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KAMENEV-TYPE OSCILLATION CRITERIA FOR SECOND-ORDER MATRIX DYNAMIC EQUATIONS ON TIME SCALES

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ABSTRACT. We consider the second order matrix valued dynamic equation $(P(t)X^{\Delta}(t))^{\Delta} + Q(t)X(\sigma(t)) = 0$ on a time scale $\mathbb{T}$, where $P, Q \in C_{rd}(\mathbb{T}, \mathbb{R}^{n\times n})$ with $P(t)$ positive definite and $Q(t)$ Hermitian on $\mathbb{T}$. Kamenev type criteria and interval criteria are established. Our results cover those for matrix differential equations and provide new oscillation criteria for matrix difference equations.

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1. INTRODUCTION

In this paper, we study the self-adjoint second order matrix dynamic equation

$$(P(t)X^{\Delta}(t))^{\Delta} + Q(t)X(\sigma(t)) = 0 \tag{1.1}$$

on a time scale $\mathbb{T}$, that is, on a nonempty closed subset $\mathbb{T}$ of $\mathbb{R}$ with the inherited Euclidean topology. *We assume throughout that $P, Q \in C_{rd}(\mathbb{T}, \mathbb{R}^{n\times n})$ with $P(t)$ positive definite and $Q(t)$ Hermitian on $\mathbb{T}$*, and any solution $X(t)$ of Eq. (1.1) is an $n \times n$ matrix solution.

Without loss of generality we assume $\sup \mathbb{T} = \infty$ since we are interested in extending oscillation criteria for the corresponding differential and difference equations, namely

$$(P(t)X'(t))' + Q(t)X(t) = 0 \tag{1.2}$$

with $\mathbb{T}$ the interval $\mathbb{R}_+ := [0, \infty)$, and

$$\Delta(P_n \Delta X_n) + Q_n X_{n+1} = 0 \tag{1.3}$$

with $\mathbb{T} = \mathbb{N}_0$, the set of nonnegative integers.

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For the scalar case, numerous oscillation criteria have been established for Eqs. (1.2) and (1.3). Many of them involve the integral or the sum of the coefficient functions and hence require information concerning the equation on the whole set $\mathbb{R}_+$ or $\mathbb{N}$. For instance, for the scalar Eq. (1.2) with $P \equiv 1$, three well-known conditions which guarantee that all solutions oscillate are as follows:

(A1) $\int_0^\infty Q(t)dt = \infty$, (Fite [15])

(A2) $\lim_{t\to\infty} \frac{1}{t} \int_0^t \int_0^s Q(r)dr\,ds = \infty$, (Wintner [27])

(A3) $-\infty < \liminf_{t\to\infty} \frac{1}{t} \int_0^t \int_0^s Q(r)dr\,ds < \limsup_{t\to\infty} \frac{1}{t} \int_0^t \int_0^s Q(r)dr\,ds \le \infty.$
(Hartman [16])

Kamenev [18] gave another condition for the oscillation of the scalar Eq. (1.2), i.e.,

(A4) $\limsup_{t\to\infty} \frac{1}{t^m} \int_0^t (t-s)^m q(s)\,ds = \infty, \quad m > 1.$

We can easily see that (A1) and (A2) imply (A4) with $m = 2$.

The Kamenev criterion has been extended by several authors. Among them, Philos [21] obtained results on oscillation by replacing the kernel function $(t-s)^m$ by a general class of functions $H(t,s)$ satisfying certain assumptions.

Based on their work, Kong [19] established interval criteria for oscillation of (1.2) utilizing the information of the equation only on a sequence of disjoint subintervals of $[0,\infty)$. The interval criteria substantially improve those by Kamenev and Philos.

The extensions of the oscillation criteria (A1)–(A4) and their improvements were found for the matrix equations (1.2) and (1.3), see [3, 9, 20, 22–26].

Recently, people have studied Eq. (1.1) on time scales which unifies those for the continuous and discrete cases. Certain oscillation results have been obtained, see [4–8, 10–14]. Among them, [4, 5] established the Kamenev-Philos type and the interval criteria for oscillation of the scalar equation (1.1). While most work done so far is for the scalar case, Erbe and Peterson [11, 12] studied the matrix equation (1.1). Their work is significant and important not only because some oscillation criteria of types (A1)–(A3) were found, but also because it set up a foundation for the further study of Eq. (1.1). However, the oscillation criteria obtained in [11, 12] are heavily dependent on the discreteness of the time scale $\mathbb{T}$. In fact, it is required that $\int_0^\infty \mu(s)\Delta s = \infty$ in [11] and there exists a sequence of right-scattered points of $\mathbb{T}$ in [12]. Therefore, these criteria do not apply to the case when $\mathbb{T} = \mathbb{R}$. We also note that no matrix form of the Kamenev-Philos type criteria and interval criteria have been established on time scales. This is due to the fact that the Riccati equation, which plays a key role in the proofs, has a different structure for Eq. (1.1) which prevents simple extensions of the existing work for scalar equations on time scales and for continuous matrix equations on the real number line.

In this paper, we will extend the work by Del Medico and Kong [4] to the general matrix equation (1.1) on time scales. By applying the Riccati equation established in [11, 12] and a "diagonalizing" technique, we will obtain the Kamenev-Philos type criteria for Eq. (1.1). Consequently, the interval criteria for Eq. (1.1) can be derived in the same way as in [4]. For the purpose of completeness and conciseness, we will give proofs for the Kamenev-Philos type criteria but state the interval criteria without proof. Interpretations of our results to the matrix difference equation (1.3) are also given. The results in this paper cover both the continuous and discrete cases.

2. PRELIMINARIES ON TIME SCALES

Before presenting the main results, we recall the following concepts related to time scales for the convenience of the reader. For further knowledge on time scales, the reader is referred to [1, 2, 7, 17] and the references therein.

Definition 2.1. Let $\mathbb{T}$ be a time scale. For $t \in \mathbb{T}$ we define the forward-jump operator σ and the backward-jump operator ρ on $\mathbb{T}$ by

$$\sigma(t) := \inf\{s \in \mathbb{T} : s > t\} \quad \text{and} \quad \rho(t) := \sup\{s \in \mathbb{T} : s < t\},$$

where $\inf \emptyset := \sup \mathbb{T}$ and $\sup \emptyset := \inf \mathbb{T}$. If $\sigma(t) > t$, t is said to be right-scattered; otherwise, it is right-dense. If $\rho(t) < t$, t is said to be left t-scattered; otherwise, it is left-dense. The graininess function $\mu : \mathbb{T} \to [0, \infty)$ is then defined by $\mu(t) := \sigma(t) - t$.

Definition 2.2. For $f : \mathbb{T} \to \mathbb{R}$ and $t \in \mathbb{T}$, (if $t = \sup \mathbb{T}$, assume t is not left-scattered), define the Δ-derivative $f^{\Delta}(t)$ of $f(t)$ to be the number, provided it exists, with the property that, for any $\epsilon > 0$, there is a neighborhood U of t such that

$$|[f(\sigma(t)) - f(s)] - f^{\Delta}(t)[\sigma(t) - s]| \leq \epsilon|\sigma(t) - s|$$

for all $s \in U$. For $m > 1$, the m-th Δ-derivative of $f(t)$ is defined by $f^{\Delta^m}(t) := (f^{\Delta^{m-1}})^{\Delta}(t)$.

We say that f is Δ-differentiable on $\mathbb{T}$ provided $f^{\Delta}(t)$ exists for all $t \in \mathbb{T}$.

It is easily seen that if $f : \mathbb{T} \to \mathbb{R}$ is continuous at $t \in \mathbb{T}$ and t is right-scattered, then

$$f^{\Delta}(t) = \frac{f(\sigma(t)) - f(t)}{\sigma(t) - t}.$$

In particular if $\mathbb{T} = \mathbb{Z}$, the set of integers, then

$$f^{\Delta}(t) = \Delta f(t) = f(t+1) - f(t).$$

If $t \in \mathbb{T}$ is right-dense and $f : \mathbb{T} \to \mathbb{R}$ is differentiable at t, then

$$f^{\Delta}(t) = f'(t) = \lim_{s \to t} \frac{f(t) - f(s)}{t - s}.$$

The following formula involving the graininess function is valid for all points of $\mathbb{T}$ at which $f^{\Delta}(t)$ exists:

$$f(\sigma(t)) = f(t) + f^{\Delta}(t)\mu(t). \tag{2.1}$$

Definition 2.3. Let $f : \mathbb{T} \to \mathbb{R}$ be a function. We say that f is rd-continuous if it is continuous at each right-dense point in $\mathbb{T}$ and $\lim_{s\to t^-} f(s)$ exists as a finite number for all left-dense points $t \in \mathbb{T}$. We denote by $C_{rd}(\mathbb{T}, \mathbb{R})$ the set of all rd-continuous functions $f : \mathbb{T} \to \mathbb{R}$.

Definition 2.4. If $F^{\Delta}(t) = f(t)$, then we define the integral of f on $[a, b] \cap \mathbb{T}$ by

$$\int_a^b f(\tau)\Delta\tau = F(b) - F(a).$$

It has been shown that if f is rd-continuous on $[a, b] \cap \mathbb{T}$, then $\int_a^b f(\tau)\Delta\tau$ exists. Since $\int_{\rho(b)}^b f(\tau)\Delta\tau = f(\rho(b))(b - \rho(b))$, we see that the single value $f(b)$ has no contribution to the integral.

The next result is one of the integration by parts formulas on time scales:

$$\int_a^b f(\sigma(t))g^{\Delta}(t)\Delta t = [f(t)g(t)]\Big|_a^b - \int_a^b f^{\Delta}(t)g(t)\Delta t. \tag{2.2}$$

Finally, if $F : \mathbb{T} \to \mathbb{R}^{n\times n}$ is a matrix valued function, then F^{Δ} and $\int_a^b F(\tau)\Delta\tau$ are defined entry-wise and the properties (2.1) and (2.2) can hence be extended to matrix valued functions accordingly.

3. KAMENEV-PHILOS TYPE CRITERIA

In this section we establish generalized Kamenev-Philos type criteria for the oscillatory behavior of Eq. (1.1). The following convention on matrices will be used: For any $A \in \mathbb{C}^{n\times n}$ we denote by A^* and $\operatorname{tr} A$ the complex conjugate transpose and the trace of A, respectively. For any $n \times n$ Hermitian matrices A and B, we say that $A > B$ [or $A \geq B$] if $A - B$ is positive definite [or positive semi-definite]. If $A \in \mathbb{C}^{n\times n}$ is Hermitian, then we let λ_i be the i-th eigenvalue of A so that

$$\lambda_{\max}(A) = \lambda_1(A) \geq \cdots \geq \lambda_n(A) = \lambda_{\min}(A).$$

Definition 3.1. A solution $X(t)$ of Eq. (1.1) is said to be prepared if the matrix $X^*(t)P(t)X^{\Delta}(t)$ is Hermitian for all $t \in \mathbb{T}$.

Eq. (1.1) is said to be nonoscillatory if it has a prepared solution $X(t)$ and $t_0 \geq 0$ such that

$$X^*(\sigma(t))P(t)X^{\Delta}(t) > 0 \quad \text{on } [t_0, \infty) \cap \mathbb{T}. \tag{3.1}$$

Otherwise, Eq. (1.1) is oscillatory.

It is known that if $X(t)$ is a prepared solution of Eq. (1.1) satisfying Eq. (3.1), then $X^{-1}(t)$ exists on $[t_0,\infty)\cap\mathbb{T}$. Our approach to the oscillation problems of Eq. (1.1) is based largely on the application of the Riccati equation established in [11, 12]. The following lemma can be found in [12], Theorem 12.

Lemma 3.1. *Suppose that $X(t)$ is a prepared solution of (1.1) on $[t_0,\infty)\cap\mathbb{T}$ satisfying (3.1). Let*

$$Z(t) = P(t)X^{\Delta}(t)X^{-1}(t). \tag{3.2}$$

Then $Z(t)$ is a Hermitian solution of the Riccati equation

$$Z^{\Delta}(t) + Z(t)[P(t)+\mu(t)Z(t)]^{-1}Z(t) + Q(t) = 0 \tag{3.3}$$

satisfying

$$P(t)+\mu(t)Z(t) > 0 \quad \text{on } [t_0,\infty)\cap\mathbb{T}. \tag{3.4}$$

We now extend the results of Del Medico and Kong [4] to the matrix equation case.

Let $\mathcal{D} = \{(t,s)\in\mathbb{T}^2 : t\ge s\ge 0\}$. For any function $f(t,s):\mathbb{T}^2\to\mathbb{R}$, denote by f_1^{Δ} and f_2^{Δ} the partial derivatives of f with respect to t and s, respectively. For $\mathcal{D}\subset\mathbb{R}$, denote by $L_{loc}(\mathcal{D})$ the space of functions which are integrable on any compact subset of $\mathcal{D}$. Define

$$\begin{aligned}\mathcal{H}^* =&\{H(t,s)\in C^1(\mathcal{D},\mathbb{R}_+) : (H_2^{\Delta}(t,\cdot))^2/H(t,\cdot)\in L([0,\rho(t)]\cap\mathbb{T}),\\ &H(t,t)=0, H(t,s)>0 \text{ and } H_2^{\Delta}(t,s)\le 0 \text{ for } t>s\ge 0\}\end{aligned}$$

and

$$\begin{aligned}\mathcal{H}_* =&\{H(t,s)\in C^1(\mathcal{D},\mathbb{R}_+) : (H_1^{\Delta}(\cdot,s))^2/H(\cdot,s)\in L_{loc}([\sigma(s),\infty)\cap\mathbb{T}),\\ &H(t,t)=0, H(t,s)>0 \text{ and } H_1^{\Delta}(t,s)\ge 0 \text{ for } t>s\ge 0\}.\end{aligned}$$

Examples of functions in these classes can be obtained in the following way: Let $G\in C^1(\mathbb{R}_+,\mathbb{R}_+)$ satisfy $G(0)=0, G'(u)>0$ for $u>0$, and $(G'(u))^2/G(u)\in L_{loc}(\mathbb{R}_+)$; $\alpha\in C(\mathbb{T},\mathbb{R}_+)$ and $\beta\in C^1(\mathbb{T},\mathbb{R}_+)$ satisfying $\alpha(t)>0$ and $\beta^{\Delta}(t)>0$ for $t\in\mathbb{T}$. Let

$$H^*(t,s)=\alpha(t)G(\beta(t)-\beta(s)) \quad\text{and}\quad H_*(t,s)=\alpha(s)G(\beta(t)-\beta(s)).$$

Then $H^*(t,s)\in\mathcal{H}^*$ and $H_*(t,s)\in\mathcal{H}_*$. In particular, $H(t,s)=(t-s)^r, r>1$, is a function in both $\mathcal{H}^*$ and $\mathcal{H}_*$.

These function classes will be used throughout this paper.

Lemma 3.2. *Let $H\in\mathcal{H}^*$. Then for any $f\in C_{rd}(\mathbb{T},\mathbb{R}^{n\times n})$ and $t\in\mathbb{T}$*

$$\int_{\rho(t)}^{t} H(t,\sigma(s))f(s)\,\Delta s = 0. \tag{3.5}$$

Proof. Obviously, (3.5) holds when $\rho(t)=t$. When $\rho(t)<t$, we see that $H(t,\sigma(\rho(t))) = H(t,t)=0$. Hence (3.5) holds in all cases. ∎

The first theorem gives oscillation conditions using functions in $\mathcal{H}^*$.

Theorem 3.1. *Let $H \in \mathcal{H}^*$. Assume that for any $t_0 \in \mathbb{T}$*

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)}\lambda_1\left\{\int_{t_0}^{\rho(t)} \left[H(t,\sigma(s))\,Q(s) - \frac{(H_2^\Delta(t,s))^2}{4H(t,\sigma(s))}\,P(s)\right]\Delta s \right.$$
$$\left. +\quad H_2^\Delta(t,\rho(t))\chi_{t-\rho(t)}P(\rho(t))\right\} = \infty, \tag{3.6}$$

where $\chi : \mathbb{R}_+ \to \mathbb{R}$ is defined by $\chi_t = \begin{cases} 0, & t=0 \\ 1, & t\in(0,\infty). \end{cases}$ Then Eq. (1.1) *is oscillatory.*

Proof. Assume Eq. (1.1) is not oscillatory. Then there exists a prepared solution $X(t)$ and a $t_0 \ge 0$ such that (3.1) holds. Let $Z(t)$ be defined by (3.2). By Lemma 3.1, $Z(t)$ is a Hermitian solution of Eq. (3.3) on $[t_0,\infty)\cap\mathbb{T}$ and satisfies (3.4).

For simplicity, in the following we let $H = H(t,s)$, $H_\sigma = H(t,\sigma(s))$, and $H_2^\Delta = H_2^\Delta(t,s)$, and omit the arguments when no confusion is raised. Let $t\in\mathbb{T}$. Multiplying (3.3), where t is replaced by s, by H_σ, and integrating it with respect to s from t_0 to t we obtain

$$\int_{t_0}^{t} H_\sigma\, Q\,\Delta s = -\int_{t_0}^{t} H_\sigma\left(Z^\Delta + Z[P+\mu Z]^{-1}Z\right)\,\Delta s. \tag{3.7}$$

Note that $H(t,t)=0$. By Lemma 3.2 and the integration by parts formula (2.2) we have that for $t\ge\sigma(t_0)$

$$\begin{aligned}\int_{t_0}^{\rho(t)} H_\sigma\, Q\,\Delta s &= \int_{t_0}^{t} H_\sigma\, Q\,\Delta s\\ &= -[H(t,s)Z(s)]_{s=t_0}^{t} + \int_{t_0}^{t}(H_2^\Delta\, Z - H_\sigma Z[P+\mu Z]^{-1}Z)\,\Delta s\\ &= H(t,t_0)Z(t_0) + \int_{\rho(t)}^{t} H_2^\Delta\, Z\,\Delta s + \int_{t_0}^{\rho(t)}(H_2^\Delta\, Z - H_\sigma Z[P+\mu Z]^{-1}Z)\,\Delta s.\end{aligned} \tag{3.8}$$

Since $H_2^\Delta(t,s)\le 0$ on $\mathcal{D}$, from (3.4)

$$\begin{aligned}\int_{\rho(t)}^{t} H_2^\Delta\, Z\Delta s &= H_2^\Delta(t,\rho(t))Z(\rho(t))(t-\rho(t))\\ &= H_2^\Delta(t,\rho(t))Z(\rho(t))\mu(\rho(t))\chi_{t-\rho(t)}\\ &\le -H_2^\Delta(t,\rho(t))\chi_{t-\rho(t)}P(\rho(t)).\end{aligned} \tag{3.9}$$

For $s\in[t_0,\rho(t))$, since $P(s)>0$, there exists $R(s)\in\mathbb{R}^{n\times n}$ such that $R(s)>0$ and $R^2(s) = P(s)$. Let $Y(s) = (R^{-1}ZR^{-1})(s)$. Then $Y(s)$ is Hermitian, and hence there exists an $n \times n$ unitary matrix $U(s)$ such that

$$(UYU^*)(s) = \Lambda(s) := \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}(s),$$

where $\lambda_i(s)$, $i = 1, \dots, n$, are the eigenvalues of $Y(s)$. Note that $Z(s) = (RYR)(s)$. From (3.4) we have that $I + \mu(s)Y(s) > 0$ on $[t_0, \infty) \cap \mathbb{T}$, i.e., $1 + \mu(s)\lambda_i(s) > 0$ on $[t_0, \infty) \cap \mathbb{T}$, $i = 1, \dots, n$. Thus, for $t \geq \sigma(t_0)$ and $s \in [t_0, \rho(t))$

$$\begin{aligned} & H_2^\Delta Z - H_\sigma Z[P + \mu Z]^{-1} Z \\ & \quad = R\left(H_2^\Delta Y - H_\sigma YR[R^2 + \mu RYR]^{-1} RY\right) R \\ & \quad = R\left(H_2^\Delta Y - H_\sigma Y[I + \mu Y]^{-1} Y\right) R \\ & \quad = RU^*\left(H_2^\Delta (UYU^*) - H_\sigma (UYU^*)[I + \mu(UYU^*)]^{-1}(UYU^*)\right) UR \\ & \quad = RU^*\left(H_2^\Delta \Lambda - H_\sigma \Lambda^2 [I + \mu\Lambda]^{-1}\right) UR \\ & \quad = RU^* DUR, \end{aligned} \tag{3.10}$$

where $D = H_2^\Delta \Lambda - H_\sigma \Lambda^2 [I + \mu\Lambda]^{-1}$ is an $n \times n$ diagonal matrix. Hence

$$D = \begin{bmatrix} d_1 & & \\ & \ddots & \\ & & d_n \end{bmatrix}(t, s),$$

where

$$d_i(t, s) = H_2^\Delta (t, s)\lambda_i(s) - \frac{H(t, \sigma(s))\lambda_i^2(s)}{1 + \mu\lambda_i(s)}, \ i = 1, \dots, n.$$

For $t \geq \sigma(t_0)$, $s \in [t_0, \rho(t))$, and $\lambda_i(s) \leq 0$

$$\begin{aligned} d_i = & H_2^\Delta \lambda_i - H_\sigma \frac{\lambda_i^2}{1 + \mu\,\lambda_i} \leq H_2^\Delta \lambda_i - H_\sigma \lambda_i^2 \\ = & - H_\sigma \left(\lambda_i - \frac{H_2^\Delta}{2H_\sigma}\right)^2 + \frac{(H_2^\Delta)^2}{4H_\sigma} \leq \frac{(H_2^\Delta)^2}{4H_\sigma}. \end{aligned}$$

For $t \geq \sigma(t_0)$, $s \in [t_0, \rho(t))$, and $\lambda_i(s) > 0$, from (2.1)

$$\begin{aligned} d_i = & H_2^\Delta \lambda_i - H_\sigma \frac{\lambda_i^2}{1 + \mu\lambda_i} = -\frac{1}{1 + \mu\lambda_i}[(H_\sigma - \mu H_2^\Delta)\lambda_i^2 - H_2^\Delta \lambda_i] \\ = & - \frac{1}{1 + \mu\lambda_i}[H\lambda_i^2 - H_2^\Delta \lambda_i] = -\frac{H}{1 + \mu\lambda_i}\left[\lambda_i^2 - \frac{H_2^\Delta}{H}\lambda_i\right] \\ = & - \frac{H}{1 + \mu\lambda_i}\left(\lambda_i - \frac{H_2^\Delta}{2H}\right)^2 + \frac{(H_2^\Delta)^2}{4H(1 + \mu\lambda_i)} \\ \leq & \frac{(H_2^\Delta)^2}{4H(1 + \mu\lambda_i)} \leq \frac{(H_2^\Delta)^2}{4H} \leq \frac{(H_2^\Delta)^2}{4H_\sigma}. \end{aligned}$$

Therefore, for all $t \geq \sigma(t_0)$ and $s \in [t_0, \rho(t))$ we have $d_i \leq \dfrac{(H_2^{\Delta})^2}{4H_\sigma}$, $i = 1, \dots, n$. Hence $D \leq \dfrac{(H_2^{\Delta})^2}{4H_\sigma} I$. By (3.10),

$$H_2^{\Delta} Z - H_\sigma Z[P + \mu Z]^{-1} Z \leq RU^* \left(\frac{(H_2^{\Delta})^2}{4H_\sigma} I \right) UR = \frac{(H_2^{\Delta})^2}{4H_\sigma} R^2 = \frac{(H_2^{\Delta})^2}{4H_\sigma} P. \tag{3.11}$$

Then from (3.8), (3.9), and (3.11) we obtain that for $t \geq \sigma(t_0)$

$$\begin{aligned} \int_{t_0}^{\rho(t)} H(t, \sigma(s))\, Q(s)\, \Delta s &\leq H(t, t_0) Z(t_0) \\ &\quad - H_2^{\Delta}(t, \rho(t)) \chi_{t-\rho(t)} P(\rho(t)) + \int_{t_0}^{\rho(t)} \frac{(H_2^{\Delta}(t, s))^2}{4H_\sigma(t, \sigma(s))} P(s)\, \Delta s. \end{aligned}$$

Hence

$$\begin{aligned} \int_{t_0}^{\rho(t)} &\left[H(t, \sigma(s))\, Q(s) - \frac{(H_2^{\Delta}(t, s))^2}{4H_\sigma(t, \sigma(s))} P(s) \right] \Delta s \\ &+ H_2^{\Delta}(t, \rho(t)) \chi_{t-\rho(t)} P(\rho(t)) \leq H(t, t_0) Z(t_0). \end{aligned}$$

This implies that

$$\begin{aligned} \lambda_1 \Bigg\{ \int_{t_0}^{\rho(t)} &\left[H(t, \sigma(s))\, Q(s) - \frac{(H_2^{\Delta}(t, s))^2}{4H_\sigma(t, \sigma(s))} P(s) \right] \Delta s \\ &+ H_2^{\Delta}(t, \rho(t)) \chi_{t-\rho(t)} P(\rho(t)) \Bigg\} \leq H(t, t_0) \lambda_1 \{ Z(t_0) \}. \end{aligned}$$

Therefore,

$$\begin{aligned} \frac{1}{H(t, t_0)} \lambda_1 \Bigg\{ \int_{t_0}^{\rho(t)} &\left[H(t, \sigma(s))\, Q(s) - \frac{(H_2^{\Delta}(t, s))^2}{4H_\sigma(t, \sigma(s))} P(s) \right] \Delta s \\ &+ H_2^{\Delta}(t, \rho(t)) \chi_{t-\rho(t)} P(\rho(t)) \Bigg\} \leq \lambda_1 \{ Z(t_0) \} < \infty, \end{aligned}$$

which contradicts (3.6) and completes the proof. □

The following corollary is a direct consequence of Theorem 3.1.

Corollary 3.1. *Let $H \in \mathcal{H}^*$. Assume that for any $t_0 \in \mathbb{T}$*

$$\begin{aligned} \limsup_{t \to \infty} \frac{1}{H(t, t_0)} \Bigg[\int_{t_0}^{\rho(t)} &\left[H(t, \sigma(s)) \operatorname{tr} Q(s) - \frac{(H_2^{\Delta}(t, s))^2}{4H(t, \sigma(s))} \operatorname{tr} P(s) \right] \Delta s \\ &+ \ H_2^{\Delta}(t, \rho(t)) \chi_{t-\rho(t)} \operatorname{tr} P(\rho(t)) \Bigg] = \infty, \end{aligned} \tag{3.12}$$

where $\chi : \mathbb{R}_+ \to \mathbb{R}$ is defined by $\chi_t = \begin{cases} 0, & t = 0 \\ 1, & t \in (0, \infty). \end{cases}$ Then Eq. (1.1) *is oscillatory.*

Proof. Note that for any Hermitian matrix A, $\lambda_1(A) \geq \mathrm{tr}A/n$, and the trace is a linear operator on the $n \times n$ matrix space. Therefore, the conclusion follows from Theorem 3.1. $\square$

In the sequel we define

$$\mathbb{T}_1 = \{s \in \mathbb{T} : s \text{ is right-dense}\} \text{ and } \mathbb{T}_2 = \{s \in \mathbb{T} : s \text{ is right-scattered}\}. \tag{3.13}$$

The corollary below is from corollary 3.1 where $H(t,s) = (t-s)^r$, $r > 1$.

Corollary 3.2. *For $t \in \mathbb{T}$, let $\mathbb{T}_1(t) = [0,t) \cap \mathbb{T}_1$ and $\mathbb{T}_2(t) = [0,t) \cap \mathbb{T}_2$. Assume there exists $r > 1$ such that*

$$\begin{aligned} \limsup_{t\to\infty} \frac{1}{t^r}\Bigg[\int_0^t (t-\sigma(s))^r \mathrm{tr}\, Q(s)\Delta s - \int_{\mathbb{T}_1(\rho(t))} \frac{r^2}{4}(t-s)^{r-2}\mathrm{tr}\, P(s)ds \\ - \sum_{\mathbb{T}_2(\rho(t))} \frac{r^2(t-s)^{2r-2}}{4(t-\sigma(s))^r}\mathrm{tr}\, P(s)\mu(s) - r(t-\rho(t))^{r-1}\mathrm{tr}\, P(\rho(t))\Bigg] = \infty. \end{aligned} \tag{3.14}$$

Then Eq. (1.1) *is oscillatory.*

Proof. Note that for any $t_0 \in \mathbb{T}$, (3.14) is equivalent to

$$\begin{aligned} \limsup_{t\to\infty} \frac{1}{(t-t_0)^r}\Bigg[\int_{t_0}^t (t-\sigma(s))^r \mathrm{tr}\, Q(s)\Delta s - \int_{\mathbb{T}_1(\rho(t)} \frac{r^2}{4}(t-s)^{r-2}\mathrm{tr}\, P(s)ds \\ - \sum_{\mathbb{T}_2(\rho(t)} \frac{r^2(t-s)^{2r-2}}{4(t-\sigma(s))^r}\mathrm{tr}\, P(s)\mu(s) - r(t-\rho(t))^{r-1}\mathrm{tr}\, P(\rho(t))\Bigg] = \infty. \end{aligned} \tag{3.15}$$

Let $H(t,s) = (t-s)^r$. Then $H \in \mathcal{H}^*$, and

$$H_2^\Delta(t,s) = \begin{cases} -r(t-s)^{r-1}, & s \in \mathbb{T}_1 \\ ((t-\sigma(s))^r - (t-s)^r)/\mu(s), & s \in \mathbb{T}_2. \end{cases}$$

Note from the Mean Value Theorem that for $t \in \mathbb{T}$ and $s \in [0,t) \cap \mathbb{T}_2$, there exists $\xi(s) \in [s, \sigma(s)]$ such that

$$0 \geq H_2^\Delta(t,s) = -r(t-\xi(s))^{r-1} \geq -r(t-s)^{r-1}. \tag{3.16}$$

Hence, for $t \in \mathbb{T}$ in both cases

$$H_2^\Delta(t,\rho(t))\chi_{t-\rho(t)} \geq -r(t-\rho(t))^{r-1}.$$

Therefore (3.15) implies (3.12) and the conclusion follows from Corollary 3.1. $\square$

The next theorem gives oscillation conditions using functions in $\mathcal{H}_*$. Note that this result does not apply to the case where all points in $\mathbb{T}$ are right-dense.

Theorem 3.2. *Let $H \in \mathcal{H}_*$ and let $\mathbb{T}_1, \mathbb{T}_2$ be defined by* (3.13)*. Then* Eq. (1.1) *is oscillatory provided there exists $\{t_n\}_{n=1}^\infty \subset \mathbb{T}_2$, $t_n \to \infty$, such that for any $t_0 \in \mathbb{T}$, one of the following holds:*

(i) $\lim_{n\to\infty} H(t_n,t_0)\mathrm{tr}\, P(t_n)/\mu(t_n) = \infty$ *and*

$$\limsup_{n\to\infty} \frac{\mu(t_n)}{H(t_n,t_0)\mathrm{tr}\, P(t_n)} \left[\int_{t_0}^{t_n} H(\sigma(s),t_0)\mathrm{tr}\, Q(s)\,\Delta s - \int_{\sigma(t_0)}^{t_n} \frac{(H_1^\Delta(s,t_0))^2}{4H(s,t_0)} \mathrm{tr}\, P(s)\,\Delta s \right] = \infty; \tag{3.17}$$

(ii) $\limsup_{n\to\infty} H(t_n,t_0)\mathrm{tr}\, P(t_n)/\mu(t_n) = \infty$ *and*

$$\lim_{n\to\infty} \frac{\mu(t_n)}{H(t_n,t_0)\mathrm{tr}\, P(t_n)} \left[\int_{t_0}^{t_n} H(\sigma(s),t_0)\mathrm{tr}\, Q(s)\,\Delta s - \int_{\sigma(t_0)}^{t_n} \frac{(H_1^\Delta(s,t_0))^2}{4H(s,t_0)} \mathrm{tr}\, P(s)\,\Delta s \right] = \infty;$$

(iii) $\limsup_{n\to\infty} H(t_n,t_0)\mathrm{tr}\, P(t_n)/\mu(t_n) < \infty$ *and*

$$\limsup_{n\to\infty} \left[\int_{t_0}^{t_n} H(\sigma(s),t_0)\mathrm{tr}\, Q(s)\,\Delta s - \int_{\sigma(t_0)}^{t_n} \frac{(H_1^\Delta(s,t_0))^2}{4H(s,t_0)} \mathrm{tr}\, P(s)\,\Delta s \right] = \infty.$$

Proof. Assume Eq. (1.1) is not oscillatory. Then there exists a prepared solution $X(t)$ and a $t_0 \geq 0$ such that (3.1) holds. Let $Z(t)$ be defined by (3.2). By Lemma 3.1, $Z(t)$ is a Hermitian solution of Eq. (3.3) on $[t_0,\infty)\cap\mathbb{T}$ and satisfies (3.4).

For simplicity, in the following we let $H = H(s,t_0)$, $H^\sigma = H(\sigma(s),t_0)$, and $H_1^\Delta = H_1^\Delta(s,t_0)$, and omit the arguments when no confusion is raised. Multiplying (3.3), where t is replaced by s, by H^σ, integrating it with respect to s from t_0 to t, and then using the integration by parts formula (2.2) we have

$$\begin{aligned} \int_{t_0}^{t} H^\sigma Q\,\Delta s &= -\int_{t_0}^{t} H^\sigma \left(Z^\Delta + Z[P+\mu Z]^{-1}Z\right)\Delta s \\ &= -H(t,t_0)Z(t) + \left(\int_{t_0}^{\sigma(t_0)} + \int_{\sigma(t_0)}^{t}\right)\left(H_1^\Delta Z - H^\sigma\left(Z[P+\mu Z]^{-1}Z\right)\right)\Delta s. \end{aligned} \tag{3.18}$$

By (2.1) we see that for $s \in [t_0,t)\cap\mathbb{T}$

$$H(\sigma(t_0),t_0) = H_1^\Delta(t_0,t_0)\mu(t_0) + H(t_0,t_0) = H_1^\Delta(t_0,t_0)\mu(t_0).$$

Hence

$$\begin{aligned} &\int_{t_0}^{\sigma(t_0)} \left(H_1^\Delta Z - H^\sigma(Z[P+\mu Z]^{-1}Z)\right)\Delta s \\ &\quad = \left(H_1^\Delta(t_0,t_0)Z(t_0) - H(\sigma(t_0),t_0)(Z[P+\mu Z]^{-1}Z)(t_0)\right)\mu(t_0) \\ &\quad = H_1^\Delta(t_0,t_0)\left(\mu Z - (\mu Z)[P+\mu Z]^{-1}(\mu Z)\right)(t_0). \end{aligned}$$

Note that when evaluated at t_0

$$\begin{aligned} \mu Z - (\mu Z)[P+\mu Z]^{-1}(\mu Z) &= ([P+\mu Z] - \mu Z)[P+\mu Z]^{-1}(\mu Z) \\ &= P[P+\mu Z]^{-1}(P+\mu Z - P) = P - P[P+\mu Z]^{-1}P \leq \chi_{\mu(t_0)}P. \end{aligned}$$

Thus we have

$$\int_{t_0}^{\sigma(t_0)} \left(H_1^\Delta Z - H^\sigma \left(Z[P+\mu Z]^{-1}Z\right)\right) \Delta s \leq H_1^\Delta(t_0,t_0)\chi_{\mu(t_0)}P(t_0). \tag{3.19}$$

We define $R(s)$, $U(s)$, and $\Lambda(s)$ as in the proof of Theorem 3.1. Then similar to (3.10) we obtain that for $t \geq \sigma(t_0)$ and $s \in [\sigma(t_0), t)$

$$H_1^\Delta Z - H^\sigma \left(Z[P+\mu Z]^{-1}Z\right) = RU^*EUR, \tag{3.20}$$

where $E = H_1^\Delta \Lambda - H\Lambda^2[I+\mu\Lambda]^{-1}$ is an $n \times n$ diagonal matrix. Hence

$$E(s,t_0) = \begin{bmatrix} e_1 & & \\ & \ddots & \\ & & e_n \end{bmatrix}(s,t_0),$$

where

$$e_i(s,t_0) = H_1^\Delta(s,t_0)\lambda_i(s) - H(\sigma(s),t_0)\frac{\lambda_i^2(s)}{1+\mu(s)\lambda_i(s)}, \ i = 1,\dots,n.$$

For $t \geq \sigma(t_0)$, $s \in [\sigma(t_0), t)$ and $\lambda_i(s) \leq 0$

$$e_i \leq H_1^\Delta \lambda_i - H^\sigma \lambda_i^2 = -H^\sigma \left(\lambda_i - \frac{H_1^\Delta}{2H^\sigma}\right)^2 + \frac{(H_1^\Delta)^2}{4H^\sigma} \leq \frac{(H_1^\Delta)^2}{4H^\sigma} \leq \frac{(H_1^\Delta)^2}{4H}.$$

For $t \geq \sigma(t_0)$, $s \in [\sigma(t_0), t)$, and $\lambda_i(s) > 0$, from (2.1)

$$\begin{aligned} e_i &= -\frac{1}{1+\mu\lambda_i}\left[\left(H^\sigma - H_1^\Delta \mu\right)\lambda_i^2 - H_1^\Delta \lambda_i\right] \\ &= -\frac{1}{1+\mu\lambda_i}\left[H\lambda_i^2 - H_1^\Delta\lambda_i\right] \\ &= -\frac{H}{1+\mu\lambda_i}\left(\lambda_i - \frac{H_1^\Delta}{2H}\right)^2 + \frac{(H_1^\Delta)^2}{4H(1+\mu\lambda_i)} \leq \frac{(H_1^\Delta)^2}{4H}. \end{aligned}$$

Therefore, for all $t \geq \sigma(t_0)$ and $s \in [\sigma(t_0), t)$ we have $e_i \leq \dfrac{(H_1^\Delta)^2}{4H}$, $i = 1,\dots,n$. Hence $E \leq \dfrac{(H_1^\Delta)^2}{4H} I$. By (3.20)

$$\begin{aligned} H_1^\Delta Z - H^\sigma(\left(Z[P+\mu Z]^{-1}Z\right) &\leq \left(RU^*\left[\frac{(H_1^\Delta)^2}{4H}I\right]UR\right) \\ &= \frac{(H_1^\Delta)^2}{4H}R^2 = \frac{(H_1^\Delta)^2}{4H}P. \end{aligned} \tag{3.21}$$

From (3.18), (3.19), and (3.21) we have

$$\int_{t_0}^{t} H^\sigma Q\,\Delta s \leq -H(t,t_0)Z(t) + H_1^\Delta(t_0,t_0)\chi_{\mu(t_0)}P(t_0) + \int_{\sigma(t_0)}^{t} \frac{(H_1^\Delta)^2}{4H}P\,\Delta s. \tag{3.22}$$

For $t \in \mathbb{T}_2$, (3.22) together with (3.4) implies

$$\int_{t_0}^{t} H^\sigma Q\,\Delta s \leq \frac{H(t,t_0)}{\mu(t)}P(t) + H_1^\Delta(t_0,t_0)\chi_{\mu(t_0)}P(t_0) + \int_{\sigma(t_0)}^{t} \frac{(H_1^\Delta)^2}{4H}P\,\Delta s.$$

Taking the trace on both sides we have that

(3.23)
$$\int_{t_0}^{t} H^\sigma \operatorname{tr} Q\,\Delta s \le \frac{H(t,t_0)}{\mu(t)}\operatorname{tr} P(t) + H_1^\Delta(t_0,t_0)\chi_{\mu(t_0)}\operatorname{tr} P(t_0) + \int_{\sigma(t_0)}^{t} \frac{(H_1^\Delta)^2}{4H}\operatorname{tr} P\,\Delta s.$$

Assume condition (i) holds. Let $t = t_n$ in (3.23) we obtain

$$\frac{\mu(t_n)}{H(t_n,t_0)\operatorname{tr} P(t_n)}\left[\int_{t_0}^{t_n} H^\sigma \operatorname{tr} Q\,\Delta s - \int_{\sigma(t_0)}^{t_n} \frac{(H_1^\Delta)^2}{4H}\operatorname{tr} P\,\Delta s\right]$$
$$\le 1 + \frac{H_1^\Delta(t_0,t_0)p(t_0)\mu(t_n)}{H(t_n,t_0)p(t_n)}\chi_{\mu(t_0)}.$$

Taking the limsup as $n \to \infty$ on both sides we have

$$\limsup_{n\to\infty} \frac{\mu(t_n)}{H(t_n,t_0)p(t_n)}\left[\int_{t_0}^{t_n} H^\sigma \operatorname{tr} Q\,\Delta s - \int_{\sigma(t_0)}^{t_n} \frac{(H_1^\Delta)^2}{4H}\operatorname{tr} P\,\Delta s\right] < \infty$$

which contradicts (3.17).

The conclusions with conditions (ii) and (iii) can be obtained easily. We omit the details. □

Corollary 3.3. *Let $\mathbb{T}_1$ and $\mathbb{T}_2$ be defined by* (3.13), *and for any $l, t \in \mathbb{T}$ let*

$$\mathbb{T}_1(l,t) = [l,t) \cap \mathbb{T}_1 \quad \text{and} \quad \mathbb{T}_2(l,t) = [l,t) \cap \mathbb{T}_2.$$

Then Eq. (1.1) *is oscillatory provided there exist $r > 1$ and $\{t_n\}_{n=1}^\infty \subset \mathbb{T}_2$, $t_n \to \infty$, such that for any $t_0 \in \mathbb{T}$, one of the following holds:*

(i) $\lim_{n\to\infty}(t^n)^r \operatorname{tr} P(t_n)/\mu(t_n) = \infty$ *and*

$$\limsup_{n\to\infty}\frac{\mu(t_n)}{(t_n)^r \operatorname{tr} P(t_n)}\left[\int_{t_0}^{t_n}(\sigma(s)-t_0)^r \operatorname{tr} Q(s)\Delta s - \int_{\mathbb{T}_1(\sigma(t_0),t_n)}\frac{r^2}{4}(s-t_0)^{r-2}\operatorname{tr} P(s)\,ds\right.$$
$$\left. - \sum_{\mathbb{T}_2(\sigma(t_0),t_n)}\frac{r^2(\sigma(s)-t_0)^{2r-2}}{4(s-t_0)^r}\operatorname{tr} P(s)\mu(s)\right] = \infty;$$

(ii) $\limsup_{n\to\infty}(t_n)^r \operatorname{tr} P(t_n)/\mu(t_n) = \infty$ *and*

$$\lim_{n\to\infty}\frac{\mu(t_n)}{(t_n)^r \operatorname{tr} P(t_n)}\left[\int_{t_0}^{t_n}(\sigma(s)-t_0)^r \operatorname{tr} Q(s)\Delta s - \int_{\mathbb{T}_1(\sigma(t_0),t_n)}\frac{r^2}{4}(s-t_0)^{r-2}\operatorname{tr} P(s)\,ds\right.$$
$$\left. - \sum_{\mathbb{T}_2(\sigma(t_0),t_n)}\frac{r^2(\sigma(s)-t_0)^{2r-2}}{4(s-t_0)^r}\operatorname{tr} P(s)\mu(s)\right] = \infty;$$

(iii) $\limsup_{n\to\infty}(t_n)^r \operatorname{tr} P(t_n)/\mu(t_n) < \infty$ *and*

$$\limsup_{n\to\infty}\left[\int_{t_0}^{t_n}(\sigma(s)-t_0)^r \operatorname{tr} Q(s)\Delta s - \int_{\mathbb{T}_1(\sigma(t_0),t_n)}\frac{r^2}{4}(s-t_0)^{r-2}\operatorname{tr} P(s)\,ds\right.$$
$$\left. - \sum_{\mathbb{T}_2(\sigma(t_0),t_n)}\frac{r^2(\sigma(s)-t_0)^{2r-2}}{4(s-t_0)^r}\operatorname{tr} P(s)\mu(s)\right] = \infty.$$

Proof. Let $H(t,s)=(t-s)^r$. Then $H\in\mathcal{H}_*$, and

$$H_1^{\Delta}(s,t_0)=\begin{cases} r(s-t_0)^{r-1}, & s\in\mathbb{T}_1\\ ((\sigma(s)-t_0)^r-(s-t_0)^r)/\mu(s), & s\in\mathbb{T}_2.\end{cases}$$

Note from the Mean Value Theorem that for $s\in[t_0,\infty)\cap\mathbb{T}_2$, there exists $\xi(s)\in[s,\sigma(s)]$ such that

$$0\le H_1^{\Delta}(s,t_0)=r(\xi(s)-t_0)^{r-1}\le r(\sigma(s)-t_0)^{r-1}.$$

Therefore the conclusion follows from Theorem 3.2. □

4. INTERVAL CRITERIA

Now, we present the analogues of the interval criteria for oscillation of Eq. (1.2) in [4] to the matrix equation (1.1) on time scales. The proofs are similar to those in [4] and hence are omitted. Further conditions for oscillation of the Kamenev type are derived from them. In this section we use the notation $\mathcal{H}=\mathcal{H}^*\cap\mathcal{H}_*$.

Theorem 4.1. *Let $a,c,b\in\mathbb{T}$ such that $a<c<b$. Assume that for some $H\in\mathcal{H}$,*

(4.1)
$$\begin{aligned}&\frac{1}{H(c,a)}\int_a^c H(\sigma(s),a)\operatorname{tr}Q(s)\Delta s+\frac{1}{H(b,c)}\int_c^b H(b,\sigma(s))\operatorname{tr}Q(s)\Delta s\\ &>\frac14\left[\frac{1}{H(c,a)}\int_{\sigma(a)}^c\frac{(H_1^{\Delta}(s,a))^2}{H(s,a)}\operatorname{tr}P(s)\Delta s+\frac{1}{H(b,c)}\int_c^{\rho(b)}\frac{(H_2^{\Delta}(b,s))^2}{H(b,\sigma(s))}\operatorname{tr}P(s)\Delta s\right]\\ &+\frac{H_1^{\Delta}(a,a)}{H(c,a)}\chi_{\mu(a)}\operatorname{tr}P(a)-\frac{H_2^{\Delta}(b,\rho(b))}{H(b,c)}\chi_{b-\rho(b)}\operatorname{tr}P(\rho(b)).\end{aligned}$$

Then every solution of Eq. (1.1) *has at least one generalized zero in (a,b).*

Theorem 4.2. Eq. (1.1) *is oscillatory provided that for any $T\ge 0$, there exists $H\in\mathcal{H}$ and $a,b,c\in\mathbb{T}$ such that $T\le a<c<b$ and (4.1) holds.*

Corollary 4.1. *Assume there exists $H\in\mathcal{H}$ such that for any $l\in\mathbb{T}$*

$$\limsup_{t\to\infty}\left[\int_l^t H(\sigma(s),l)\operatorname{tr}Q(s)\Delta s-\int_{\sigma(l)}^t\frac{(H_1^{\Delta}(s,l))^2}{4H(s,l)}\operatorname{tr}P(s)\Delta s-H_1^{\Delta}(l,l)\chi_{\mu(l)}\operatorname{tr}P(l)\right]>0$$

and

$$\limsup_{t\to\infty}\left[\int_l^t H(t,\sigma(s))\operatorname{tr}Q(s)\Delta s-\int_l^{\rho(t)}\frac{(H_2^{\Delta}(t,s))^2}{4H(t,\sigma(s))}\operatorname{tr}P(s)\Delta s+H_2^{\Delta}(t,\rho(t))\chi_{t-\rho(t)}\operatorname{tr}P(\rho(t))\right]>0.$$

Then Eq. (1.1) *is oscillatory.*

Corollary 4.2. *Let* $\mathbb{T}_1$ *and* $\mathbb{T}_2$ *be defined by* (3.13), *and for any* $l, t \in \mathbb{T}$ *let*

$$\mathbb{T}_1(l,t) = [l,t) \cap \mathbb{T}_1 \quad \text{and} \quad \mathbb{T}_2(l,t) = [l,t) \cap \mathbb{T}_2.$$

Assume there exists $r > 1$ *such that for any* $l \geq 0$

$$\limsup_{t\to\infty} \left[\int_l^t (\sigma(s) - l)^r \operatorname{tr} Q(s) \Delta s - \int_{\mathbb{T}_1(\sigma(l),t)} \frac{r^2}{4}(s-l)^{r-2} \operatorname{tr} P(s) ds \right.$$
$$\left. - \sum_{\mathbb{T}_2(\sigma(l),t)} \frac{r^2(\sigma(s)-l)^{2r-2}}{4(s-l)^r} \operatorname{tr} P(s)\mu(s) - r \operatorname{tr} P(l)\mu^{r-1}(l) \right] > 0$$

and

$$\limsup_{t\to\infty} \left[\int_l^t (t - \sigma(s))^r \operatorname{tr} Q(s) \Delta s - \int_{\mathbb{T}_1(l,\rho(t))} \frac{r^2}{4}(t-s)^{r-2} \operatorname{tr} P(s) ds \right.$$
$$\left. - \sum_{\mathbb{T}_2(l,\rho(t))} \frac{r^2(t-s)^{2r-2}}{4(t-\sigma(s))^r} \operatorname{tr} P(s)\mu(s) - r \operatorname{tr} P(\rho(t))(t-\rho(t))^{r-1} \right] > 0.$$

Then Eq. (1.1) *is oscillatory.*

Remark 4.1. It can be easily seen that Corollaries 4.1 and 4.2 are improvements of Theorems 3.1, 3.2, Corollaries 3.1 and 3.2 for many cases.

Remark 4.2. By applying our theorems and corollaries in Sections 3 and 4 to the case where $\mathbb{T} = \mathbb{R}_+$, we can obtain criteria for oscillation of Eq. (1.2) which cover and generalize those by Philos [21] and Kong [19].

5. APPLICATIONS TO DIFFERENCE EQUATIONS

Here, we apply the results in Sections 3 and 4 to obtain the Kamenev type and interval criteria for oscillation for the difference equation (1.3).

The first two theorems are direct consequences of Corollaries 3.1 and 3.2 with $\mathbb{T} = \mathbb{N}$.

Theorem 5.1. *Assume there exists* $r > 1$ *such that*

$$\limsup_{n\to\infty} \frac{1}{n^r} \left[\sum_{k=0}^{n-1} (n-k-1)^r \operatorname{tr} Q_k - \sum_{k=0}^{n-2} \frac{r^2(n-k)^{2r-2}}{4(n-k-1)^r} \operatorname{tr} P_k - r \operatorname{tr} P_{n-1} \right] = \infty.$$

Then Eq. (1.3) *is oscillatory.*

Theorem 5.2. Eq. (1.3) *is oscillatory provided there exists* $r > 1$ *such that for any* $n_0 \in \mathbb{N}$, *one of the following holds:*

(i) $\lim_{n\to\infty} n^r \operatorname{tr} P_n = \infty$ *and*

$$\limsup_{n\to\infty} \frac{1}{n^r \operatorname{tr} P_n}\left[\sum_{k=n_0}^{n-1}(k+1-n_0)^r \operatorname{tr} Q_k - \sum_{k=n_0+1}^{n-1}\frac{r^2(k+1-n_0)^{2r-2}}{4(k-n_0)^r}\operatorname{tr} P_k\right] = \infty;$$

(ii) $\limsup_{n\to\infty} n^r \operatorname{tr} P_n = \infty$ *and*

$$\lim_{n\to\infty} \frac{1}{n^r \operatorname{tr} P_n}\left[\sum_{k=n_0}^{n-1}(k+1-n_0)^r \operatorname{tr} Q_k - \sum_{k=n_0+1}^{n-1}\frac{r^2(k+1-n_0)^{2r-2}}{4(k-n_0)^r}\operatorname{tr} P_k\right] = \infty;$$

(iii) $\lim_{n\to\infty} n^r \operatorname{tr} P_n < \infty$ *and*

$$\limsup_{n\to\infty} \left[\sum_{k=n_0}^{n-1}(k+1-n_0)^r \operatorname{tr} Q_k - \sum_{k=n_0+1}^{n-1}\frac{r^2(k+1-n_0)^{2r-2}}{4(k-n_0)^r}\operatorname{tr} P_k\right] = \infty.$$

The next result is an interval criterion for Eq. (1.3) derived from Theorem 4.2.

Theorem 5.3. *Assume that for any $K > 0$, there exists $r > 1$ and $l, m \in \mathbb{N}$, $K \le l < m$, satisfying that*

$$\sum_{k=l}^{m-1}\left[(k+1-l)^r \operatorname{tr} Q_k + (k-l)^r \operatorname{tr} Q_{2m-k-1}\right]$$
$$> \frac{r^2}{4}\sum_{k=l+1}^{m-1}\left[\frac{(k+1-l)^{2r-2}}{(k-l)^r}(\operatorname{tr} P_k + \operatorname{tr} P_{2m-k-1})\right] + (\operatorname{tr} P_l + \operatorname{tr} P_{2m-l-1}).$$

Then Eq. (1.3) *is oscillatory.*

As a result of the improved Kamenev criterion given by Corollary 4.2 with $\mathbb{T} = \mathbb{N}$, we have the following.

Theorem 5.4. *Assume there exists $r > 1$ such that for every $n_0 \in \mathbb{N}$*

$$\limsup_{n\to\infty} \left[\sum_{k=n_0}^{n-1}(k+1-n_0)^r \operatorname{tr} Q_k - \sum_{k=n_0+1}^{n-1}\frac{r^2(k+1-n_0)^{2r-2}}{4(k-n_0)^r}\operatorname{tr} P_k - r\operatorname{tr} P_{n_0}\right] > 0$$

and

$$\limsup_{n\to\infty} \left[\sum_{k=n_0}^{n-1}(n-k-1)^r \operatorname{tr} Q_k - \sum_{k=n_0}^{n-2}\frac{r^2(n-k)^{2r-2}}{4(n-k-1)^r}\operatorname{tr} P_k - r\operatorname{tr} P_{n-1}\right] > 0.$$

Then Eq. (1.3) *is oscillatory.*

For the case where $P \equiv I$, we summarize the results in Theorems 5.1, 5.2, and 5.4 below.

Theorem 5.5. *Let $P \equiv I$. Then* Eq. (1.3) *is oscillatory provided there exists an $r > 1$ such that for any $n_0 \in \mathbb{N}$, one of the following holds:*

(i) $\limsup_{n\to\infty} \frac{1}{n^r} \sum_{k=0}^{n-1} (n-k-1)^r \operatorname{tr} Q_k = \infty$;

(ii) $\limsup_{n\to\infty} \frac{1}{n^r} \sum_{k=n_0}^{n-1} (k+1-n_0)^r \operatorname{tr} Q_k = \infty$;

(iii) $\limsup_{n\to\infty} \frac{1}{n^{r-1}} \sum_{k=n_0}^{n-1} (k+1-n_0)^r \operatorname{tr} Q_k > \frac{r^2}{4(r-1)}$ *and*

$$\limsup_{n\to\infty} \frac{1}{n^{r-1}} \sum_{k=n_0}^{n-1} (n-k-1)^r \operatorname{tr} Q_k > \frac{r^2}{4(r-1)}.$$

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