omega_2 = 1000$. Manager 2 prefers much more final market share to profit than manager 1. From analysis in scenario 1, intuitively manager 2 should use bigger control/advertising than manager 2 to attract expensive-product-buyers. We can see this in *Figure* 6. At the same time, we can see a little migration of median-product-buyer from company 1 to company 2 and little migration of cheap-product-buyers from company 2 to company 1, which make sense because expensive-product-buyers are what manager 2 wants.

Scenario 3: $\omega_1 = 1000, \omega_2 = 1$. Manager 1 prefers much more final market share to profit than manager 2. From *Figure* 7, the observation is that manager 1 uses bigger control/advertising initially to get more customers in, which leads all

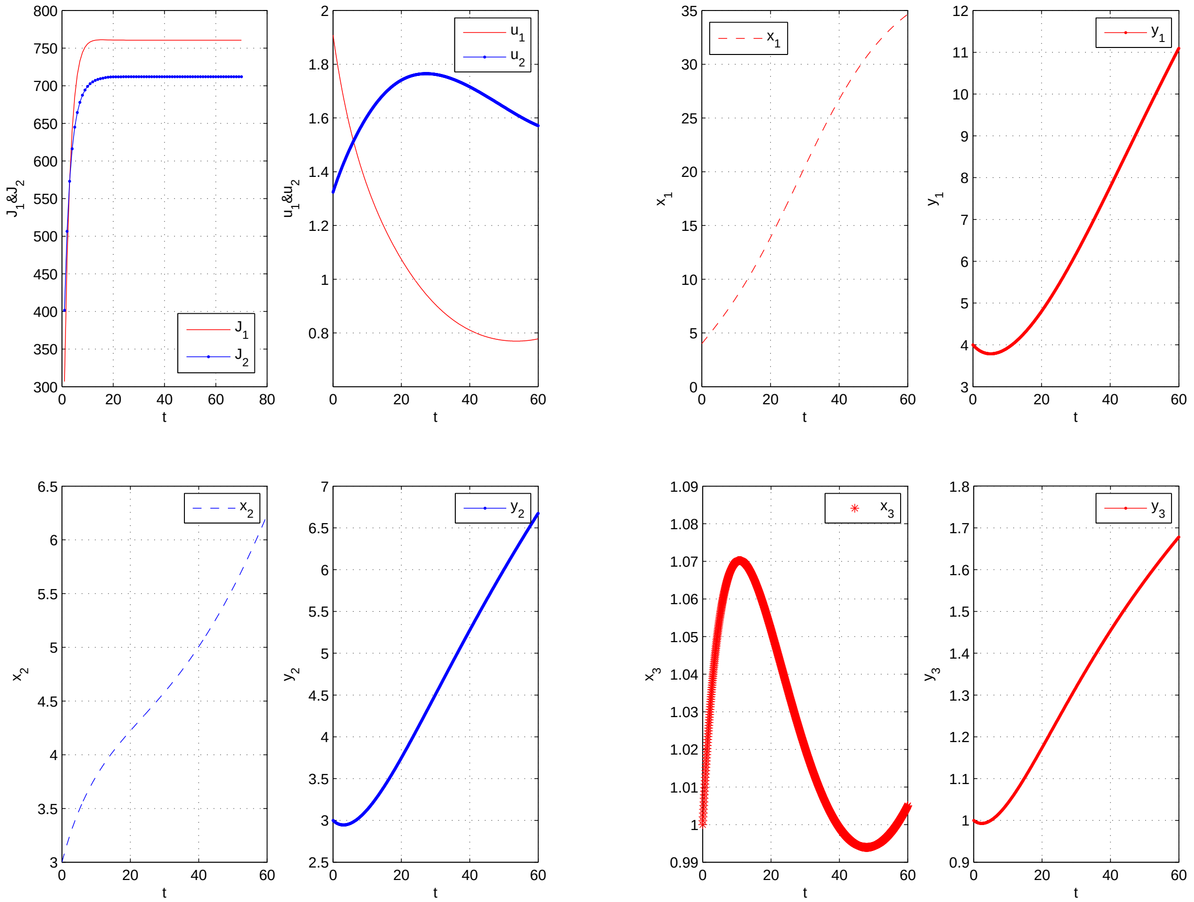

FIGURE 5. $\omega_1 = \omega_2 = 1000$

of three kinds of customers to migrate to company 1. Manager 2 can not let this migration continue more and more, so he increases his control/advertising so that he attracts back some customers. Most of the time, control/advertising of company 1 is bigger because it seeks larger final market share.

Scenario 4: $\omega_1 = 1, \omega_2 = 1$. Both managers prefer profit to the final market share to the same extent. From *Figure* 8, we can still see that at first manager 1 uses bigger control to bring investors in and then decrease control; on the other hand, after the customers migrate to company 1, manager 2 increases his control and bring some customers back. Both of them decrease controls gradually, because they care only about the profit.

Above all, we can see in this open-loop game for market, no matter in what situation, manager 1 always predominates manager 2 because of his company's good conditions. On the one hand, Nash equilibrium strategy for manager 1 is always at first using bigger control/advertising to bring customers in and afterwards decrease control/advertising gradually, which will not lead great migration of customers to the other company. This is because once these customers stay with bigger company

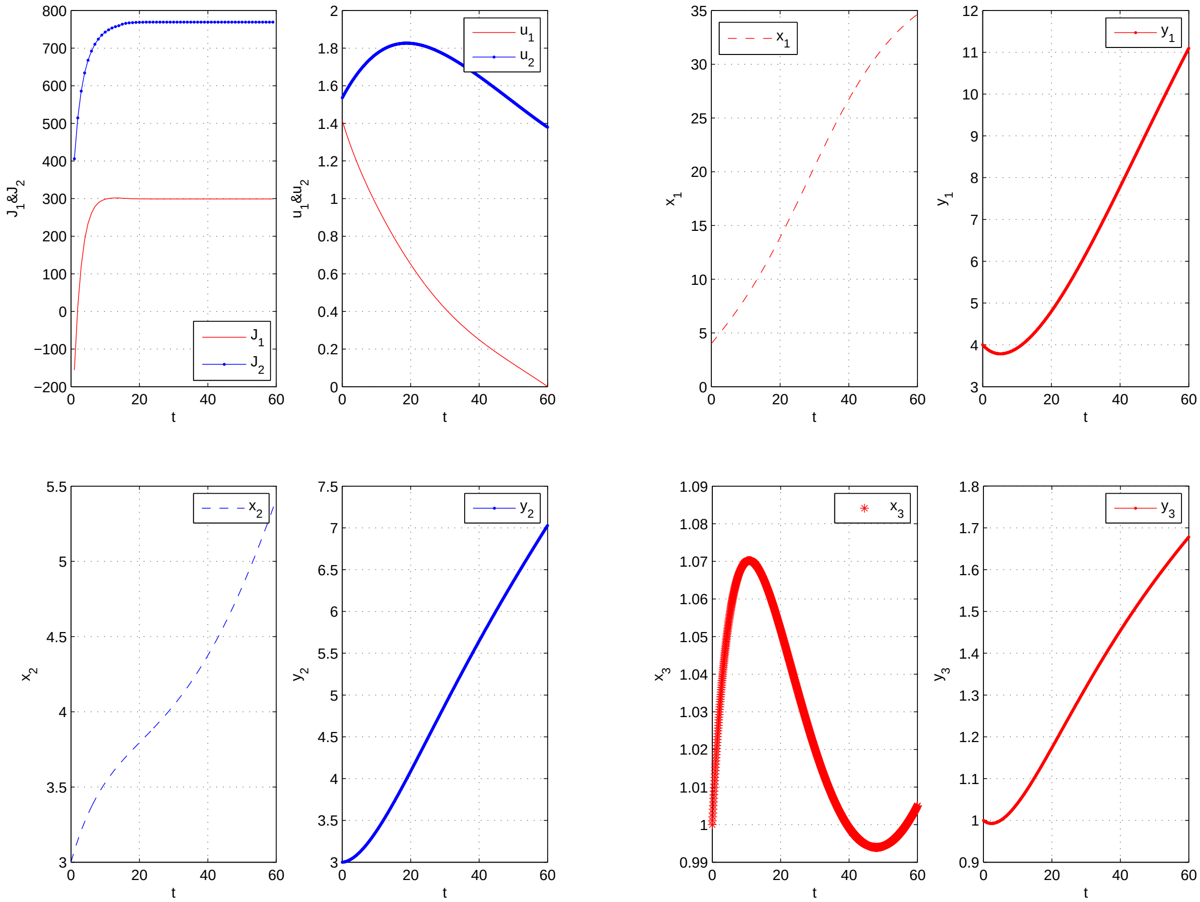

FIGURE 6. $\omega_1 = 1, \omega_2 = 1000$

they do not want to migrate except for the much greater pull from another company. From the numerical result the company 1's control fully exploits this preference in customer's heart. Further, manager 2 recognizes his conditions are not as good as manager 1, so if he wants to get more final market share, he should increase his control/advertising and keep his control/advertising at some level similar or bigger than manager 2. This is because once he decrease it, the customers will prefer to migrate to the bigger company.

7. CONCLUSION

The above open-loop model of market competition deals with the competition when a new product comes into the market. It is beyond the general market competition based on Vidale-Wolfe dynamics. It gives us insight into the complexity of competition in the marketing. It gives us a justifiable explanation about successful use of advertising between two companies at the beginning of product life cycle. We note that the algorithm presented is especially efficient to solve open-loop differential game even if we cannot get the explicit expression for control u from the Hamilton

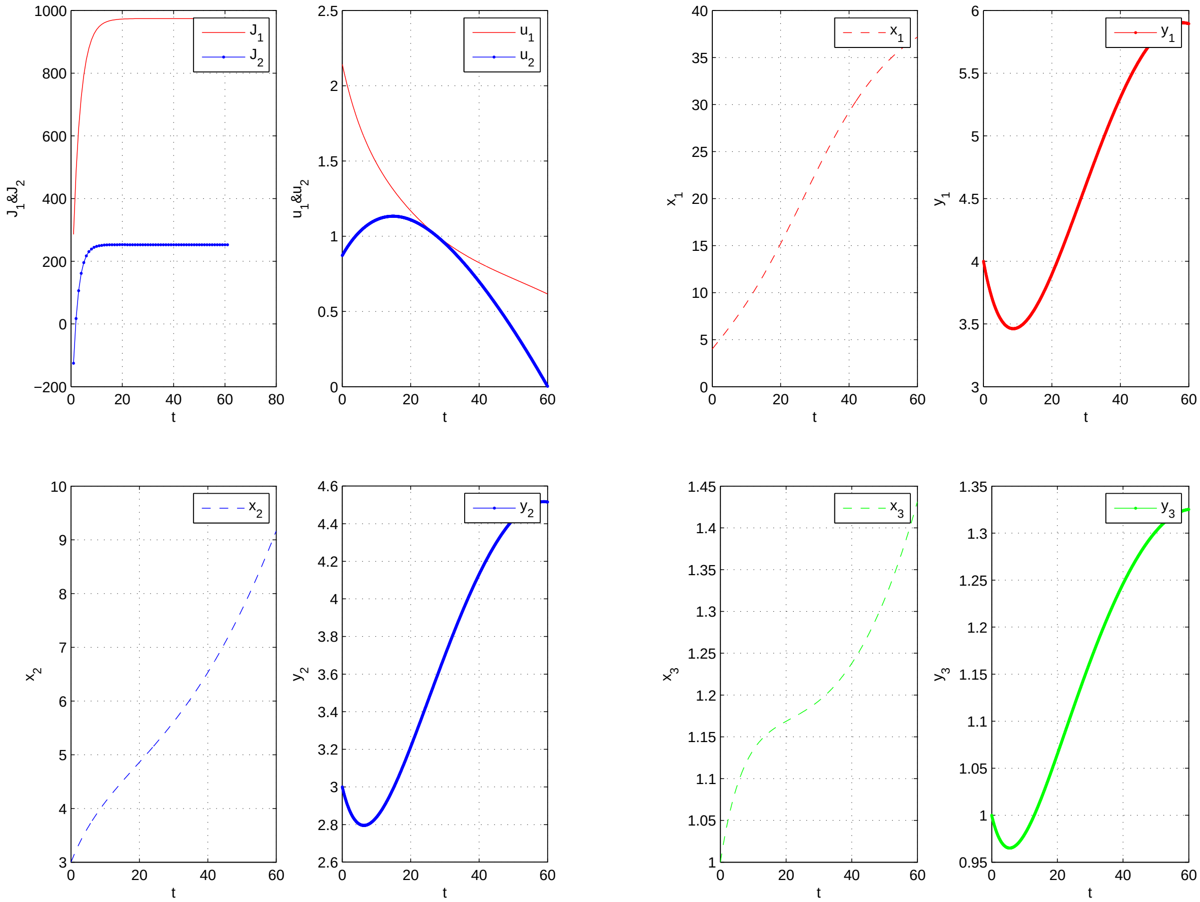

Figure 7. $\omega_1 = 1000, \omega_2 = 1$

function. This is significant since we are put in a position to deal with realistic problems. The solution has been proven to be correct based on the practical experience. Although this model is open-loop and need more extension, it give us one way to further research into this field.

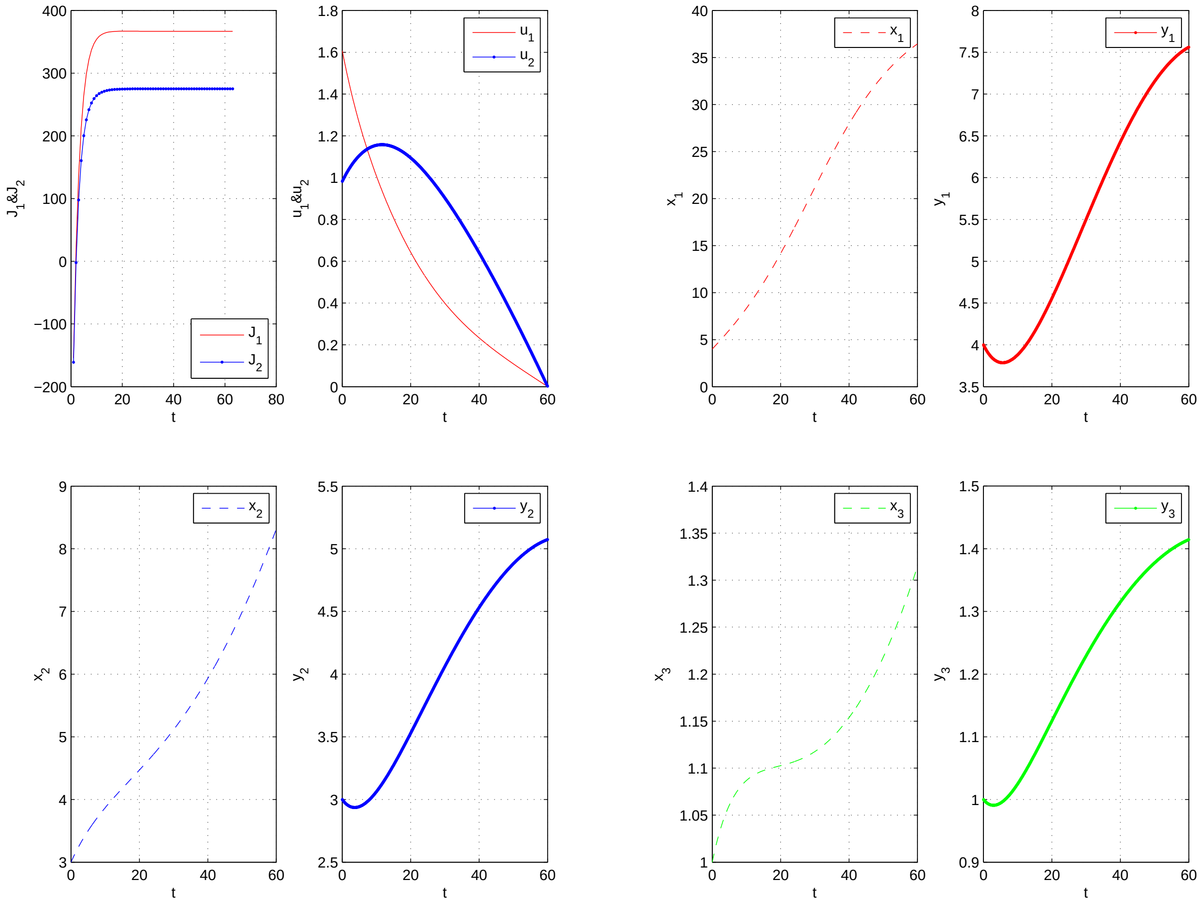

FIGURE 8. $\omega_1 = 1, \omega_2 = 1$

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