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SOLVABILITY ANALYSIS OF PERIODIC SOLUTIONS FOR nTH ORDER DIFFERENTIAL EQUATIONS

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ABSTRACT. In this paper, by using an initial value problem method and a global inverse function theorem, we give some existence and uniqueness results of periodic solution for a class of nth-order nonlinear ordinary differential equations.

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1. INTRODUCTION

We consider the following nth-order differential equation:

$$x^{(n)}(t) + f(t, x(t), x'(t), \dots, x^{(n-1)}(t)) = 0, \tag{1.1}$$

where $f(t, y_0, \dots, y_{n-1}) : \mathbb{R} \times \mathbb{R}^n \longrightarrow \mathbb{R}$ is continuous, 2π-periodic with respect to t and $\frac{\partial f}{\partial y_i}$ is continuous for $t \in [0, 2\pi]$, $y_i \in \mathbb{R}$, $i = 0, 1, \dots, n-1$. We study the existence and uniqueness of (1.1) subject to the periodic conditions:

$$x^{(i)}(0) = x^{(i)}(2\pi), \quad i = 0, \dots, n-1. \tag{1.2}$$

During the past 20 years, there has been a huge amount of work [9, 3, 12, 11, 13, 19, 10, 20] concerning periodic solutions for the even order Duffing equation

$$x^{(2n)} + \sum_{i=0}^{n-1} c_i x^{(2i)} + f(t, x) = 0, \tag{1.3}$$

where $f : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ is continuous and 2π-periodic with respect to t and c_i, $i = 0, \dots, n-1$, are constants.

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Recently, Cong et al. in [4] used the continuation theorem of Mawhin [16] and Schauder's fixed point theorem [17] to deduce an existence uniqueness result for 2π-periodic solutions of the following $(2n+1)$th order differential equation

$$x^{(2n+1)} + \sum_{i=0}^{n-1} c_i x^{(2i+1)} + f(t,x) = 0. \tag{1.4}$$

In [14], Liu et al. established the existence of 2π-periodic solutions for equation (1.1), by applying the theory of topological degree [16, 15]. For more details on the study of odd-order differential equations and arbitrary order differential equations, see [6, 5, 7, 8] and the references therein.

The purpose of this paper is to provide a constructive proof for the existence and uniqueness of periodic solutions of equation (1.1) with an initial value problem method, which can be implemented on the computer. Our techniques are different from those mentioned above.

We now state some lemmas. Consider an initial value problem

$$\boldsymbol{x}'(t) = g(t,\boldsymbol{x}(t)), \quad \boldsymbol{x}(0) = \boldsymbol{x}_0, \qquad t \in [0,2\pi]. \tag{1.5}$$

Lemma 1.1 (see [2])**.** *Assume that $g(t,\boldsymbol{x}(t)) \in C^1([0,2\pi]\times\mathbb{R}^n,\mathbb{R}^n)$ and $\frac{\partial g(t,\boldsymbol{x}(t))}{\partial \boldsymbol{x}}$ are continuous on $[0,2\pi]\times\mathbb{R}^n$. Let the solution $\boldsymbol{x} = \boldsymbol{x}(t,t_0,\boldsymbol{x}_0)$ of (1.5) exist for $t\in[0,2\pi]$ and let*

$$H(t,t_0,\boldsymbol{x}_0) = \frac{\partial g(t,\boldsymbol{x}(t,t_0,\boldsymbol{x}_0))}{\partial \boldsymbol{x}}. \tag{1.6}$$

Then

$$U(t,t_0,\boldsymbol{x}_0) = \frac{\partial \boldsymbol{x}(t,t_0,\boldsymbol{x}_0)}{\partial \boldsymbol{x}_0} \tag{1.7}$$

exists and is the solution of

$$V' = H(t,t_0,\boldsymbol{x}_0)V, \tag{1.8}$$

where $U(t_0,t_0,\boldsymbol{x}_0)$ is an $n\times n$ unit matrix.

Lemma 1.2 (see [18])**.** *Assume that $\mathcal{V}$ is a Hilbert space. Let $\mathcal{L}\in C^1(\mathcal{V},\mathcal{V})$ with $\mathcal{L}'(\boldsymbol{x})$ everywhere invertible, $\forall \boldsymbol{x}\in\mathcal{V}$. Then $\mathcal{L}$ is a global diffeomorphism onto $\mathcal{V}$ if*

$$\int_0^{+\infty} \inf_{\boldsymbol{x}\in\mathcal{V},\|\boldsymbol{x}\|\le s} \frac{1}{\|[\mathcal{L}'(\boldsymbol{x})]^{-1}\|}\,\mathrm{d}s = +\infty. \tag{1.9}$$

Furthermore, if $\mathcal{V}=\mathbb{R}^n$, then $\mathcal{L}$ is a global diffeomorphism onto $\mathbb{R}^n$ if

$$\int_0^{+\infty} \inf_{\boldsymbol{x}\in\mathbb{R}^n,\|\boldsymbol{x}\|= s} \frac{1}{\|[\mathcal{L}'(\boldsymbol{x})]^{-1}\|}\,\mathrm{d}s = +\infty. \tag{1.10}$$

2. EXISTENCE AND UNIQUENESS

We write $x_1 = x, x_2 = x', \ldots, x_n = x^{(n-1)}$. Denote

$$\boldsymbol{x} = (x_1, x_2, \ldots, x_n)^{\mathrm{T}}.$$

Then (1.1) can be written as

$$\boldsymbol{x}' = G(t, \boldsymbol{x}(t)), \quad \boldsymbol{x}(0) = \boldsymbol{x}(2\pi), \tag{2.1}$$

where $G(t, \boldsymbol{x}) = (x_2, \ldots, x_n, -f(t, \boldsymbol{x}))^{\mathrm{T}}$. Also we obtain

$$\frac{\partial G(t, \boldsymbol{x})}{\partial \boldsymbol{x}} = \begin{pmatrix} 0 & 1 & 0 & \cdots & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & \cdots & \cdots & 0 & 1 \\ -\frac{\partial f}{\partial x_1} & \cdots & \cdots & \cdots & \cdots & -\frac{\partial f}{\partial x_n} \end{pmatrix}. \tag{2.2}$$

Consider an initial value problem

$$\boldsymbol{x}' = G(t, \boldsymbol{x}), \quad \boldsymbol{x}(0) = \boldsymbol{\alpha}, \tag{2.3}$$

where $\boldsymbol{\alpha} \in \mathbb{R}^n$. Define the C^1 mappings $h(\boldsymbol{\alpha}) : \mathbb{R}^n \to \mathbb{R}^n$ and $H(\boldsymbol{\alpha}) : \mathbb{R}^n \to \mathbb{R}^n$ as follows:

$$h(\boldsymbol{\alpha}) = \boldsymbol{x}(2\pi, \boldsymbol{\alpha}), \quad H(\boldsymbol{\alpha}) = \boldsymbol{\alpha} - h(\boldsymbol{\alpha}). \tag{2.4}$$

Then, finding periodic solutions of problem (1.1), (1.2) is equivalent to finding fixed points of $h(\boldsymbol{\alpha})$ or zeros of $H(\boldsymbol{\alpha})$. Denote the non-negative orthant by $\mathbb{R}_+ = \{s \in \mathbb{R}, s \geq 0\}$, and the positive orthant by $\mathbb{R}_{++} = \{s \in \mathbb{R}, s > 0\}$.

Theorem 2.1. *Suppose that the initial value problem* (2.3) *has a solution* $\boldsymbol{x}(t, \alpha)$ *on* $[0, 2\pi]$. *If the inverse of* $\int_0^{2\pi} \frac{\partial G(t, \boldsymbol{x}(t, \boldsymbol{\alpha}))}{\partial \boldsymbol{\alpha}} \, \mathrm{d}t$ *exists and there is a function* $\Theta(s) : \mathbb{R}_+ \to \mathbb{R}_{++}$ *such that*

$$\left\| \left(\int_0^{2\pi} \frac{\partial G(t, \boldsymbol{x}(t, \boldsymbol{\alpha}))}{\partial \boldsymbol{\alpha}} \, \mathrm{d}t \right)^{-1} \right\| \leq \frac{1}{\Theta(\|\boldsymbol{x}\|)}, \quad \int_0^{\infty} \Theta(s) \, \mathrm{d}s = +\infty, \tag{2.5}$$

then problem (1.1), (1.2) *has a unique solution.*

Proof. Consider the variation equation of (2.3) with respect to $\boldsymbol{\alpha}$

$$z' = \frac{\partial G(t, \boldsymbol{x}(t, \boldsymbol{\alpha}))}{\partial \boldsymbol{x}} z, \tag{2.6}$$

where $z(0) = I_{n \times n}$. By Lemma 1.1, we know that the matrix $\frac{\partial \boldsymbol{x}(t, \boldsymbol{\alpha})}{\partial \boldsymbol{\alpha}}$ is a solution of (2.6). Thus by integrating (2.6) with respect to t, we get

$$\begin{aligned} \frac{\partial \boldsymbol{x}(t, \boldsymbol{\alpha})}{\partial \boldsymbol{\alpha}} &= I + \int_0^t \frac{\partial G(s, \boldsymbol{x}(s, \boldsymbol{\alpha}))}{\partial \boldsymbol{x}} \cdot \frac{\partial \boldsymbol{x}(s, \boldsymbol{\alpha})}{\partial \boldsymbol{\alpha}} \, \mathrm{d}s \\ &= I + \int_0^t \frac{\partial G(s, \boldsymbol{x}(s, \boldsymbol{\alpha}))}{\partial \boldsymbol{\alpha}} \, \mathrm{d}s. \end{aligned} \tag{2.7}$$

On the other hand, from (2.4), we obtain

$$H'(\boldsymbol{\alpha}) = I - h'(\boldsymbol{\alpha}) = I - \frac{\partial \boldsymbol{x}(2\pi, \boldsymbol{\alpha})}{\partial \boldsymbol{\alpha}}. \tag{2.8}$$

Then, it follows from (2.7) that

$$H'(\boldsymbol{\alpha}) = -\int_0^{2\pi} \frac{\partial G(s, \boldsymbol{x}(s, \boldsymbol{\alpha}))}{\partial \boldsymbol{\alpha}} \, \mathrm{d}s. \tag{2.9}$$

By (2.5), we have

$$\begin{aligned} \int_0^\infty \inf_{\|\boldsymbol{\alpha}\|=s} \frac{1}{\|(H'(\boldsymbol{\alpha}))^{-1}\|} \, \mathrm{d}s &= \int_0^\infty \inf_{\|\boldsymbol{\alpha}\|=s} \frac{1}{\left\| \left(\int_0^{2\pi} \frac{\partial G(t, \boldsymbol{x}(t, \boldsymbol{\alpha}))}{\partial \boldsymbol{\alpha}} \, \mathrm{d}t \right)^{-1} \right\|} \, \mathrm{d}s \\ &\geq \int_0^\infty \Theta(s) \, \mathrm{d}s = +\infty. \end{aligned} \tag{2.10}$$

This, together with Lemma 1.2, implies that $H(\boldsymbol{\alpha})$ is a homeomorphism from $\mathbb{R}^n$ onto $\mathbb{R}^n$. By the norm-coerciveness theorem in [17, Theorem 5.3.8], we have

$$\lim_{\|\boldsymbol{\alpha}\| \to \infty} \|H(\boldsymbol{\alpha})\| = \infty. \tag{2.11}$$

Now define a mapping

$$\mathcal{M}(\boldsymbol{\alpha}, \tau, \boldsymbol{\alpha}_0) = H(\boldsymbol{\alpha}) - (1-\tau)H(\boldsymbol{\alpha}_0), \quad \tau \in [0,1], \quad \boldsymbol{\alpha}, \boldsymbol{\alpha}_0 \in \mathbb{R}^n. \tag{2.12}$$

If τ is small enough, then the initial value problem

$$\frac{\mathrm{d}\boldsymbol{\alpha}}{\mathrm{d}\tau} = -[H'(\boldsymbol{\alpha})]^{-1} H(\boldsymbol{\alpha}_0), \quad \boldsymbol{\alpha}(0) = \boldsymbol{\alpha}_0, \tag{2.13}$$

has a solution $\boldsymbol{\alpha}(\tau)$ satisfying $\mathcal{M}(\boldsymbol{\alpha}, \tau, \boldsymbol{\alpha}_0) = 0$, and

$$H'(\boldsymbol{\alpha}) \frac{d\boldsymbol{\alpha}}{d\tau} + H(\boldsymbol{\alpha}_0) = 0. \tag{2.14}$$

From the initial condition $\boldsymbol{\alpha}(0) = \boldsymbol{\alpha}_0$, we obtain

$$H(\boldsymbol{\alpha}) + H(\boldsymbol{\alpha}_0)\tau = H(\boldsymbol{\alpha}_0), \tag{2.15}$$

for τ small enough. We see from (2.15) that $\|\boldsymbol{\alpha}(\tau)\|$ is bounded. Indeed if $\|\boldsymbol{\alpha}(\tau)\|$ is unbounded, by (2.11) we have that $\|H(\boldsymbol{\alpha}(\tau))\|$ is also unbounded, which leads to a contradiction of (2.15). Therefore there exists a solution of (2.13) on $0 \le \tau \le 1$ and (2.15) holds. From (2.12), we have $\mathcal{M}(\boldsymbol{\alpha}(1), 1, \boldsymbol{\alpha}_0) = 0$, that is $h(\boldsymbol{\alpha}(1)) = \boldsymbol{\alpha}(1)$. The uniqueness follows since H is a homeomorphism. The proof is complete. $\square$

Remark 2.2. In fact, the solution $\boldsymbol{\alpha}(\tau)$, $0 \le \tau \le 1$ of the initial value problem (2.13) is a homotopy path of the periodic solution of problem (1.1). Therefore Theorem 2.1 provides a numerical method for finding periodic solutions which is not dependent on $\boldsymbol{\alpha}_0$ and has global convergence (see [1]). Note that (2.13) is equivalent to the initial value problem

$$\frac{\mathrm{d}\mathcal{M}(\boldsymbol{\alpha}(s), \tau(s), \boldsymbol{\alpha}_0)}{\mathrm{d}s} = 0, \qquad \tau(0) = 0, \tag{2.16}$$

which is parameterized with respect to arc length s (see [1]). We first choose an initial point $\boldsymbol{\alpha}_0$, then we get a solution $\boldsymbol{\alpha} = \boldsymbol{\alpha}(\tau(s))$ by solving the initial value problem (2.16). This solution leads to $\boldsymbol{\alpha}^* = \boldsymbol{\alpha}(1)$, which is the initial point of the periodic solution of (1.1). Then we can obtain the periodic solution of (1.1) by computing the numerical solution to problem (2.3).

Remark 2.3. The estimate (2.5) for the inverse of $\int_0^{2\pi} \frac{\partial G(t,\boldsymbol{x}(t,\boldsymbol{\alpha}))}{\partial \boldsymbol{\alpha}}\,\mathrm{d}t$ in Theorem 2.1 is not easy to establish in a practical setting. Thus we remark that finding a suitable condition to replace the estimate (2.5) would be a worthwhile study.

3. FURTHER DISCUSSION

As indicated in Remark 2.3, it is not easy to establish the estimate (2.5). In this section, we propose some weaker conditions to replace the estimate (2.5) in Theorem 2.1.

Consider the linear operator $\mathcal{L} : D(\mathcal{L}) \to U$ where

$$\mathcal{L}\boldsymbol{x} = -\boldsymbol{x}',$$

and a continuously Fréchet differentiable operator $N : D(\mathcal{L}) \to U$ which is defined by

$$(N(\boldsymbol{x}))(t) = G(t, \boldsymbol{x}), \quad t \in [0, 2\pi]. \tag{3.1}$$

Then (2.1) is reformulated as

$$\mathcal{L}\boldsymbol{x} + N(\boldsymbol{x}) = 0, \quad \boldsymbol{x} \in D(\mathcal{L}). \tag{3.2}$$

Let $U = L_n^2[0, 2\pi]$ be the set of all vector-valued functions $\boldsymbol{x}(t) = (x_i(t))_{n\times 1}$ on $[0, 2\pi]$ such that $x_i \in L^2[0, 2\pi]$ for $i = 1, \ldots, n$. Then U is a Hilbert space with the following inner product:

$$\langle \boldsymbol{x}, \boldsymbol{y} \rangle = \int_0^{2\pi} \boldsymbol{x}^{\mathrm{T}}(t)\boldsymbol{y}(t)dt, \tag{3.3}$$

and we denote by $\|\cdot\|$ the norm induced by this inner product. Also

$$\begin{aligned} D(\mathcal{L}) = \Big\{ \boldsymbol{x}(t) = (x_1(t), \ldots, x_n(t))^{\mathrm{T}} \;\Big|\; & \boldsymbol{x}(0) = \boldsymbol{x}(2\pi); \\ & x_i(t) \text{ absolutely continuous and } x_i'(t) \in L^2[0, 2\pi] \Big\}, \end{aligned} \tag{3.4}$$

and then $\mathcal{L}$ is a closed skew-adjoint operator on $D(\mathcal{L})$.

Theorem 3.1. *Assume that* $(\mathcal{L} + N'(\boldsymbol{x}))^{-1}$ *in system* (3.2) *exists for all* $\boldsymbol{x} \in D(\mathcal{L})$, *and there is a function* $\Theta(s) : \mathbb{R}_+ \to \mathbb{R}_{++}$ *such that*

$$\left\| (\mathcal{L} + N'(\boldsymbol{x}))^{-1} \right\| \le \frac{1}{\Theta(\|\boldsymbol{x}\|)}, \quad \int_0^{\infty} \inf_{\|\boldsymbol{x}\| \le s} \Theta(\|\boldsymbol{x}\|)\, \mathrm{d}s = +\infty, \tag{3.5}$$

where $N'(\boldsymbol{x}) = \frac{\partial G(t,\boldsymbol{x})}{\partial \boldsymbol{x}}$. *Then problem* (1.1), (1.2) *has a unique solution.*

Proof. By (3.5), we have

$$\int_0^\infty \inf_{\|\boldsymbol{x}\|\le s} \frac{1}{\|(\mathcal{L}+Q(\boldsymbol{x}))^{-1}\|}\,\mathrm{d}s \;\ge\; \int_0^\infty \inf_{\|\boldsymbol{x}\|\le s} \Theta(\|\boldsymbol{x}\|)\,\mathrm{d}s = +\infty. \tag{3.6}$$

Then by Lemma 1.2, the system (3.2) has a solution, that is to say, problem (1.1), (1.2) has a unique solution. The proof is complete. □

If we assume that $\Theta(s)$ is a nonincreasing function we have

$$\int_0^\infty \inf_{\|\boldsymbol{x}\|\le s} \Theta(\|\boldsymbol{x}\|)\,\mathrm{d}s = \int_0^\infty \Theta(s)\,\mathrm{d}s.$$

Then we get the following corollary.

Corollary 3.2. *Assume that* $(\mathcal{L}+N'(\boldsymbol{x}))^{-1}$ *in system* (3.2) *exists for all* $x \in D(\mathcal{L})$, *and there is a nonincreasing function* $\Theta(s):\mathbb{R}_+ \to \mathbb{R}_{++}$ *with*

$$\left\|(\mathcal{L}+N'(\boldsymbol{x}))^{-1}\right\| \le \frac{1}{\Theta(\|\boldsymbol{x}\|)}, \quad \int_0^\infty \Theta(s)\,\mathrm{d}s = +\infty. \tag{3.7}$$

Then problem (1.1), (1.2) *has a unique solution.*

Remark 3.3. We should note that Theorem 3.1 and Corollary 3.2 give two conditions (3.5) and (3.7), which avoids the condition (2.5).

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