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EXISTENCE RESULTS FOR SOLUTIONS OF BVPS OF SECOND ORDER IMPULSIVE DIFFERENTIAL EQUATIONS ON A HALF LINE

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ABSTRACT. This paper deals with a class of boundary value problems of the second order impulsive differential equations on a half line. Sufficient conditions are established for the existence of at least one solution of these problems. Our method is based upon the fixed point theorem in Banach spaces. An Example is presented to illustrate the efficiency of the obtained results.

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1. INTRODUCTION

Since the differential equations with impulses are a basic tool for studying evolution processes that are subject to abrupt changes in their states, the theory of impulsive differential equations provides a general framework for mathematical modelling of many real world phenomena (refer to [1,2]).

In recent years, remarked progress has been made in the theory of impulsive differential equations. The solvability of the boundary value problems on a finite interval for second-order impulsive differential equations (IBVP for short) has been studied in many papers. The methods used in these papers are upper and lower solution methods and monotone iterative techniques, see [3–5] and the references therein; the coincidence degree theory of Mawhin, see papers [6,7]; the fixed point theorems in cones in Banach spaces, see [8–16] and the references therein.

The Boundary value problems on infinite interval occur in many applications and received much attention, see [17]. The existence of positive solutions or solutions for nth-order nonlinear impulsive integro-differential equations in Banach spaces were investigated in [18–22] and the references therein. Due to the fact that an infinite interval is non-compact, the discussion about BVPs on half line is more complicated. There is only a small amount of work dedicated to the existence of solutions of boundary value problems of impulsive differential equations on a half line.

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This paper is motivated by [23]. Lian and Ge studied the following three-point boundary value problems for second order differential equations

$$\begin{cases} x'' + f(t, x(t), x'(t)) = 0, \quad t \in (0, +\infty), \\ x(0) = \alpha x(\eta), \quad \lim_{t\to+\infty} x'(t) = 0, \end{cases} \tag{1.1}$$

where $\alpha \neq 1$, $\eta \in (0, +\infty)$, $f : [0, +\infty) \times R^2 \to R$ is an S-Carathéodory function. It was proved in [23] that if there exist functions $p, q, r \in L^1[0, \infty)$ with $tp, tq, tr \in L^1[0, +\infty)$ such that

$$|f(t, x, y)| \le p(t)|x| + q(t)|y| + r(t), \quad t \in [0, +\infty), \quad (x, y) \in R^2 \tag{1.2}$$

and

$$\begin{aligned} &\eta P + P_1 + Q < 1, \qquad \alpha < 0, \\ &\frac{\alpha\eta}{1-\alpha} P + P_1 + Q < 1, \qquad 0 \le \alpha < 1, \\ &\max\left\{\frac{\alpha\eta}{\alpha-1} P + P_1 + Q,\ \frac{\eta}{\alpha-1} P + \frac{\alpha}{\alpha-1} P_1\right\} < 1, \qquad \alpha > 1, \end{aligned}$$

where

$$P = \int_0^{+\infty} p(s)ds, \quad P_1 = \int_0^{+\infty} sp(s)ds, \quad Q = \int_0^{+\infty} q(s)ds,$$

then BVP(1.1) has at least one solution.

First,the growth condition (1.2) imposed on f is an at most linear growth condition. The question is that under what conditions BVP(1.1) has solutions without the assumption (1.2).

Second, it is interesting, important and necessary to establish the sufficient conditions for the existence of solutions of the following boundary value problem of impulsive differential equations on half line

$$\begin{cases} x'' + f(t, x(t), x'(t)) = 0, \quad t \in (0, +\infty), \quad t \neq t_k, \quad k \in N, \\ \Delta x(t_k) = I_k(x(t_k), x'(t_k)), \quad k \in N, \\ \Delta x'(t_k) = J_k(x(t_k), x'(t_k)), \quad k \in N, \\ x(0) = \alpha x(\eta), \\ \lim_{t\to+\infty} x'(t) = 0, \end{cases} \tag{1.3}$$

where

- $\alpha \neq 1$, $\eta \in (0, +\infty)$;
- $f : [0, +\infty) \times R^2 \to R$ is an S-Carathéodory function;
- I_k, J_k are continuous functions.

Let $J = [0, +\infty)$, $0 < t_1 < \cdots < t_n < \cdots$, R be the real number set. Define the sets

$$L^1[0, +\infty) = \{x : [0, +\infty) \to R \text{ is absolutely integrable on } [0, +\infty)\}$$

with the normal $||x||_{L_1} = \int_0^{+\infty} |x(t)|dt$,

$$PC^0(J,R) = \left\{ x : J \to R : \begin{array}{l} x \text{ is continuous everywhere except} \\ \text{at the points } t = t_k \in J \\ \text{and } x(t_k^+) \text{ and } x(t_k^-) \text{ exist in } R \\ \text{with } x(t_k^-) = x(t_k), k \in N, \\ \text{there exists the limit } \lim_{t\to+\infty} \frac{|x(t)|}{1+t} \end{array} \right\}$$

with the normal

$$\|x\|_\infty = \sup_{t\in J} \frac{|x(t)|}{1+t}$$

and

$$PC^1(J,R) = \left\{ x : J \to R : \begin{array}{l} x \text{ is differentiable everywhere except} \\ \text{at the points } t = t_k \in J \\ \text{and } x(t_k^+) \text{ and } x(t_k^-) \text{ exist in } R \\ \text{with } x(t_k^-) = x(t_k), k \in N, \\ x' \text{ is continuous everywhere except} \\ \text{at the points } t = t_k \in J \\ \text{and } x'(t_k^+) \text{ and } x'(t_k^-) \text{ exist in } R \text{ with} \\ x'(t_k^-) = x'(t_k) = \lim_{\Delta t\to 0^-} \frac{x(t_k+\Delta t)-x(t_k)}{\Delta t}, \\ k \in N, \text{ there exist the limits} \\ \lim_{t\to+\infty} \frac{|x(t)|}{1+t}, \ \lim_{t\to+\infty} x'(t). \end{array} \right\}$$

For each $x \in PC^1(J,R)$, the norm

$$||x|| = \max\left\{ \sup_{t\in J} \frac{|x(t)|}{1+t}, \ \ ||x'||_\infty = \sup_{t\in J} |x'(t)| \right\}$$

is defined. Then $PC^0(J,R)$ and $PC^1(J,R)$ are Banach spaces.

A function $f : J \times R^2 \to D$ is called an impulsive S-Carathéodory function if

* $f(\bullet, x, y)$ is measurable on $[0, +\infty)$ for each $u = (x, y) \in R^2$;
* $f(t, \bullet, \bullet)$ is continuous a.e. $t \in [0, +\infty)$ and there exist the limits

$$\lim_{t\to t_k^-} f(t,x,y) = f(t_k,x,y), \quad \lim_{t\to t_k^+} f(t,x,y);$$

* for each $r > 0$, there exists a function $\phi \in L^1[0,+\infty)$ with $t\phi \in L^1[0,+\infty)$ and $\phi(t) > 0$ for all $t \in (0,\infty)$ such that

$$\max\{|x|,|y|\} \le r \text{ implies that } |f(t,x,y)| \le \phi(t), \quad \text{a.e. } t \in [0,+\infty).$$

In this paper, we establish existence results for solutions of the following infinite-point boundary value problems of second order impulsive differential equations on the half line (IBVP for short)

$$(1.4)\qquad \begin{cases} x''(t) = f(t, x(t), x'(t))), \quad \text{a.e. } t \in [0, +\infty), \quad t \neq t_k, \quad k \in N, \\ \Delta x(t_k) = I_k(x(t_k), x'(t_k)), \quad k \in N, \\ \Delta x'(t_k) = J_k(x(t_k), x'(t_k)), \quad k \in N, \\ x(0) - \sum_{i=1}^{+\infty} \alpha_i x(\eta_i) = 0, \\ \lim_{t \to +\infty} x'(t) = 0, \end{cases}$$

where

- $N = \{1, 2, \dots, n, \dots\}$;
- $0 = t_0 < t_1 < \cdots < t_m < t_{m+1} < \cdots$;
- $0 < \eta_1 < \cdots < \eta_m < \cdots$;
- $\alpha_i \in R$ with $\sum_{m=1}^{+\infty} \alpha_m \neq 1$ and $\sum_{m=1}^{+\infty} |\alpha_m| < +\infty$;
- f is an impulsive S-Carathéodory function;
- $\Delta x(t_k) = x(t_k^+) - x(t_k)$, $\Delta x'(t_k) = x'(t_k^+) - x'(t_k)$ for all $k \in N$, I_k, J_k are continuous functions.

IBVP(4) is a generalized form of BVP(1.3). By a solution of IBVP(1.4) we mean a function $x \in PC^1(J, R)$ that satisfies all equations in (1.4).

To derive our results, the well known fixed point theorem, Shaefer′s fixed point theorem, is used, and we do not rely on the existence of upper and lower solutions.

The main results in this paper are established in section 2. Examples are presented to illustrate the main results in section 3.

2. MAIN RESULTS

In this section, we establish the main results. We set the following assumptions which should be used.

(A_1) J_k is continuous and the inequality

$$J_k(u, v)v \geq 0$$

holds for all $u, v \in R$ and $k \in N$.

(A_2) Assume that $\sum_{m=1}^{+\infty} \alpha_m \neq 1$ and $\sum_{m=1}^{+\infty} |\alpha_m| < +\infty$, I_k is continuous and there exist constants $a_k > 0$ such that $|I_k(u, v)| \leq a_k |u|$ for all $u, v \in R$ and $k \in N$, $\sum_{i=1}^{+\infty} a_i$ is convergent and

$$1 > \sum_{n=1}^{+\infty} a_n + \frac{\sum_{n=1}^{+\infty} |\alpha_n| \sum_{0 < t_k < \eta_n} a_k (1 + t_k)}{\left|1 - \sum_{n=1}^{+\infty} \alpha_n\right|}$$

holds.

(B) f is an impulsive S-Carathéodory function, and there exist impulsive S-Carathéodory functions $g : [0, +\infty) \times R^2 \to R$, $h_1 : [0, +\infty) \times R \to R$, $h_2 : [0, +\infty) \times R \to R$,

$r \in L^[0,+\infty)$ and nonnegative functions r_1, r_2 with $(1+t)r_1(t), r_2(t) \in L^1[0,+\infty)$ such that

$$f(t,u,v) = g(t,u,v) + h_1(t,u) + h_2(t,v) + r(t)$$

$$g(t,u,v)v \le 0$$

and

$$|h_1(t,u)| \le r_1(t)|u|, \quad |h_2(t,v)| \le r_2(t)|v|$$

hold for all $(t,u,v) \in [0,+\infty) \times R^2$.

Lemma 2.1. *Let X be a Banach space and C be a convex subset of X. Suppose $L : X \to X$ is completely continuous. Define the*

$$F(L) = \{x \in C : x = \lambda Lx, \quad \textit{for aome } \lambda \in [0,1]\}.$$

Then either

- *$F(L)$ is unbounded or*
- *the operator L has fixed point in X.*

Proof. The proof can be found in [12]. □

Lemma 2.2. *Assume that $\sum_{m=1}^{+\infty} \alpha_m \ne 1$ and $\sum_{m=1}^{+\infty} |\alpha_m| < +\infty$. Suppose that $v, tv \in L^1[0,+\infty)$ and $a_k, b_k \in R(k \in N)$ with $\sum_{i=1}^{+\infty} b_k, \sum_{i=1}^{+\infty} a_k$ and $\int_0^{+\infty} \sum_{s \le t_k < +\infty} b_k ds$ converging. Then*

$$\begin{cases} x''(t) = v(t), \quad t \in [0,+\infty), \quad t \ne t_k, \quad k \in N, \\ \Delta x(t_k) = a_k, \quad k \in N, \\ \Delta x'(t_k) = b_k, \quad k \in N, \\ x(0) - \sum_{i=1}^{+\infty} \alpha_i x(\eta_i) = 0, \\ \lim_{t \to +\infty} x'(t) = 0 \end{cases} \tag{2.1}$$

has unique solution

$$\begin{aligned} x(t) \;=\; & \frac{\sum_{n=1}^{+\infty} \alpha_n \sum_{0<t_k<\eta_n} a_k - \sum_{n=1}^{+\infty} \alpha_n \int_0^{\eta_n} \sum_{s \le t_k < +\infty} b_k ds}{1 - \sum_{n=1}^{+\infty} \alpha_n} \\ & - \frac{\sum_{n=1}^{+\infty} \alpha_n \int_0^{\eta_n} \left(\int_s^{+\infty} v(u)du \right) ds}{1 - \sum_{n=1}^{+\infty} \alpha_n} \\ & + \sum_{0<t_k<t} a_k - \int_0^t \left(\int_s^{+\infty} v(u)du \right) ds - \int_0^t \sum_{s \le t_k < +\infty} b_k ds. \end{aligned}$$

Proof. Since $v \in L^1[0,+\infty)$ and that $\sum_{i=1}^{+\infty} b_k$ converges, one gets by integrating $x''(t) = v(t)$ from t to $+\infty$ that

$$x'(t) = -\int_t^{+\infty} v(u)du - \sum_{t \le t_k < +\infty} b_k.$$

It follows from $tv \in L^1[0,+\infty)$ and $\sum_{i=1}^{+\infty} a_k$ converges and $\int_0^{+\infty}\sum_{s\le t_k<+\infty} b_k ds$ converges that

$$x(t) = x(0) + \sum_{0<t_k<t} a_k - \int_0^t \left(\int_s^{+\infty} v(u)du\right) ds - \int_0^t \sum_{s\le t_k<+\infty} b_k ds.$$

Since $x(0) - \sum_{i=1}^{+\infty} \alpha_i x(\eta_i) = 0$ and $\sum_{i=1}^{+\infty}\alpha_i \neq 1$, we get that

$$\begin{aligned} x(0) &= \frac{\sum_{n=1}^{+\infty}\alpha_n \sum_{0<t_k<\eta_n} a_k - \sum_{n=1}^{+\infty}\alpha_n \int_0^{\eta_n}\sum_{s\le t_k<+\infty} b_k ds}{1-\sum_{n=1}^{+\infty}\alpha_n} \\ &\quad - \frac{\sum_{n=1}^{+\infty}\alpha_n \int_0^{\eta_n}\left(\int_s^{+\infty} v(u)du\right)ds}{1-\sum_{n=1}^{+\infty}\alpha_n}. \end{aligned}$$

Hence

$$\begin{aligned} x(t) &= \frac{\sum_{n=1}^{+\infty}\alpha_n \sum_{0<t_k<\eta_n} a_k - \sum_{n=1}^{+\infty}\alpha_n \int_0^{\eta_n}\sum_{s\le t_k<+\infty} b_k ds}{1-\sum_{n=1}^{+\infty}\alpha_n} \\ &\quad - \frac{\sum_{n=1}^{+\infty}\alpha_n \int_0^{\eta_n}\left(\int_s^{+\infty} v(u)du\right)ds}{1-\sum_{n=1}^{+\infty}\alpha_n} \\ &\quad + \sum_{0<t_k<t} a_k - \int_0^t \left(\int_s^{+\infty} v(u)du\right) ds - \int_0^t \sum_{s\le t_k<+\infty} b_k ds. \end{aligned}$$

The proof is complete. □

Let

$$X = \left\{ x \in PC^1([0,+\infty), R) : x(0) - \sum_{i=1}^{+\infty}\alpha_i x(\eta_i) = 0,\ \lim_{t\to+\infty} x'(t) = 0 \right\}.$$

Define the nonlinear operator L by

$$\begin{aligned} (Lx)(t) &= \frac{\sum_{n=1}^{+\infty}\alpha_n \sum_{0<t_k<\eta_n} I_k(x(t_k),x'(t_k))}{1-\sum_{n=1}^{+\infty}\alpha_n} \\ &\quad - \frac{\sum_{n=1}^{+\infty}\alpha_n \int_0^{\eta_n}\sum_{s\le t_k<+\infty} J_k(x(t_k),x'(t_k))ds}{1-\sum_{n=1}^{+\infty}\alpha_n} \\ &\quad + \sum_{0<t_k<t} I_k(x(t_k),x'(t_k)) - \int_0^t \sum_{s\le t_k<+\infty} J_k(x(t_k),x'(t_k))ds \\ &\quad - \frac{\sum_{n=1}^{+\infty}\alpha_n \int_0^{\eta_n}\left(\int_s^{+\infty} f(u,x(u),x'(u))du\right)ds}{1-\sum_{n=1}^{+\infty}\alpha_n} \\ &\quad - \int_0^t \left(\int_s^{+\infty} f(u,x(u),x'(u))du\right) ds, \quad x \in X. \end{aligned}$$

Lemma 2.3. *Suppose that f is an impulsive S-Carathéodory function, I_k, J_k are continuous. It is easy to show that*

- *x is a solution of IBVP(1.4) if and only if x is a solution of the operator equation $x = Lx$ in X;*
- *$Lx \in X$ for all $x \in X$;*
- *L is completely continuous.*

Proof. The proof is similar to that of the proof of Lemmas 2.4 and 3.1 in [20]. □

Theorem 2.4. *Suppose that $(A_1), (A_2)$ and (B) hold. Then IBVP(1.4) has at least one solution if*

$$\frac{1}{2} > \frac{1 + \frac{\sum_{n=1}^{+\infty} |\alpha_n| \eta_n}{\left|1 - \sum_{n=1}^{+\infty} \alpha_n\right|}}{1 - \left(\sum_{n=1}^{+\infty} a_n + \frac{\sum_{n=1}^{+\infty} |\alpha_n| \sum_{0<t_k<\eta_n} a_k(1+t_k)}{\left|1-\sum_{n=1}^{+\infty}\alpha_n\right|}\right)} \int_0^{+\infty} r_1(s)(1+s)ds + \int_0^{+\infty} r_2(s)ds.$$

Proof. Consider the set

$$\Omega_0 = \{x \in X : x = \lambda Lx \text{ for some } \lambda \in [0,1]\}.$$

For $x \in \Omega_0$. Then one gets that

$$\begin{cases} x''(t) = \lambda f(t, x(t), x'(t))), \quad \text{a.e. } t \in [0, +\infty), \quad t \neq t_k, \quad k \in N, \\ \Delta x(t_k) = \lambda I_k(x(t_k), x'(t_k)), \quad k \in N, \\ \Delta x'(t_k) = \lambda J_k(x(t_k), x'(t_k)), \quad k \in N, \\ x(0) - \sum_{i=1}^{+\infty} \alpha_i x(\eta_i) = 0, \\ \lim_{t\to+\infty} x'(t) = 0. \end{cases} \tag{2.2}$$

Then

$$x''(t)x'(t) = \lambda f(t, x(t), x'(t)))x'(t), \quad \text{a.e. } t \in [0, +\infty), \quad t \neq t_k, \quad k \in N. \tag{2.3}$$

It follows from (2.3) that

$$\int_t^{+\infty} x''(s)x'(s)ds = \lambda \int_t^{+\infty} f(s, x(s), x'(s)))x'(s)ds.$$

One sees that

$$\begin{aligned} &-\frac{1}{2}[x'(t)]^2 - \frac{1}{2}\sum_{t\le t_k<+\infty}\left[\frac{1}{2}[x'(t_k^+)]^2 - \frac{1}{2}[x'(t_k)]^2\right]^2 \\ &= \frac{1}{2}[x'(+\infty)]^2 - \frac{1}{2}[x'(t)]^2 - \frac{1}{2}\sum_{t\le t_k<+\infty}\left[\frac{1}{2}[x'(t_k^+)]^2 - \frac{1}{2}[x'(t_k)]^2\right]^2 \\ &= \lambda\left(\int_t^{+\infty} g(t, x(t), x'(t))x'(t)dt\right. \\ &\quad \left.+ \int_t^{+\infty} h_1(t, x(t))x'(t)dt + \int_t^{+\infty} h_2(t, x'(t))x'(t)dt + \int_t^{+\infty} r(t)x'(t)dt\right). \end{aligned}$$

Since

$$[x'(t_k^+)]^2 - [x'(t_k)]^2 \quad = \quad [x'(t_k^+) - x'(t_k)][x'(t_k^+) + x'(t_k)]$$

$$\begin{aligned}
&= \Delta x'(t_k)[2x'(t_k)+\Delta x'(t_k)] \\
&= \lambda J_k(x(t_k),x'(t_k))\left(2x'(t_k)+\lambda J_k(x(t_k),x'(t_k))\right) \\
&\geq 2\lambda J_k(x(t_k),x'(t_k))x'(t_k) \\
&\geq 0,
\end{aligned}$$

we get that

$$\begin{aligned}
\frac{1}{2}[x'(t)]^2 &= -\frac{1}{2}\sum_{t\leq t_k<+\infty}\left[\frac{1}{2}[x'(t_k^+)]^2-\frac{1}{2}[x'(t_k)]^2\right]^2 \\
&\quad -\lambda\left(\int_t^{+\infty} g(t,x(t),x'(t))x'(t)dt\right. \\
&\quad +\int_t^{+\infty} h_1(t,x(t))x'(t)dt+\int_t^{+\infty} h_2(t,x'(t))x'(t)dt \\
&\quad \left.+\int_t^{+\infty} r(t)x'(t)dt\right) \\
&\leq -\lambda\left(\int_t^{+\infty} g(t,x(t),x'(t))x'(t)dt\right. \\
&\quad -\int_t^{+\infty} h_1(t,x(t))x'(t)dt-\int_t^{+\infty} h_2(t,x'(t))x'(t)dt \\
&\quad \left.-\int_t^{+\infty} r(t)x'(t)dt\right) \\
&\leq -\lambda\left(\int_t^{+\infty} h_1(t,x(t))x'(t)dt+\int_t^{+\infty} h_2(t,x'(t))x'(t)dt\right. \\
&\quad \left.+\int_t^{+\infty} r(t)x'(t)dt\right) \\
&\leq \int_t^{+\infty} |h_1(t,x(t))||x'(t)|dt+\int_t^{+\infty} |h_2(t,x'(t))||x'(t)|dt \\
&\quad +\int_t^{+\infty} |r(s)||x'(s)|dt \\
&\leq \int_t^{+\infty} r_1(t)|x(t)||x'(t)|dt+\int_t^{+\infty} r_2(t)|x'(t)||x'(t)|dt \\
&\quad +\int_t^{+\infty} |r(s)||x'(s)|dt \\
&\leq ||x||_\infty||x'||_\infty\int_t^{+\infty} r_1(s)(1+s)ds+||x'||_\infty^2\int_t^{+\infty} r_2(s)ds \\
&\quad +||x'||_\infty\int_t^{+\infty} |r(s)|ds.
\end{aligned}$$

Since

$$|x(\eta_n)-x(0)| = \left|\sum_{0<t_k<\eta_n}\Delta x(t_k)+\int_0^{\eta_n} x'(s)ds\right|$$

$$
\begin{aligned}
&\leq \lambda \sum_{0<t_k<\eta_n} a_k(1+t_k)\frac{|x(t_k)|}{1+t_k} + \eta_n||x'||_\infty \\
&\leq ||x||_\infty \sum_{0<t_k<\eta_n} a_k(1+t_k) + \eta_n||x'||_\infty,
\end{aligned}
$$

we get that

$$
\begin{aligned}
\left|\frac{x(t)}{1+t}\right| &= \frac{1}{1+t}\left|x(0) + \sum_{0<t_k<t}\Delta x(t_i) + \int_0^t x'(s)ds\right| \\
&\leq \frac{1}{1+t}\left|\frac{x(0) - \sum_{n=1}^{+\infty}\alpha_n x(0)}{1-\sum_{n=1}^{+\infty}\alpha_n}\right| \\
&\quad + \frac{1}{1+t}\left|\sum_{0<t_k<t}\Delta x(t_i) + \frac{1}{1+t}\int_0^t x'(s)ds\right| \\
&\leq \frac{1}{1+t}\left|\frac{\sum_{n=1}^{+\infty}\alpha_n\left(x(\eta_n) - x(0)\right)}{1-\sum_{n=1}^{+\infty}\alpha_n}\right| \\
&\quad + \frac{1}{1+t}\left|\sum_{0<t_k<t}\Delta x(t_i) + \int_0^t x'(s)ds\right| \\
&\leq \frac{1}{1+t}\frac{\sum_{n=1}^{+\infty}|\alpha_n|\left(||x||_\infty\sum_{0<t_k<\eta_n}a_k(1+t_k) + \eta_n||x'||_\infty\right)}{\left|1-\sum_{n=1}^{+\infty}\alpha_n\right|} \\
&\quad + \frac{\lambda}{1+t}\sum_{0<t_k<t}|I_k(x(t_k),x'(t_k))| + \frac{1}{1+t}\int_0^t |x'(s)|ds \\
&\leq \frac{\sum_{n=1}^{+\infty}|\alpha_n|\left(||x||_\infty\sum_{0<t_k<\eta_n}a_k(1+t_k) + \eta_n||x'||_\infty\right)}{\left|1-\sum_{n=1}^{+\infty}\alpha_n\right|} \\
&\quad + \frac{1}{1+t}\sum_{0<t_k<t}a_k(1+t_k)\frac{|x(t_k)|}{1+t_k} + \frac{t}{1+t}||x'||_\infty \\
&\leq \frac{\sum_{n=1}^{+\infty}|\alpha_n|\left(||x||_\infty\sum_{0<t_k<\eta_n}a_k(1+t_k) + \eta_n||x'||_\infty\right)}{\left|1-\sum_{n=1}^{+\infty}\alpha_n\right|} \\
&\quad + \sum_{0<t_k<t}a_k\frac{|x(t_k)|}{1+t_k} + ||x'||_\infty \\
&\leq \left(\sum_{n=1}^{+\infty}a_n + \frac{\sum_{n=1}^{+\infty}|\alpha_n|\sum_{0<t_k<\eta_n}a_k(1+t_k)}{\left|1-\sum_{n=1}^{+\infty}\alpha_n\right|}\right)||x||_\infty \\
&\quad + \left(1 + \frac{\sum_{n=1}^{+\infty}|\alpha_n|\eta_n}{\left|1-\sum_{n=1}^{+\infty}\alpha_n\right|}\right)||x'||_\infty.
\end{aligned}
$$

It follows that

$$||x||_\infty \leq \frac{1+\frac{\sum_{n=1}^{+\infty}|\alpha_n|\eta_n}{\left|1-\sum_{n=1}^{+\infty}\alpha_n\right|}}{1-\left(\sum_{n=1}^{+\infty}a_n+\frac{\sum_{n=1}^{+\infty}|\alpha_n|\sum_{0<t_k<\eta_n}a_k(1+t_k)}{\left|1-\sum_{n=1}^{+\infty}\alpha_n\right|}\right)}||x'||_\infty.$$

Then

$$\begin{aligned}\frac{1}{2}[x'(t)]^2 &\leq \frac{1+\frac{\sum_{n=1}^{+\infty}|\alpha_n|\eta_n}{\left|1-\sum_{n=1}^{+\infty}\alpha_n\right|}}{1-\left(\sum_{n=1}^{+\infty}a_n+\frac{\sum_{n=1}^{+\infty}|\alpha_n|\sum_{0<t_k<\eta_n}a_k(1+t_k)}{\left|1-\sum_{n=1}^{+\infty}\alpha_n\right|}\right)}||x'||_\infty^2\int_t^{+\infty}r_1(s)(1+s)ds\\ &\quad +||x'||_\infty^2\int_t^{+\infty}r_2(s)ds+||x'||_\infty\int_t^{+\infty}|r(s)|ds.\end{aligned}$$

It follows that

$$\begin{aligned}\frac{1}{2}||x'||_\infty &\leq \frac{1+\frac{\sum_{n=1}^{+\infty}|\alpha_n|\eta_n}{\left|1-\sum_{n=1}^{+\infty}\alpha_n\right|}}{1-\left(\sum_{n=1}^{+\infty}a_n+\frac{\sum_{n=1}^{+\infty}|\alpha_n|\sum_{0<t_k<\eta_n}a_k(1+t_k)}{\left|1-\sum_{n=1}^{+\infty}\alpha_n\right|}\right)}||x'||_\infty^2\int_t^{+\infty}r_1(s)(1+s)ds\\ &\quad +||x'||_\infty^2\int_t^{+\infty}r_2(s)ds+||x'||_\infty\int_t^{+\infty}|r(s)|ds\\ &\leq \frac{1+\frac{\sum_{n=1}^{+\infty}|\alpha_n|\eta_n}{\left|1-\sum_{n=1}^{+\infty}\alpha_n\right|}}{1-\left(\sum_{n=1}^{+\infty}a_n+\frac{\sum_{n=1}^{+\infty}|\alpha_n|\sum_{0<t_k<\eta_n}a_k(1+t_k)}{\left|1-\sum_{n=1}^{+\infty}\alpha_n\right|}\right)}||x'||_\infty^2\int_0^{+\infty}r_1(s)(1+s)ds\\ &\quad +||x'||_\infty^2\int_0^{+\infty}r_2(s)ds+||x'||_\infty\int_0^{+\infty}|r(s)|ds.\end{aligned}$$

Since

$$\frac{1}{2} > \frac{1+\frac{\sum_{n=1}^{+\infty}|\alpha_n|\eta_n}{\left|1-\sum_{n=1}^{+\infty}\alpha_n\right|}}{1-\left(\sum_{n=1}^{+\infty}a_n+\frac{\sum_{n=1}^{+\infty}|\alpha_n|\sum_{0<t_k<\eta_n}a_k(1+t_k)}{\left|1-\sum_{n=1}^{+\infty}\alpha_n\right|}\right)}\int_0^{+\infty}r_1(s)(1+s)ds+\int_0^{+\infty}r_2(s)ds,$$

we get that there exists a constant $M>0$ such that $||x'||_\infty \leq M$. Then

$$||x||_\infty \leq \frac{1+\frac{\sum_{n=1}^{+\infty}|\alpha_n|\eta_n}{\left|1-\sum_{n=1}^{+\infty}\alpha_n\right|}}{1-\left(\sum_{n=1}^{+\infty}a_n+\frac{\sum_{n=1}^{+\infty}|\alpha_n|\sum_{0<t_k<\eta_n}a_k(1+t_k)}{\left|1-\sum_{n=1}^{+\infty}\alpha_n\right|}\right)}M =: M_1.$$

It follows that $||x|| \leq \max\{M, M_1\}$ for each $x \in \Omega_0$.

Let $\Omega = \{x \in X : ||x|| < \max\{M, M_1\} + 1\}$. Lemma 2.3 implies that that $L : X \to X$ is completely continuous. It is easy to see that Ω is an open bounded subset of X and $0 \in \Omega \subset X$ and $x \neq \lambda Lx$ for all $x \in X \cap \partial\Omega$ and $\lambda \in [0, 1]$, then Lemma 2.1 implies that there is at least one $x \in \Omega$ such that $x = Lx$. So Lemma 2.3 implies that x is a solution of IBVP(1.4). It follows from above discussion that the proof is complete. $\square$

3. EXAMPLES

In this section, we present an example to illustrate the main results.

Example 3.1. Consider the IBVP

$$(3.1)\qquad \begin{cases} x''(t) = -\frac{[x'(t)]^{2m+1}}{1+[x(t)]^2} + r_1(t)x(t) + r_2(t)x'(t) + r(t), \\ \\ qquad \qquad \text{a.e. } t \in (0,+\infty), \quad t \neq t_k, \quad k \in N, \\ \Delta x(t_k) = a_k x(t_k), \quad k \in N, \\ \Delta x'(t_k) = b_k [x(t_k)]^3, \quad k \in N, \\ x(0) = \alpha x(\eta), \\ \lim_{t\to+\infty} x'(t) = 0, \end{cases}$$

where $\alpha \in R$ with $\alpha \neq 1$, $\eta \in (0,+\infty)$, $a_k \in R(k \in N)$ with $\sum_{i=1}^{+\infty} |a_k|$ converging, $b_k \geq 0(k \in N)$, m is a nonnegative integer, $0 < t_1 < \cdots < t_m < \cdots$, $r \in PC^0[0,+\infty)$, $(1+s)r_1 \in L^1[0,+\infty)$, $r_2 \in L^1[0,+\infty)$.

It is easy to see that

(i) Since $J_k(x,y) = b_k y^3$ and $b_k \geq 0$, we get that (A_1) holds.

(ii) Choose $\eta_1 = \eta < \eta_2 < \cdots < \cdots$ and $\alpha_1 = \alpha$ and $\alpha_i = 0$ for all $i \geq 2$, we see that $\sum_{i=1}^{+\infty} \alpha_i \neq 1$, $I_k(x,y) = a_k x$ implies that $|I_k(x,y)| \leq |a_k||x|$. $\sum_{i=1}^{+\infty} |a_k|$ is convergent.

$$\sum_{n=1}^{+\infty} |a_n| + \frac{\sum_{n=1}^{+\infty} |\alpha_n| \sum_{0<t_k<\eta_n} |a_k|(1+t_k)}{\left|1-\sum_{n=1}^{+\infty} \alpha_n\right|} = \sum_{n=1}^{+\infty} |a_n| + \frac{|\alpha| \sum_{0<t_k<\eta} |a_k|(1+t_k)}{|1-\alpha|}.$$

If

$$\sum_{n=1}^{+\infty} |a_n| + \frac{|\alpha| \sum_{0<t_k<\eta} |a_k|(1+t_k)}{|1-\alpha|} < 1,$$

then (A_2) holds.

(iii) It is easy to see that (B) holds with

$$f(t,x,y) = -\frac{y^{2m+1}}{1+x^2} + r_1(t)x + r_2(t)y + r(t)$$

$$g(t,x,y) = -\frac{y^{2m+1}}{1+x^2}, \quad h_1(t,x) = r_1(t)x, \quad h_2(t,y) = r_2(t)y.$$

It follows from Theorem 2.4 that IBVP(3.1) has at least one solution if

$$\sum_{n=1}^{+\infty} |a_n| + \frac{|\alpha| \sum_{0<t_k<\eta} |a_k|(1+t_k)}{|1-\alpha|} < 1,$$

and

$$\frac{1}{2} > \frac{1+\frac{|\alpha|\eta}{|1-\alpha|}}{1-\left(\sum_{n=1}^{+\infty} a_n + \frac{|\alpha|\sum_{0<t_k<\eta} a_k(1+t_k)}{|1-\alpha|}\right)} \int_0^{+\infty} |r_1(s)|(1+s)ds + \int_0^{+\infty} |r_2(s)|ds.$$

Remark 3.1. When $a_k = b_k = 0$, IBVP in Example 3.1 becomes

$$(3.2)\qquad \begin{cases} x''(t) = -\frac{[x'(t)]^{2m+1}}{1+[x(t)]^2} + r_1(t)x(t) + r_2(t)x'(t) + r(t), \quad \text{a.e. } t \in (0,+\infty), \\ x(0) = \alpha x(\eta), \\ \lim_{t\to+\infty} x'(t) = 0, \end{cases}$$

where $\alpha \in R$ with $\alpha \neq 1$, $\eta \in (0,+\infty)$, m is a nonnegative integer, $r \in PC^0[0,+\infty)$, $(1+s)r_1 \in L^1[0,+\infty), r_2 \in L^1[0,+\infty)$.

It is easy to see that BVP(3.2) can not be solved by theorems in [20] since condition (1.2) does not hold.

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