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COMBINATION OF LIAPUNOV AND RETRACT METHODS IN THE INVESTIGATION OF THE ASYMPTOTIC BEHAVIOR OF SOLUTIONS OF SYSTEMS OF DISCRETE EQUATIONS

JOSEF DIBLÍK AND IRENA HLAVIČKOVÁ

Brno University of Technology, Brno, Czech Republic
diblik.j@fce.vutbr.cz, diblik@feec.vutbr.cz
hlavicka@feec.vutbr.cz

ABSTRACT. This contribution is devoted to a discussion of the asymptotic behavior of solutions of systems of first order nonlinear difference equations. We show that under appropriate conditions there exists at least one solution of the system considered the graph of which stays in a prescribed domain. The domains we work with are the so called polyfacial sets. In literature, retract and Liapunov type approaches are known as excellent asymptotic analysis tools. We present a method which connects both these techniques. Thanks to this, the achieved result can be applied to a substantially wider range of equations. The main result is applied to study a linear system of difference equations as well as to investigate a nonlinear system similar to the discrete scalar equation of Bernoulli's type. Results are illustrated by detailed examples.

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1. INTRODUCTION

The asymptotic behavior of solutions of discrete equations was studied (under different assumptions) e.g. in [2]–[23]. Two important tools are very often used in asymptotic analysis, referred to as the retract and Liapunov type approaches. Investigations based on these tools have been performed, e.g., in [6] and [8]. Paper [6] deals with a system where all the boundary points of the studied domain are the so called points of strict egress and the retract principle is used there. In [8], the case of the purely Liapunov type set is investigated. In the paper presented we give a method which connects both these techniques. Comparing the improvements introduced by the new technique with the above cited methods, we underline the connection of both these methods into one tool. The domains we work with are the so called polyfacial sets and we simultaneously admit both of the above mentioned properties for them: part of the boundary points are points of strict egress and the remaining part of the boundary together with some inner points have the character typical for Liapunov type sets.

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The simplest version of such "hybrid" case was studied by both authors in [11], considering a system of two equations. The presented paper brings a non-trivial generalization of that result to any finite number of equations. Thanks to this, the achieved result can be applied to a substantially wider range of equations and its flexibility is demonstrated on applications.

Throughout this paper, we use the following notation: for integers s, q, $s \le q$ we define

$$\mathbb{Z}_s^q := \{s, s+1, \dots, q\}$$

where possibilities $s = -\infty$ or $q = \infty$ are admitted, too.

We investigate the asymptotic behavior for $k \to \infty$ of the solutions of the system of n difference equations

$$\Delta u(k) = F(k, u(k)) \tag{1.1}$$

where $k \in \mathbb{Z}_a^\infty$, $a \in \mathbb{N} = \{0, 1, 2, \dots\}$ is fixed, $u = (u_1, \dots, u_n)$, $\Delta u(k) = u(k+1) - u(k)$ and $F = (F_1, \dots, F_n)$ is a mapping from $\mathbb{Z}_a^\infty \times \mathbb{R}^n$ to $\mathbb{R}^n$.

We recall some known facts. The *solution* of system (1.1) is defined as an infinite sequence of number vectors

$$\{u(a), u(a+1), u(a+2), \dots\}$$

with $u = (u_1, \dots, u_n)$ such that for any $k \in \mathbb{Z}_a^\infty$, equality (1.1) holds. The *existence and uniqueness* of the solution of system (1.1) with a prescribed initial condition

$$u(a) = u^a \in \mathbb{R}^n \tag{1.2}$$

on $\mathbb{Z}_a^\infty$ is obvious. The sequence $\{(k, u(k))\}$, $k \in \mathbb{Z}_a^\infty$, is called the *graph* of the solution $u = u(k)$ of initial problem (1.1), (1.2). If for every fixed $k \in \mathbb{Z}_a^\infty$ the right hand side $F(k, u)$ is continuous with respect to its argument u, then the solution of initial problem (1.1), (1.2) *depends continuously* on the initial data.

The paper is organized as follows - in Section 2 we pose the problem formulated and necessary auxiliary notions. These are used in Section 3 where the main result regarding the existence at least one solution of the system considered the graph of which stays in a prescribed domain is proved. Following two section deal with its application. In Section 4 asymptotic behavior of solutions of a linear nonhomogeneous system is investigated and in Section 5 nonlinear system of Bernoulli's type is treated. All results are illustrated by detailed examples. In the last Section 6, some comparisons with the latest results are given. Moreover, some comments on the flexibility, severity and application of the method presented are included.

2. DESCRIPTION OF THE PROBLEM CONSIDERED AND AUXILIARY NOTIONS

2.1. **Polyfacial Sets.** Let $b_i, c_i\colon \mathbb{Z}_a^\infty \to \mathbb{R}$, $i = 1, \dots, n$, be functions such that $b_i(k) < c_i(k)$ for each $k \in \mathbb{Z}_a^\infty$.
Define the sets

$$\Omega(k) := \{(k, u_1, \dots, u_n)\colon u_i \in \mathbb{R}, b_i(k) < u_i < c_i(k), i = 1, \dots, n\}$$

and the set $\Omega \subset \mathbb{Z}_a^\infty \times \mathbb{R}^n$ as

$$\Omega := \bigcup_{k \in \mathbb{Z}_a^\infty} \Omega(k)$$

Such set Ω is called a *polyfacial set*.
Our aim is to find sufficient conditions which *guarantee the existence of at least one solution* $u = u^*(k)$, $k \in \mathbb{Z}_a^\infty$, of system (1.1) satisfying

$$(k, u^*(k)) \in \Omega(k) \tag{2.1}$$

for every $k \in \mathbb{Z}_a^\infty$.
Since the asymptotic behavior of a solution satisfying (2.1) is in some sense conditioned by the shape of the set Ω, we call the asymptotic behavior of such solutions *compulsory* asymptotic behavior.

In [8] the above described problem is solved via Liapunov type technique. Here we will combine this technique with the retract type technique which was used in [6]. Before we start, define some basic notions that will be used.

2.2. **Consequent Points.**

Definition 2.1. Define the mapping $\mathcal{C}\colon \mathbb{Z}_a^\infty \times \mathbb{R}^n \to \mathbb{Z}_a^\infty \times \mathbb{R}^n$ as

$$\mathcal{C}\colon (k, u) \mapsto (k + 1, u + F(k, u)).$$

For any point $M = (k, u) \in \mathbb{Z}_a^\infty \times \mathbb{R}^n$, the point $\mathcal{C}(M)$ is called the *first consequent point* of the point M.

This means that if a point M lies on the graph of some solution of system (1.1), then its first consequent point $\mathcal{C}(M)$ is the next point on this graph.

Definition 2.2. For any $r \in \mathbb{N}$ define the *r-th consequent point* of a point $M \in \mathbb{Z}_a^\infty \times \mathbb{R}^n$ as

$$\mathcal{C}^r(M) = \mathcal{C}\left(\mathcal{C}^{r-1}(M)\right) \quad \text{for } r \geq 1$$

and

$$\mathcal{C}^0(M) = M.$$

2.3. **Liapunov type polyfacial sets.** We say that a polyfacial set Ω is of *Liapunov type* with respect to the discrete system (1.1) if

$$b_i(k+1) < u_i + F_i(k,u) < c_i(k+1)$$

for every $i = 1, \dots, n$ and every $(k,u) \in \Omega$. Such sets were used in [8].
The geometrical meaning of this property is this: If a point $M = (k,u)$ lies inside the set $\Omega(k)$, then its first consequent point $\mathcal{C}(M)$ stays inside $\Omega(k+1)$.
In this contribution we will deal with sets that need not be of Liapunov type, but they will have a similar property only with respect to a part of the indices. We give the relevant definition below.

Definition 2.3. We say that a polyfacial set Ω is of *Liapunov type with respect to the j-th variable* and to the discrete system (1.1) if for every $(k,u) \in \Omega$

$$b_j(k+1) < u_j + F_j(k,u) < c_j(k+1). \tag{2.2}$$

The geometrical meaning is that if $M = (k,u) \in \Omega(k)$, then the u_j-coordinate of its first consequent point stays between $b_j(k+1)$ and $c_j(k+1)$, meanwhile the other coordinates of $\mathcal{C}(M)$ may be arbitrary.

2.4. **Points of strict egress and their geometrical sense.** An important role in the application of the retract type technique is played by the so called strict egress points.
Before we define these points, let us describe the boundary of the set Ω in detail. As one can easily see,

$$\partial\Omega = \left(\bigcup_{j=1}^{n} \Omega_B^j\right) \cup \left(\bigcup_{j=1}^{n} \Omega_C^j\right)$$

with

$$\Omega_B^j := \{(k,u)\colon k \in \mathbb{Z}_a^\infty, u_j = b_j(k), b_i(k) \le u_i \le c_i(k), i = 1, \dots, n, i \ne j\}$$

and

$$\Omega_C^j := \{(k,u)\colon k \in \mathbb{Z}_a^\infty, u_j = c_j(k), b_i(k) \le u_i \le c_i(k), i = 1, \dots, n, i \ne j\}.$$

In accordance with [6, Lemmas 1,2], a point $(k,u) \in \partial\Omega$ is a *point of the type of strict egress* for the polyfacial set Ω with respect to discrete system (1.1) if and only if for some $j \in \{1, \dots, n\}$

$$u_j = b_j(k) \quad \text{and} \quad F_j(k,u) < b_j(k+1) - b_j(k), \tag{2.3}$$

or

$$u_j = c_j(k) \quad \text{and} \quad F_j(k,u) > c_j(k+1) - c_j(k). \tag{2.4}$$

Geometrically these inequalities mean the following:

If a point $M = (k, u) \in \partial\Omega$ is a point of the type of strict egress, then the first consequent point $\mathcal{C}(M) \notin \overline{\Omega}$.

2.5. **Retract and retraction.** If $A \subset B$ are any two sets of a topological space and $\pi\colon B \to A$ is a continuous mapping from B onto A such that $\pi(p) = p$ for every $p \in A$, then π is said to be a *retraction* of B onto A. If there exists a retraction of B onto A, A is called a *retract* of B.

3. MAIN RESULT

Let Ω be a polyfacial set. In this part we will solve our problem supposing that for some fixed set of indices $I_L \subseteq \{1, \dots, n\}$ the set Ω is of Liapunov type with respect to the $j-$th variable for $j \in I_L$ and for the remaining indices the corresponding parts of the set $\partial\Omega$ consist of points of the type of strict egress.

Theorem 3.1 (Main Result)**.** *Suppose that $F\colon \mathbb{Z}_a^\infty \times \mathbb{R}^n \to \mathbb{R}^n$ is a continuous mapping and that the set Ω is a polyfacial set. Let there be a set of indices $I_e \subseteq \{1, \dots, n\}$ such that if $M \in \Omega_B^i \cup \Omega_C^i$ for some $i \in I_e$, then M is a point of strict egress for the set Ω with respect to system* (1.1)*. Further suppose that for any $j \in I_L := \{1, \dots, n\} \setminus I_e$ the set Ω is of the Liapunov type with respect to the j-th variable. Then there exists a solution $u = u^*(k)$, $k \in \mathbb{Z}_a^\infty$, of* (1.1) *satisfying the relation*

$$(k, u^*(k)) \in \Omega(k), \text{ i.e. } b_i(k) < u_i^*(k) < c_i(k), \quad i = 1, \dots, n, \tag{3.1}$$

for every $k \in \mathbb{Z}_a^\infty$.

Before we prove this theorem, let us make a few comments. The statement of the theorem says that under given assumptions, there exists an initial condition $u(a) = u^a$ such that the coordinates of the appropriate solution $u(k)$ are bounded from below by the functions $b_i(k)$ and from above by the functions $c_i(k)$. A result of this type is important when we deal with a situation where the initial conditions are adjustable, e.g. when we study whether there is a chance how to set the initial state of some system so that the solution stays bounded. In the case that the initial condition is fixly given, Theorem 3.1 does not give us any useful result.

Further, Theorem 3.1 does not give us any recipe how to find the functions $b_i(k)$ and $c_i(k)$. This is a situation similar to the Liapunov second (or direct) method in the theory of stability of ordinary differential equations. There, too, the existence of a function with certain properties (the so called Liapunov function) is supposed, but the theorem does not tell us how to obtain this function for a given equation. However, for some particular classes of equations, the Liapunov function can be found. Similarly, for some special forms of system (1.1), functions $b_i(k)$ and $c_i(k)$ can be found, as will be shown in sections 4 and 5.

Proof. First consider the case $I_e = \emptyset$, i.e. $I_L = \{1, \dots, n\}$. Then the set Ω is of Liapunov type with respect to discrete system (1.1). In this case for any initial condition $u(a) = u^a$ such that $(a, u^a) \in \Omega(a)$, the solution given by this condition satisfies (3.1). The proof is very simple and it can be found in [8]. We omit this proof here referring the reader to that source.

The situation for $I_e = \{1, \dots, n\}$ and $I_L = \emptyset$ was discussed in [6], [7], [9] and [10]. There it is proved that if the boundary of the set Ω consists only of the points of strict egress, then the solution with property (3.1) has to exist. Again, we omit this proof.

Now we focus our attention to the "hybrid" case. Let $I_e \neq \emptyset$ and $I_L \neq \emptyset$ and denote m the number of elements of the set I_e. Without the loss of generality we may suppose that $I_e = \{1, \dots, m\}$.

For the reader's convenience, the proof is divided into several parts.

a) General scheme of the proof.
Suppose that the initial data $(a, u^*(a)) \in \Omega(a)$, generating a solution $u = u^*(k)$ of system (1.1) with property (3.1) do not exist. We will prove that under this assumption there exists a retraction $\mathcal{R}$ (which will be a composition of two auxiliary mappings P and Q defined below) of a set which is topologically equivalent to a closed m-dimensional ball onto its boundary, which cannot happen.

Denote $u_e = (u_1, \dots, u_m)$ and $u_L = (u_{m+1}, \dots, u_n)$. Then $u = (u_e, u_L)$.

In the following we will work with the sets

$$\Omega_e(k) := \{(k, u_e) \colon b_i(k) < u_i < c_i(k), i = 1, \dots, m\}, \quad k \in \mathbb{Z}_a^\infty.$$

The boundaries and closures of the sets $\Omega_e(k)$ will be always taken in the corresponding space

$$S_e(k) := \{(k, u_e) \colon u_e \in \mathbb{R}^m\}, \quad k \in \mathbb{Z}_a^\infty. \tag{3.2}$$

Further, if the r-th consequent point of some point $M \in \mathbb{Z}_a^\infty \times \mathbb{R}^n$ is the point $N = (k, u_e, u_L)$, i.e. if $\mathcal{C}^r(M) = N$, then denote $\mathcal{C}_e^r(M) := (k, u_e)$.

Now we will construct a retraction $\mathcal{R}$ which will map the set $\Omega_e(a)$ onto the set $\partial\Omega_e(a)$.

b) The leaving index k^* and its computation.
For the construction of the mapping $\mathcal{R}$, the moment when the graph of the solution leaves the prescribed set Ω will be essential. This moment will be described with help of the so called leaving index.

Choose $u_L^a = (u_{m+1}^a, \dots, u_n^a)$ with $u_i^a \in (b_i(a), c_i(a))$, $i = m+1, \dots, n$ arbitrarily but fixed. Now consider the point $M = (a, u^a) = (a, u_e^a, u_L^a)$ such that

$M_e := (a, u_e^a) \in \overline{\Omega_e(a)}$. The initial condition $u(a) = (u_e(a), u_L(a)) = (u_e^a, u_L^a)$ defines the corresponding solution $u = u(k) = (u_e(k), u_L(k))$ of system (1.1) and as u_L^a is fixed, the solution is now given just by the choice of u_e^a.

If $M_e \in \partial\Omega_e(a)$, then (due to the strict egress property)

$$\mathcal{C}_e(M) \notin \overline{\Omega_e(a+1)} \quad \text{and hence} \quad \mathcal{C}(M) \notin \overline{\Omega(a+1)}.$$

If $M_e \in \Omega_e(a)$, then from the supposition that no initial data from $\Omega(a)$ give a solution whose graph would stay in Ω we conclude that there exists an integer $s > 0$ such that

$$\mathcal{C}^r(M) \in \Omega(a+r), \quad r = 0, \dots, s-1$$

and

$$\mathcal{C}^s(M) \notin \Omega(a+s),$$

i.e. there exists an $i \in \{1, \dots, n\}$ such that the inequalities

$$b_i(a+s) < u_i(a+s) < c_i(a+s), \tag{3.3}$$

do not hold.

As the set Ω is of Liapunov type with respect to the variables with indices from I_L and $\mathcal{C}^{s-1}(M) \in \Omega(a+s-1)$, the validity of (3.3) cannot be violated for any index $i \in I_L$. Thus, it has to be for some $i \in I_e$ and hence

$$\mathcal{C}_e^s(M) \notin \Omega_e(a+s).$$

Moreover, if $\mathcal{C}_e^s(M) \in \partial\Omega_e(a+s)$ then we have (again due to the strict egress property) $\mathcal{C}_e^{s+1}(M) \notin \overline{\Omega_e(a+s+1)}$.

Define now the *leaving index* k^* as

$$k^* = \begin{cases} s-1 & \text{if } M_e \in \Omega_e(a) \text{ and } \mathcal{C}_e^s(M) \notin \overline{\Omega_e(a+s)}, \\ s & \text{if } M_e \in \Omega_e(a) \text{ and } \mathcal{C}_e^s(M) \in \partial\Omega_e(a+s), \\ 0 & \text{if } M_e \in \partial\Omega_e(a). \end{cases}$$

The value k^* characterizes the last moment for which the solution stays in the set $\overline{\Omega}$. Altogether, we have

$$\mathcal{C}_e^{k^*}(M) \in \overline{\Omega_e(a+k^*)} \quad \text{and} \quad \mathcal{C}_e^{k^*+1}(M) \notin \overline{\Omega_e(a+k^*+1)}.$$

As the value of k^* depends on the chosen initial point M, we could write $k^* = k^*(M)$ but we will mostly omit the argument M, unless it is necessary.

c) Connecting set and connecting function.
To construct the retraction $\mathcal{R}$, we will introduce a "connection" of the sets $\Omega_e(k)$, $k \in \mathbb{Z}_a^\infty$.

First extend the functions b_i, c_i, $i = 1, \dots, m$ onto the whole interval $[a, \infty)$:

$$\begin{aligned} b_i(t) &:= b_i([t]) + (b_i([t]+1) - b_i([t])) \cdot (t - [t]), \\ c_i(t) &:= c_i([t]) + (c_i([t]+1) - c_i([t])) \cdot (t - [t]), \end{aligned}$$

$[t]$ being the integer part of t. Note that b_i, c_i are now piecewise linear continuous functions of a real variable $t \in [a, \infty)$ such that $b_i(t) < c_i(t)$ for every t and that the original values of $b_i(k), c_i(k)$ for $k \in \mathbb{Z}_a^\infty$ are preserved.

The set connecting all the sets $\Omega_e(k)$, $k \in \mathbb{Z}_a^\infty$ can now be defined as

$$V_{a,\infty} := \{(t, u_e)\colon a \le t < \infty, b_i(t) \le u_i \le c_i(t), i = 1, \dots, m\}.$$

Further, define the u-boundary of this set as

$$\partial_u V_{a,\infty} := \partial V_{a,\infty} \setminus \{(a, u_e)\colon b_i(a) < u_i < c_i(a), i = 1 \dots, m\},$$

$\partial V_{a,\infty}$ being the "classical" boundary of the set $V_{a,\infty}$ in the space $\mathbb{R} \times \mathbb{R}^m$.

It is easy to see that the set $V_{a,\infty}$ can be written as

$$V_{a,\infty} = \bigcup_{k \in \mathbb{Z}_a^\infty} V_{k,k+1}$$

with

$$V_{k,k+1} := \{(t, u_e)\colon k \le t \le k+1, b_i(t) \le u_i \le c_i(t), i = 1, \dots, m\}.$$

For our further considerations, it is vital to notice that every set $V_{k,k+1}$ is convex. This is because the functions $b_i(t)$ and $c_i(t)$ are linear on each interval $[k, k+1]$ and thus the set $V_{k,k+1}$ is an intersection of half-spaces.
As for the boundary of $V_{k,k+1}$, it can be decomposed to two parts, the u-boundary and the t-boundary, where

$$\begin{aligned} \partial_u V_{k,k+1} &:= \partial_u V_{a,\infty} \cap V_{k,k+1}, \\ \partial_t V_{k,k+1} &:= \{(t, u_e)\colon t = k \text{ or } t = k+1, b_i(t) < u_i < c_i(t), i = 1, \dots, m\}. \end{aligned}$$

Now we will find a continuous function $\mathcal{V}\colon [a, \infty) \times \mathbb{R}^m \to \mathbb{R}$ with help of which the set $V_{a,\infty}$ can be described as

$$V_{a,\infty} = \{(t, u_e)\colon a \le t < \infty, \mathcal{V}(t, u_e) \le 0\} \tag{3.4}$$

and its u-boundary as

$$\partial_u V_{a,\infty} = \{(t, u_e)\colon a \le t < \infty, \mathcal{V}(t, u_e) = 0\}.$$

For finding such a function, notice that the cut of the set $V_{a,\infty}$ through any hyperplane $t = t^*$, $t^* \in [a, \infty)$, is the m-dimensional interval

$$\{(t^*, u_e)\colon b_i(t^*) \le u_i \le c_i(t^*), i = 1, \dots, m\}. \tag{3.5}$$

Now recall the trivial fact that the inequality

$$\max_{i=1,\dots,m} |x_i| \le 1$$

defines the m-dimensional interval $\{x \in \mathbb{R}^m\colon -1 \le x_i \le 1, i = 1, \dots, m\}$.
Using a simple transformation, we get that the interval (3.5) can be described as the set of all points (t^*, u_e) for which the inequality

$$\max_{i=1,\dots,m} \left(\left| u_i - \frac{b_i(t^*) + c_i(t^*)}{2} \right| \Big/ \left(\frac{c_i(t^*) - b_i(t^*)}{2} \right) \right) \le 1$$

holds. The function $\mathcal{V}$ from (3.4) is therefore (after a small simplification)

$$\mathcal{V}(t, u_e) = \max_{i=1,\dots,m} \frac{|2u_i - b_i(t) - c_i(t)|}{c_i(t) - b_i(t)} - 1. \tag{3.6}$$

This function is continuous, because a maximum of continuous functions is continuous, too.

The function $\mathcal{V}$ defined by (3.6) will be called the *connecting function* for the sets $\Omega_e(k)$, $k \in \mathbb{Z}_a^\infty$.

d) Auxiliary mapping $\boldsymbol{P}$ and its continuity.
Define the value of the mapping $P\colon \overline{\Omega_e(a)} \to \partial_u V_{a,\infty}$ for the point $M_e \in \overline{\Omega_e(a)}$ as the intersection of the line segment with the end points $\mathcal{C}_e^{k^*}(M)$ and $\mathcal{C}_e^{k^*+1}(M)$ with $\partial_u V_{a+k^*,a+k^*+1}$ (see Figure 1).

Prove that the mapping P is well defined on $\overline{\Omega_e(a)}$. We have to consider two cases: either $\mathcal{C}_e^{k^*}(M) \in \partial\Omega_e(a + k^*)$ or $\mathcal{C}_e^{k^*}(M) \in \Omega_e(a + k^*)$.

If $\mathcal{C}_e^{k^*}(M) \in \partial\Omega_e(a + k^*)$, then for some $i \in \{1, \dots, m\}$, the u_i-coordinate of $\mathcal{C}_e^{k^*}(M)$ equals to $c_i(a + k^*)$ or to $b_i(a + k^*)$. Then, by the strict egress property, the u_i-coordinate of $\mathcal{C}^{k^*+1}(M)$ is greater than $c_i(a + k^* + 1)$ or less then $b_i(a + k^* + 1)$, respectively. Thus any point (t, u_e) of the line segment connecting $\mathcal{C}_e^{k^*}(M)$ and $\mathcal{C}_e^{k^*+1}(M)$, except the point $\mathcal{C}_e^{k^*}(M)$ itself, cannot fulfill the condition $u_i \le c_i(t)$ or $u_i \ge b_i(t)$, respectively, and therefore there is just one intersection of this line segment with $\partial_u V_{a+k^*,a+k^*+1}$, namely, the point $\mathcal{C}_e^{k^*}(M)$ itself. This reasoning also shows that $P(M_e) = M_e$ for $M_e \in \partial\Omega_e(a)$.

If $\mathcal{C}_e^{k^*}(M) \in \Omega_e(a+k^*)$, then it lies on the t-boundary of the convex set $V_{a+k^*,\,a+k^*+1}$. In general, for a line segment $\overline{AB}$, where A lies on the boundary of a convex set and B is outside this set, for the intersection of $\overline{AB}$ with the boundary of the set there are three possibilities: it could be one point (the point A itself), two points (A and one more point) or a line segment beginning at A. From the construction of the sets $V_{k,k+1}$ and from the position of the points $\mathcal{C}_e^{k^*}(M)$ and $\mathcal{C}_e^{k^*+1}(M)$ it is clear that here only the second case comes into question and that there is one intersection of the considered line segment with the u-boundary of the set $V_{a+k^*,\,a+k^*+1}$.

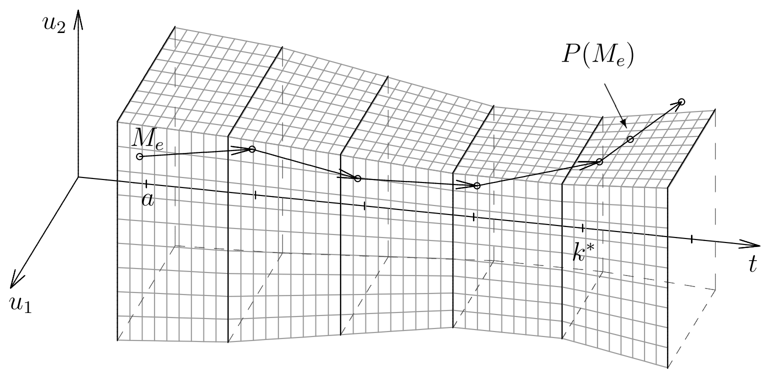

FIGURE 1. Mapping P

Prove that the mapping P is continuous. Let $\{M_j\}_{j=1}^{\infty}$ be any sequence with $M_j = (a, u^a_{e,j}, u^a_L)$ (recall that u^a_L is the fixly chosen part of the initial condition) such that $M_{e,j} := (a, u^a_{e,j}) \in \overline{\Omega_e(a)}$ and $M_{e,j} \to M_e$ (or, equivalently $M_j \to M$) for $j \to \infty$. We will show that $P(M_{e,j}) \to P(M_e)$. Because of the continuity of the mapping F

$$\mathcal{C}^i_e(M_j) \to \mathcal{C}^i_e(M) \quad \text{for any fixed } i \in \mathbb{N}. \tag{3.7}$$

We have to consider two cases:

I) $\mathcal{C}^{k^*}_e(M) \in \Omega_e(a+k^*)$,
II) $\mathcal{C}^{k^*}_e(M) \in \partial\Omega_e(a+k^*)$.

I) In this case also $\mathcal{C}^{k^*}_e(M_j) \in \Omega_e(a+k^*)$ and $\mathcal{C}^{k^*+1}_e(M_j) \notin \overline{\Omega_e(a+k^*+1)}$ for all j sufficiently large. That means that the leaving index $k^*(M_j)$ is the same as k^* given by M and thus we deal with the same set $V_{a+k^*,\,a+k^*+1}$.

Denote $\tilde{\ell}$ the line segment with the end points $\mathcal{C}^{k^*}_e(M)$, $\mathcal{C}^{k^*+1}_e(M)$ and ℓ_j the line segment with the end points $\mathcal{C}^{k^*}_e(M_j)$, $\mathcal{C}^{k^*+1}_e(M_j)$ and consider their parametrizations

$$\tilde{\ell}\colon \begin{cases} t = k^* + s, & s \in [0,1] \\ u_e = \varphi(s) \end{cases} \qquad\qquad \ell_j\colon \begin{cases} t = k^* + s, & s \in [0,1] \\ u_e = \varphi_j(s) \end{cases}$$

where

$$\begin{aligned} \varphi(s) &= \mathcal{C}^{k^*}_e(M) + \big(\mathcal{C}^{k^*+1}_e(M) - \mathcal{C}^{k^*}_e(M)\big)\cdot s, \\ \varphi_j(s) &= \mathcal{C}^{k^*}_e(M_j) + \big(\mathcal{C}^{k^*+1}_e(M_j) - \mathcal{C}^{k^*}_e(M_j)\big)\cdot s. \end{aligned}$$

As for $j \to \infty$ the end points of ℓ_j converge to the end points of $\tilde{\ell}$ and φ and φ_j are linear mappings, it is obvious that

$$\varphi_j(s) \to \varphi(s) \quad \text{for } j \to \infty \text{ for any fixed } s \in [0,1]\,. \tag{3.8}$$

As the u-boundary of the set $V_{a+k^*,a+k^*+1}$ is described by the equation $\mathcal{V}(t,u)=0$ for $t \in [a+k^*, a+k^*+1]$, the value of the parameter s for the point of intersection of the line segment $\tilde{\ell}$ with the boundary of the set $V_{a+k^*,a+k^*+1}$ can be obtained as the solution of the equation

$$\tilde{v}(s)=0, \quad \text{where} \quad \tilde{v}(s) := \mathcal{V}(k^*+s, \varphi(s)),$$

meanwhile for ℓ_j it is the solution of

$$v_j(s)=0, \quad \text{where} \quad v_j(s) := \mathcal{V}(k^*+s, \varphi_j(s)).$$

From the above considerations it follows that each of the equations has just one solution on the interval $[0,1]$. Denote these solutions $\tilde{s}$ (for the equation $\tilde{v}(s)=0$) and s_j (for $v_j(s)=0$). Remark that $\tilde{v}(s)<0$ for $0 \le s < \tilde{s}$ because the function $\mathcal{V}$ is negative for the inner points of $V_{a,\infty}$ and the beginning point of the line segment $\tilde{\ell}$, i.e. $\mathcal{C}_e^{k^*}(M)$, lies inside the set $V_{a,\infty}$. Further, $\tilde{v}(s)>0$ for $\tilde{s} < s \le 1$ because $\mathcal{V}$ is positive for the outer points $V_{a,\infty}$ and the end point of $\tilde{\ell}$, i.e. $\mathcal{C}_e^{k^*+1}(M)$, lies outside $V_{a,\infty}$. A similar thing holds for the functions v_j.

Show that $s_j \to \tilde{s}$ for $j \to \infty$. As the function $\mathcal{V}$ is continuous and because of (3.8), we can see that

$$v_j(s) \to \tilde{v}(s) \quad \text{for any fixed } s \in [0,1]\,.$$

Choose $\varepsilon > 0$ arbitrarily small. Then $\tilde{v}(\tilde{s}-\varepsilon) < 0$ and $\tilde{v}(\tilde{s}+\varepsilon) > 0$. Thus, for sufficiently large j, $v_j(\tilde{s}-\varepsilon) < 0$ and $v_j(\tilde{s}+\varepsilon) > 0$ and hence $\tilde{s}-\varepsilon < s_j < \tilde{s}+\varepsilon$. That gives $|s_j - \tilde{s}| < \varepsilon$ for j sufficiently large, i.e. $s_j \to \tilde{s}$. From this and from (3.8) we get that $P(M_{e,j}) \to P(M_e)$ for $j \to \infty$.

II) In this case there can be $\mathcal{C}_e^{k^*}(M_j) \in \Omega_e(a+k^*)$ for some members of the sequence $\{M_j\}$ and $\mathcal{C}_e^{k^*}(M_j) \notin \Omega_e(a+k^*)$ for the others. In general there could be three subsequences $\{M_{k_j}\}$, $\{M_{l_j}\}$ and $\{M_{m_j}\}$ such that

$$\mathcal{C}_e^{k^*}(M_{k_j}) \in \Omega_e(a+k^*), \quad \mathcal{C}_e^{k^*}(M_{l_j}) \in \partial\Omega_e(a+k^*) \qquad \text{and} \quad \mathcal{C}_e^{k^*}(M_{m_j}) \notin \overline{\Omega_e(a+k^*)}.$$

If $k^*>0$, we have to consider all three subsequences. If $k^*=0$, the subsequence $\{M_{m_j}\}$ can be left out from our considerations because $\mathcal{C}_e^{0}(M_{m_j}) \notin \overline{\Omega_e(a)}$ would mean that $M_{e,m_j} \notin \overline{\Omega_e(a)}$. As the sequence $\{M_j\}_{j=1}^{\infty}$ consist just of points from $\overline{\Omega(a)}$, this cannot happen.

For the subsequence $\{M_{k_j}\}$ the situation is very similar to that in the case I). Now the root of the equation $\tilde{v}(s)=0$ is equal to zero (the intersection of $\tilde{\ell}$ and $\partial_u V_{a+k^*,a+k^*+1}$ is the left end point of $\tilde{\ell}$) and we have to prove that $s_{k_j} \to 0$. Again, choose $\varepsilon > 0$ arbitrarily small. Then $\tilde{v}(\varepsilon) > 0$ and thus $v_{k_j}(\varepsilon) > 0$ for j sufficiently large. Further, as the left end point of ℓ_{k_j} lies inside $\Omega_e(a+k^*)$, $v_{k_j}(0)<0$ and thus $0<s_{k_j}<\varepsilon$, i.e. $s_{k_j}\to 0$.

For the subsequence $\{M_{l_j}\}$ there is no problem, too. If $\mathcal{C}_e^{k^*}(M_{l_j})$ belongs to $\partial_u V_{a+k^*,a+k^*+1}$, then $P(M_{e,l_j})=\mathcal{C}_e^{k^*}(M_{l_j})$ and $P(M_e)=\mathcal{C}_e^{k^*}(M)$ and hence (because of (3.7))

$$P(M_{e,l_j})\to P(M_e).$$

As for the subsequence $\{M_{m_j}\}$, the leaving index $k^*(M_{m_j})$ is different from k^* given by M because $\mathcal{C}_e^{k^*}(M_{m_j})$ is already out of $\overline{\Omega_e(a+k^*)}$. For j sufficiently large,

$$k^*(M_{m_j})=k^*-1$$

because $\mathcal{C}_e^{k^*-1}(M)\in\Omega_e(a+k^*-1)$ and thus also $\mathcal{C}_e^{k^*-1}(M_{m_j})\in\Omega_e(a+k^*-1)$. Now, $P(M_e)$ can be also seen as the point of intersection of the line segment given by the end points $\mathcal{C}_e^{k^*-1}(M)$, $\mathcal{C}_e^{k^*}(M)$ with $\partial_u V_{a+k^*-1,a+k^*}$. The desired convergence $P(M_{e,m_j})\to P(M_e)$ can be proved with help of similar considerations as in the case of the subsequence $\{M_{k_j}\}$.

We have shown that $P(M_{e,k_j})\to P(M_e)$, $P(M_{e,l_j})\to P(M_e)$ and also $P(M_{e,m_j})\to P(M_e)$, and thus $P(M_{e,j})\to P(M_e)$.

e) Auxiliary mapping Q.

Define an auxiliary mapping $\tilde{Q}\colon [a,\infty)\times\mathbb{R}^m\to S_e(a)$, where $S_e(a)$ is the space specified with $k=a$ by (3.2), as $\tilde{Q}\colon (t,u_1,\dots,u_m)\mapsto(a,y_1,\dots,y_m)$ with

$$y_i=b_i(a)+\frac{c_i(a)-b_i(a)}{c_i(t)-b_i(t)}\,(u_i-b_i(t)),\quad i=1,\dots,m.$$

This mapping is obviously continuous. Consider a point $N=(t,u_1,\dots,u_m)$ such that $N\in\partial_u V_{a,\infty}$. For such point $b_i(t)\le u_i\le c_i(t)$ for $i=1,\dots,m$ and $u_j=b_j(t)$ or $u_j=c_j(t)$ for some $j\in\{1,\dots,m\}$. That gives that for the point $\tilde{Q}(N)=(a,y_1,\dots,y_m)$, $b_i(a)\le y_i\le c_i(a)$ for $i=1,\dots,m$ and $y_j=b_j(a)$ or $y_j=c_j(a)$, respectively, which means that $\tilde{Q}(N)\in\partial\Omega_e(a)$.

If $N\in\partial\Omega_e(a)$, i.e. $t=a$, then $y_i=u_i$ for $i=1,\dots,m$, i.e. $\tilde{Q}(N)=N$.

Thus if we define the mapping Q as the restriction of $\tilde{Q}$ to the set $\partial_u V_{a,\infty}$, then Q is a retraction of the set $\partial_u V_{a,\infty}$ onto the set $\partial\Omega_e(a)$.

f) The resulting retraction $\mathcal{R}$.

In the previous parts of the proof, we have constructed a continuous mapping $P\colon\Omega_e(a)\to\partial_u V_{a,\infty}$ such that $P(M_e)=M_e$ for $M_e\in\partial\Omega_e(a)$. Further, we have defined a retraction $Q\colon\partial_u V_{a,\infty}\to\partial\Omega_e(a)$.

Finally, define the mapping $\mathcal{R}$ as the composition of mappings P and Q, $\mathcal{R}:=Q\circ P$. Mapping $\mathcal{R}$ is a retraction of the set $\overline{\Omega_e(a)}$ onto the set $\partial\Omega_e(a)$. As such a retraction cannot exist, we have come to a contradiction and the supposition that there is no solution satisfying (3.1) cannot hold. $\square$

Example 3.2. Consider the nonlinear system

$$
\begin{aligned}
\Delta u_1(k) &= F_1(k,u_1(k),u_2(k),u_3(k)) &:=& -\frac{3}{2}\,k+\sqrt{u_1^2(k)+\frac{1}{16}\,u_3^2(k)},\\
\Delta u_2(k) &= F_2(k,u_1(k),u_2(k),u_3(k)) &:=& \frac{1}{k}\sqrt{u_1^2(k)+\frac{1}{3}\,u_2^2(k)},\\
\Delta u_3(k) &= F_3(k,u_1(k),u_2(k),u_3(k)) &:=& \frac{1}{k}\sqrt{u_2^2(k)+\frac{1}{4}\,u_3^2(k)}
\end{aligned}
\tag{3.9}
$$

with $k\in\mathbb{Z}_4^\infty$. We prove that system (3.9) has a solution $u=u^*(k)=(u_1^*(k),u_2^*(k),u_3^*(k))$ such that for every $k\in\mathbb{Z}_4^\infty$

$$
k<u_1^*(k)<2k,\quad k<u_2^*(k)<3k,\quad k<u_3^*(k)<4k. \tag{3.10}
$$

Let us explain the trick how these bounds for the solution were determined: We suppose that there exists a solution $u=u(k)$ of system (3.9) such that the asymptotical representations of its coordinates (for $k\to\infty$) are

$$
u_i(k)\sim a_ik
$$

where a_i, $i=1,2,3$, are suitable coefficients. Moreover, for such functions we suppose $\Delta u_i(k)\sim a_i$, $i=1,2,3$. Then, substituting all these representations into (3.9), we get

$$
a_1\sim-\frac{3}{2}\,k+\sqrt{a_1^2k^2+\frac{1}{16}\,a_3^2k^2}, \tag{3.11}
$$

$$
a_2\sim\frac{1}{k}\sqrt{a_1^2k^2+\frac{1}{3}\,a_2^2k^2}, \tag{3.12}
$$

$$
a_3\sim\frac{1}{k}\sqrt{a_2^2k^2+\frac{1}{4}\,a_3^2k^2}. \tag{3.13}
$$

From relations (3.12) and (3.13) we can easily see that

$$
a_2^2\sim\frac{3}{2}\,a_1^2,\quad a_3^2\sim\frac{4}{3}\,a_2^2, \tag{3.14}
$$

and thus $a_3^2\sim 2a_1^2$. Substituting the last equivalence into (3.11), we get

$$
a_1\sim-\frac{3}{2}\,k+\sqrt{a_1^2k^2+\frac{1}{8}a_1^2k^2}
$$

and, after simplification,

$$
a_1\sim-\frac{3}{2}\,k+a_1k\sqrt{\frac{9}{8}}
$$

which leads to

$$
a_1\sim\frac{3k/2}{k\,\sqrt{9/8}-1}\sim\frac{3/2}{(3\sqrt{2})/4}=\sqrt{2}.
$$

Using (3.14), we get

$$
a_2\sim\sqrt{3},\quad a_3\sim 2.
$$

So we can expect that there exists a solution $u = (u_1(k), u_2(k), u_3(k))$ which is asymptotically equivalent to $(k\sqrt{2}, k\sqrt{3}, 2k)$. We finish our explanation with a recommendation that the functions $b_i(k)$, $i = 1,2,3$ should be less than the corresponding coordinates of the last vector and the functions $c_i(k)$, $i = 1,2,3$ should be greater. According to it, we will examine, e.g., the functions (compare (3.10)):

$$b_i(k) := k,\ i = 1,2,3, \quad c_1(k) := 2k, \quad c_2(k) := 3k, \quad c_3(k) := 4k.$$

These functions were chosen so that $b_i(k) < a_i k < c_i(k)$ for $i = 1,2,3$ and so that all the conditions of Theorem 3.1 were satisfied (a bit of guessing and trying was needed).

Now prove that all the conditions of Theorem 3.1 with $I_e = \{1\}$ and $I_L = \{2,3\}$, functions $b_i(k)$ and $c_i(k)$ defined by (3.14), and the sets $\Omega(k)$, $k \in \mathbb{Z}_4^\infty$, defined by

$$\Omega(k) := \{(k, u_1, u_2, u_3)\colon k < u_1 < 2k, k < u_2 < 3k, k < u_3 < 4k\}.$$

are fulfilled.

The functions $F_i(k, u_1, u_2, u_3)$, $i = 1,2,3$ are obviously continuous with respect to u_1, u_2 and u_3. Show that all the points of the sets Ω_B^1 and Ω_C^1 are points of strict egress. Indeed, if

$$u_1 = b_1(k) = k \quad \text{and} \quad k < u_2 < 3k, \quad k < u_3 < 4k, \tag{3.15}$$

then inequality (2.3) with $j = 1$ is satisfied. For the left-hand side of inequality (2.3), under the assumptions (3.15), the following estimate holds:

$$\begin{aligned} F_1(k, u_1, u_2, u_3) = -\frac{3}{2}k + \sqrt{k^2 + \frac{1}{16}u_3^2} < -\frac{3}{2}k + \sqrt{k^2 + \frac{1}{16}16k^2} \\ = k\left(-\frac{3}{2} + \sqrt{2}\right) < 0, \end{aligned}$$

meanwhile the right-hand side is

$$b_1(k+1) - b_1(k) = k + 1 - k = 1$$

and hence (2.3) is fulfilled.

Similarly, for

$$u_1 = c_1(k) = 2k \quad \text{and} \quad k < u_2 < 3k, \quad k < u_3 < 4k,$$

the left hand-side of inequality (2.4) (again for $j = 1$) is

$$\begin{aligned} F_1(k, u_1, u_2, u_3) = -\frac{3}{2}k + \sqrt{4k^2 + \frac{1}{16}u_3^2} > -\frac{3}{2}k + \sqrt{4k^2 + \frac{1}{16}k^2} \\ = k\left(-\frac{3}{2} + \sqrt{\frac{65}{16}}\right), \end{aligned}$$

the right-hand side is

$$c_1(k+1) - c_1(k) = 2k + 2 - 2k = 2$$

and as $\left(-3/2 + \sqrt{65/16}\right) \cdot k > 2$ for $k > 3$, inequality (2.4) holds for $k \in \mathbb{Z}_4^\infty$.

Now verify that the set Ω is of the Liapunov type with respect to the second and third variable, i.e. that for every (k, u_1, u_2, u_3) such that $k < u_1 < 2k$, $k < u_2 < 3k$, $k < u_3 < 4k$ inequalities (2.2) with $i = 2, 3$ are fulfilled.

Let us find the upper and lower estimate for the expression $u_2 + F_2(k, u_1, u_2, u_3)$:

$$u_2 + F_2(k, u_1, u_2, u_3) = u_2 + \frac{1}{k}\sqrt{u_1^2 + \frac{1}{3}\,u_2^2} < 3k + \frac{1}{k}\sqrt{4k^2 + \frac{1}{3}\,9k^2} = 3k + \sqrt{7}$$

and

$$u_2 + F_2(k, u_1, u_2, u_3) = u_2 + \frac{1}{k}\sqrt{u_1^2 + \frac{1}{3}\,u_2^2} > k + \frac{1}{k}\sqrt{k^2 + \frac{1}{3}\,k^2} = k + \sqrt{\frac{4}{3}}\,.$$

Altogether we have

$$k + \sqrt{4/3} < u_2 + F_2(k, u_1, u_2, u_3) < 3k + \sqrt{7}$$

and since

$$b_2(k+1) = k + 1 < k + \sqrt{4/3} \quad \text{and} \quad 3k + \sqrt{7} < 3k + 3 = c_2(k+1) \quad \text{for } k \in \mathbb{N},$$

inequalities (2.2) with $i = 2$ hold for $k \in \mathbb{Z}_4^\infty$.

Now do the same for $i = 3$:

$$u_3 + F_3(k, u_1, u_2, u_3) = u_3 + \frac{1}{k}\sqrt{u_2^2 + \frac{1}{4}\,u_3^2} < 4k + \frac{1}{k}\sqrt{9k^2 + \frac{1}{4}\,16k^2} = 4k + \sqrt{13}$$

and

$$u_3 + F_3(k, u_1, u_2, u_3) = u_3 + \frac{1}{k}\sqrt{u_2^2 + \frac{1}{4}\,u_3^2} > k + \frac{1}{k}\sqrt{k^2 + \frac{1}{4}\,k^2} = k + \sqrt{\frac{5}{4}}\,.$$

Inequalities (2.2) with $i = 3$ hold because

$$b_3(k+1) = k + 1 < k + \sqrt{5/4} \quad \text{and} \quad 4k + \sqrt{13} < 4k + 4 = c_3(k+1) \quad \text{for } k \in \mathbb{N}.$$

Thus all the conditions of Theorem 3.1 are fulfilled and there exists a solution of system (3.9) for which inequalities (3.10) hold.

4. ASYMPTOTIC BEHAVIOR OF SOLUTIONS OF A LINEAR NONHOMOGENEOUS DISCRETE SYSTEM

The main goal of our investigation here will be to establish the asymptotic formulae for the behaviour of a solution $u = (u_1, \ldots, u_n)^{\mathrm{T}}$ of a linear nonhomogeneous system of discrete equations. Suppose that the considered system has the form

$$\Delta u(k) = A(k)u(k) + g(k), \quad k \in \mathbb{Z}_a^\infty \tag{4.1}$$

where $A(k) = (a_{ij}(k))_{i,j=1}^n$ is an $n \times n$ real matrix, $a_{ij} \colon \mathbb{Z}_a^\infty \to \mathbb{R}$, $i, j = 1, \dots, n$ and $g(k) = (g_1(k), \dots, g_n(k))^{\mathrm{T}}$, $g \colon \mathbb{Z}_a^\infty \to \mathbb{R}^n$.

In this case there is a way how to find the bounding functions $b_i(k)$ and $c_i(k)$ from Theorem 3.1:

In the following we suppose $\det A(k) \neq 0$ for every $k \in \mathbb{Z}_a^\infty$. Let us define the vector

$$\omega(k) = -A^{-1}(k)g(k), \quad k \in \mathbb{Z}_a^\infty \tag{4.2}$$

where $A^{-1}(k)$ is the inverse of matrix $A(k)$. This means that the vector $\omega(k)$ is the unique solution of the system of linear equations $A(k)\omega(k) + g(k) = \theta$ ($\theta = (0, \dots, 0)^{\mathrm{T}}$), i.e. it is the root of the right-hand side of system (4.1). We will suppose that there exists a solution $u(k)$ of system (4.1) which is close to ω, say

$$u(k) = \omega(k) + f(k), \quad k \in \mathbb{Z}_a^\infty, \tag{4.3}$$

where $f(k) = (f_1(k), \dots, f_n(k))^{\mathrm{T}}$ is a "perturbation" of $\omega(k)$. Now we will try to find an approximate expression of $f(k)$. Substituting (4.3) for $u(k)$ into (4.1), we get

$$\Delta(\omega(k) + f(k)) = A(k)(\omega(k) + f(k)) + g(k). \tag{4.4}$$

Considering the definition of ω, i.e. (4.2), equation (4.4) can be rewritten as

$$\Delta\omega(k) + \Delta f(k) = A(k)f(k).$$

We expect $\Delta f(k)$ to be "small" compared to the terms $A(k)f(k)$ and $\Delta\omega(k)$ and we will omit it. Thus we get

$$\Delta\omega(k) \sim A(k)f(k)$$

as $k \to \infty$. Hence, we will define the vector $f(k)$, $k \in \mathbb{Z}_a^\infty$, as

$$f(k) = A^{-1}(k)\Delta\omega(k). \tag{4.5}$$

Now we will show that under certain conditions, there exists a solution $u(k)$ of system (4.1) which is bounded by the functions $\omega(k)$ and $\omega(k) + f(k)$.

Theorem 4.1. *Let us suppose that for every $k \in \mathbb{Z}_a^\infty$:*

1) $\det A(k) \neq 0$.
2) $f_i(k) > 0$, $\Delta f_i(k) \leq 0$, *and* $\Delta\omega_i(k) \geq 0$ *for* $i = 1, \dots, p$, $0 \leq p \leq n$.
3) $f_i(k) < 0$, $\Delta f_i(k) \leq 0$, *and* $\Delta\omega_i(k) \geq 0$, *for* $i = p+1, \dots, n$.
4) $a_{ij}(k) \leq 0$ *for* $i = 1, \dots, n$, $j = 1, \dots, p$ *and* $i \neq j$.
5) $a_{ij}(k) \geq 0$ *for* $i = 1, \dots, n$, $j = p+1, \dots, n$ *and* $i \neq j$.
6) $\sum_{j=1, j\neq i}^n |a_{ij}(k)| > 0$ *for every* $i = 1, \dots, n$.
7) $f_i(k) + a_{ii}(k)f_i(k) - \Delta\omega_i(k) \leq 0$ *and* $a_{ii}(k)f_i(k) + f_i(k+1) \leq 0$ *for* $i = p+1, \dots, n$

Then there exists a solution $u = u^*(k)$, $k \in \mathbb{Z}_a^\infty$, of (4.1) *such that*

$$\begin{aligned} &\omega_i(k) < u_i^*(k) < \omega_i(k) + f_i(k), \quad && i = 1, \dots, p, \\ &\omega_i(k) + f_i(k) < u_i^*(k) < \omega_i(k), && i = p+1, \dots, n. \end{aligned} \tag{4.6}$$

Proof. We will apply Theorem 3.1 with $I_e = \{1, \dots, p\}$, $I_L = \{p+1, \dots, n\}$,

$$F_i(k, u) = \sum_{j=1}^n a_{ij}(k)u_j + g_i(k), \quad i = 1, \dots, n,$$

$$\begin{aligned} &b_i(k) = \omega_i(k), && c_i(k) = \omega_i(k) + f_i(k), \quad && i = 1, \dots, p, \\ &b_i(k) = \omega_i(k) + f_i(k), \quad && c_i(k) = \omega_i(k), && i = p+1, \dots, n. \end{aligned}$$

First, let us point out some facts that will be useful in our next considerations.

Due to (4.2), the functions F_i can be rewritten as

$$F_i(k, u) = \sum_{j=1}^n a_{ij}(k)u_j - \sum_{j=1}^n a_{ij}(k)\omega_j(k) = \sum_{j=1}^n a_{ij}(k)(u_j - \omega_j(k)),$$

$i = 1, \dots, n$, $k \in \mathbb{Z}_a^\infty$. Further, due to assumptions 4), 5) and 6) of the Theorem, if

$$b_j(k) < u_j < c_j(k), \quad j = 1, \dots, n,$$

then

$$\sum_{j=1, j\neq i}^n a_{ij}(k)(u_j - \omega_j(k)) < 0, \quad i = 1, \dots, n \tag{4.7}$$

and

$$\sum_{j=1, j\neq i}^n a_{ij}(k)(u_j - \omega_j(k) - f_j(k)) > 0, \quad i = 1, \dots, n. \tag{4.8}$$

Now we will verify that all the assumptions of Theorem 3.1 are satisfied. Start with proving that all the points of the sets Ω_B^i and Ω_C^i, $i = 1, \dots, p$, are points of strict egress.

Due to (2.3), we have to show that if $u_i = b_i(k) = \omega_i(k)$ and $b_j(k) < u_j < c_j(k)$ for every $j = 1, \dots, n$, $j \neq i$, then

$$\sum_{j=1}^n a_{ij}(k)(u_j - \omega_j(k)) < b_i(k+1) - b_i(k) = \Delta\omega_i(k).$$

This inequality is equivalent to the inequality

$$\sum_{j=1}^n a_{ij}(k)(u_j - \omega_j(k)) - \Delta\omega_i(k) < 0$$

which can be rewritten as

$$\sum_{j=1, j\neq i}^n a_{ij}(k)(u_j - \omega_j(k)) + a_{ii}(k)(\omega_i(k) - \omega_i(k)) - \Delta\omega_i(k) < 0.$$

But this certainly holds because of (4.7) and the assumption that $\Delta\omega_i(k) \geq 0$ (see assumption 2) of the Theorem).

Similarly, due to (2.4), we have to prove that if $u_i = c_i(k) = \omega_i(k) + f_i(k)$ and $b_j(k) < u_j < c_j(k)$ for every $j = 1, \dots, n$, $j \neq i$, then

$$\sum_{j=1}^{n} a_{ij}(k)(u_j - \omega_j(k)) > c_i(k+1) - c_i(k) = \Delta\omega_i(k) + \Delta f_i(k) \tag{4.9}$$

which is equivalent to

$$\sum_{j=1}^{n} a_{ij}(k)(u_j - \omega_j(k)) - \Delta\omega_i(k) - \Delta f_i(k) > 0.$$

The left-hand side can be estimated as

$$\sum_{j=1}^{n} a_{ij}(k)(u_j - \omega_j(k)) - \Delta\omega_i(k) - \Delta f_i(k) > \sum_{j=1}^{n} a_{ij}(k) f_j(k) - \Delta\omega_i(k) - \Delta f_i(k).$$

Thanks to (4.5), $\sum_{j=1}^{n} a_{ij}(k) f_j(k) = \Delta\omega_i(k)$, and thus

$$\sum_{j=1}^{n} a_{ij}(k)(u_j - \omega_j(k)) - \Delta\omega_i(k) - \Delta f_i(k) > -\Delta f_i(k).$$

As $\Delta f_i(k) \geq 0$ due to assumption 2) of the Theorem, inequality (4.9) holds.

Now let us prove that the set Ω is of Liapunov type with respect to the i-th variable for every $i \in \{p+1, \dots, n\}$. According to (2.2), we have to show that if $b_j(k) < u_j < c_j(k)$ for $j = 1, \dots, n$, then for $i = p+1, \dots, n$

$$b_i(k+1) < u_i + F_i(k, u) < c_i(k+1). \tag{4.10}$$

To verify this property, we will show that inequalities (4.10) hold for both $u_i = b_i(k)$ and $u_i = c_i(k)$ and that the function

$$G(u_i) = u_i + F_i(k, u_1, \dots, u_{i-1}, u_i, u_{i+1}, \dots, u_n)$$

is monotone for every fixed $k \in \mathbb{Z}_a^\infty$ and every fixed $u_1, \dots, u_{i-1}, u_{i+1}, \dots, u_n$ with $b_j(k) < u_j < c_j(k)$, $j \neq i$. Then inequalities (4.10) have to hold for any u_i between $b_i(k)$ and $c_i(k)$.

Start with $u_i = b_i(k) = \omega_i(k) + f_i(k)$. First we will show that $u_i + F_i(k, u) < c_i(k+1)$ which gives

$$\omega_i(k) + f_i(k) + \sum_{j=1}^{n} a_{ij}(k)(u_j - \omega_j(k)) - \omega_i(k+1) < 0. \tag{4.11}$$

We will rearrange the left-hand side of inequality (4.11) and then, using (4.7), find its upper estimate:

$$\omega_i(k) + f_i(k) + \sum_{j=1}^{n} a_{ij}(k)(u_j - \omega_j(k)) - \omega_i(k+1)$$
$$= f_i(k) + a_{ii}(k)(\omega_i(k) + f_i(k) - \omega_i(k)) + \sum_{j=1, j\neq i}^{n} a_{ij}(k)(u_j - \omega_j(k)) - \Delta\omega_i(k)$$
$$< f_i(k) + a_{ii}(k) f_i(k) - \Delta\omega_i(k).$$

Due to assumption 7), we have

$$u_i + F(k, u_i) - c_i(k+1) < f_i(k) + a_{ii}(k) f_i(k) - \Delta\omega_i(k) \le 0.$$

Thus inequality (4.11) holds.

Now let us show that for $u_i = b_i(k)$ also $u_i + F_i(k, u) > b_i(k+1)$. That gives

$$\omega_i(k) + f_i(k) + \sum_{j=1}^{n} a_{ij}(k)(u_j - \omega_j(k)) - \omega_i(k+1) - f_i(k+1) > 0. \tag{4.12}$$

We will proceed analogously as before. This time we will use relations (4.5), (4.8) and the fact that $\Delta f_i(k) \le 0$.

$$\omega_i(k) + f_i(k) + \sum_{j=1}^{n} a_{ij}(k)(u_j - \omega_j(k)) - \omega_i(k+1) - f_i(k+1)$$
$$= \sum_{j=1}^{n} a_{ij}(k)(u_j - \omega_j(k)) - \Delta\omega_i(k) - \Delta f_i(k) = \sum_{j=1}^{n} a_{ij}(k)(u_j - \omega_j(k) - f_j(k))$$
$$- \Delta f_i(k) = a_{ii}(k)(\omega_i(k) + f_i(k) - \omega_i(k) - f_i(k))$$
$$+ \sum_{j=1, j\neq i}^{n} a_{ij}(k)(u_j - \omega_j(k) - f_j(k)) - \Delta f_i(k) > 0.$$

That proves the validity of (4.12).

Now we will investigate the upper end point, $u_i = c_i(k) = \omega_i(k)$. Inequality $u_i + F_i(k, u) < c_i(k+1)$ is here equivalent with the inequality

$$\omega_i(k) + \sum_{j=1}^{n} a_{ij}(k)(u_j - \omega_j(k)) - \omega_i(k+1) < 0. \tag{4.13}$$

Using (4.7) and the fact that $\Delta\omega_i(k) \ge 0$, we get

$$\omega_i(k) + \sum_{j=1}^{n} a_{ij}(k)(u_j - \omega_j(k)) - \omega_i(k+1) = a_{ii}(k)(\omega_i(k) - \omega_i(k))$$
$$+ \sum_{j=1, j\neq i}^{n} a_{ij}(k)(u_j - \omega_j(k)) - \Delta\omega_i(k) < -\Delta\omega_i(k) \le 0$$

and (4.13) is proved.

Finally, let us show that for $u_i = c_i(k)$ also $u_i + F_i(k,u) > b_i(k+1)$. That is equivalent with

$$\omega_i(k) + \sum_{j=1}^{n} a_{ij}(k)(u_j - \omega_j(k)) - \omega_i(k+1) - f_i(k+1) > 0. \tag{4.14}$$

Using (4.5) and (4.8), we get

$$\begin{aligned}
&\omega_i(k) + \sum_{j=1}^{n} a_{ij}(k)(u_j - \omega_j(k)) - \omega_i(k+1) - f_i(k+1) \\
&\qquad = \sum_{j=1}^{n} a_{ij}(k)(u_j - \omega_j(k)) - \Delta\omega_i(k) - f_i(k+1) \\
&= \sum_{j=1}^{n} a_{ij}(k)(u_j - \omega_j(k) - f_j(k)) - f_i(k+1) = a_{ii}(k)(\omega_i(k) - \omega_i(k) - f_i(k)) \\
&\qquad + \sum_{j=1, j\neq i}^{n} a_{ij}(k)(u_j - \omega_j(k) - f_j(k)) - f_i(k+1) > -a_{ii}(k) f_i(k) - f_i(k+1).
\end{aligned}$$

Looking at assumption 7) of the Theorem, we can see that inequality (4.14) is fulfilled.

Noticing that the function

$$\begin{aligned}
G(u_i) = u_i + \sum_{j=1}^{n} a_{ij}(k)(u_j - \omega_j(k)) &= (1 + a_{ii}(k))u_i - a_{ii}(k)\omega_i(k) \\
&\qquad + \sum_{j=1, j\neq i}^{n} a_{ij}(k)(u_j - \omega_j(k))
\end{aligned}$$

is a linear function of its argument u_i, we can straightly say that this function is monotone for any fixly chosen $k \in \mathbb{Z}_a^\infty$ and $u_1, \dots, u_{i-1}, u_{i+1}, \dots, u_n$.

All conditions of Theorem 3.1 are satisfied. From its conclusion it follows that there exists at least one solution $u = u^*(k)$ satisfying the inequalities (4.6). This completes the proof. $\square$

Example 4.2. Let us consider the linear discrete system of equations

$$\begin{aligned}
\Delta u_1(k) &= k u_1(k) + \frac{1}{\sqrt{k}} u_2(k) - k^2 - 1, \\
\Delta u_2(k) &= -k u_1(k) - \frac{2}{\sqrt{k}} u_2(k) + k^2 + 2.
\end{aligned} \tag{4.15}$$

We will show that for $k \in \mathbb{Z}_4^\infty$, all the assumptions of Theorem 4.1 are fulfilled with $p = 1$. In this case

$$A = \begin{pmatrix} k & 1/\sqrt{k} \\ -k & -2/\sqrt{k} \end{pmatrix}, \quad g = \begin{pmatrix} -k^2 - 1 \\ k^2 + 2 \end{pmatrix}.$$

After a simple calculation, we get

$$\det A(k) = -\sqrt{k}, \quad A^{-1}(k) = \begin{pmatrix} 2/k & 1/k \\ -\sqrt{k} & -\sqrt{k} \end{pmatrix},$$

$$\begin{pmatrix} \omega_1(k) \\ \omega_2(k) \end{pmatrix} = -A^{-1}(k)g(k) = \begin{pmatrix} k \\ \sqrt{k} \end{pmatrix}.$$

Further,

$$\Delta\omega_1(k) = 1, \quad \Delta\omega_2(k) = \sqrt{k+1} - \sqrt{k},$$

$$\begin{pmatrix} f_1(k) \\ f_2(k) \end{pmatrix} = A^{-1}(k)\Delta\omega(k) = \begin{pmatrix} 2/k & 1/k \\ -\sqrt{k} & -\sqrt{k} \end{pmatrix} \begin{pmatrix} 1 \\ \sqrt{k+1} - \sqrt{k} \end{pmatrix}$$
$$= \begin{pmatrix} (2 + \sqrt{k+1} - \sqrt{k})/k \\ -\sqrt{k}(1 + \sqrt{k+1} - \sqrt{k}) \end{pmatrix}.$$

Assumptions 1), 4), 5) and 6) are obviously fulfilled.

As for assumption 2), it is clear that $f_1(k) > 0$ and $\Delta\omega_1(k) > 0$. Remind that $\sqrt{k+1} - \sqrt{k}$ monotonely tends to 0 for $k \to \infty$. Thanks to this, the function in the numerator of $f_1(k)$ is decreasing and thus also $f_1(k)$ is decreasing. This fact guarantees that $\Delta f_1(k) < 0$.

Conditions $f_2(k) < 0$ and $\Delta\omega_2(k) \geq 0$ from assumption 3) hold for every $k \in \mathbb{N}$. Calculating $\Delta f_2(k)$, after some simplifications we get

$$\Delta f_2(k) = 1 - \sqrt{k+1}(\sqrt{k+2} - \sqrt{k}) - \sqrt{k+1} + \sqrt{k} < 0.$$

It remains to verify the validity of assumption 7): $f_i(k) + a_{ii}(k)f_i(k) - \Delta\omega_i(k) \leq 0$ with $i = 2$ gives

$$f_2(k)(1 + a_{22}(k)) - \Delta\omega_2(k)$$
$$= -\sqrt{k}(1 + \sqrt{k+1} - \sqrt{k})(1 - 2/\sqrt{k}) - (\sqrt{k+1} - \sqrt{k}) \leq 0.$$

This inequality holds for $k \geq 4$ because for such k, $1 - 2/\sqrt{k} \geq 0$.

The last desired inequality is $a_{22}(k)f_2(k) + f_2(k+1) \leq 0$ which gives

$$(-2/\sqrt{k})(-\sqrt{k})(1 + \sqrt{k+1} - \sqrt{k}) - \sqrt{k+1}(1 + \sqrt{k+2} - \sqrt{k+1}) \leq 0.$$

Simplifying the left-hand side, we get

$$2(1 + \sqrt{k+1} - \sqrt{k}) - \sqrt{k+1}(1 + \sqrt{k+2} - \sqrt{k+1}) \leq 0$$

which holds for $k \geq 4$.

All the assumptions of Theorem 4.1 are fulfilled and thus there exists a solution $u = u^*(k)$, $k \in \mathbb{Z}_4^\infty$, of system (4.15) such that $b_i(k) < u_i^*(k) < c_i(k)$, $i = 1, 2$. In our case, $b_2(k) = \omega_2(k) + f_2(k) = \sqrt{k} - \sqrt{k}(1 + \sqrt{k+1} - \sqrt{k}) = -\sqrt{k}(\sqrt{k+1} - \sqrt{k})$. That means that the solution $u = u^*(k)$ satisfies the inequalities

$$k < u_1^*(k) < k + (2 + \sqrt{k+1} - \sqrt{k})/k,$$
$$-\sqrt{k}(\sqrt{k+1} - \sqrt{k}) < u_2^*(k) < \sqrt{k}.$$

5. SYSTEM OF BERNOULLI'S TYPE

In this section, the result of Theorem 3.1 is applied to the investigation of the asymptotic behavior of solutions of the system

$$\begin{aligned} \Delta u_1(k) &= u_2(k)\left(-\gamma_1(k)u_1(k) + \beta_1(k)\right), \\ \Delta u_2(k) &= u_1(k)\left(\ \ \gamma_2(k)u_2(k) - \beta_2(k)\right) \end{aligned} \tag{5.1}$$

where $\gamma_i, \beta_i \colon \mathbb{Z}_a^\infty \to \mathbb{R}^+ := (0, \infty)$, $i = 1, 2$. This system is similar to the scalar equation

$$\Delta u(k) = u(k)\left(\gamma(k)u(k) - \beta(k)\right)$$

which is called the equation of Bernoulli's type or (with regard to terminology used in [1]) Verhulst's equation with opposite coefficients.

For system (5.1), again we can find the bounding functions $b_i(k)$ and $c_i(k)$ from Theorem 3.1. We will construct them in a similar way as in the case of the linear system (4.1):

Suppose that there exists a solution $u(k)$ of system (5.1) which is "close" to the unique root $\omega(k) = (\omega_1(k), \omega_2(k))^T$ of the system

$$-\gamma_1(k)\omega_1(k) + \beta_1(k) = 0, \tag{5.2}$$

$$\gamma_2(k)\omega_2(k) - \beta_2(k) = 0 \tag{5.3}$$

derived from the right-hand side of (5.1). In our case

$$\omega_i(k) = \frac{\beta_i(k)}{\gamma_i(k)}, \quad i = 1, 2. \tag{5.4}$$

It means

$$u(k) = \omega(k) + f(k), \quad k \in \mathbb{Z}_a^\infty, \tag{5.5}$$

where $f(k) = (f_1(k), f_2(k))^{\mathrm{T}}$ is a "perturbation" of ω. Substituting (5.5) for $u(k)$ into (5.1) and using (5.2), (5.3), we get

$$\Delta(\omega_1(k) + f_1(k)) = (\omega_2(k) + f_2(k))\left(-\gamma_1(k)f_1(k)\right), \tag{5.6}$$

$$\Delta(\omega_2(k) + f_2(k)) = (\omega_1(k) + f_1(k))\gamma_2(k)f_2(k). \tag{5.7}$$

Supposing that $\Delta f_i(k)$, $i = 1, 2$, are "small" compared to the remaining terms in (5.6), (5.7), we assume that

$$\Delta\omega_1(k) \sim -\omega_2(k)\gamma_1(k)f_1(k),$$
$$\Delta\omega_2(k) \sim \quad \omega_1(k)\gamma_2(k)f_2(k)$$

if $k \to \infty$.

Hence, we will define $f_1(k)$ and $f_2(k)$, $k \in \mathbb{Z}_a^\infty$, as

$$f_1(k) = -\frac{\Delta\omega_1(k)}{\omega_2(k)\gamma_1(k)}\,, \quad f_2(k) = \frac{\Delta\omega_2(k)}{\omega_1(k)\gamma_2(k)}\,.$$

Now we will show that under certain conditions, there exists a solution $u(k)$ of system (5.1) which is bounded by the coordinates of the functions $\omega(k)$ and $\omega(k) + f(k)$.

Theorem 5.1. *Let the functions $\gamma_i, \beta_i \colon \mathbb{Z}_a^\infty \to \mathbb{R}^+$, $i = 1, 2$, be given.*

Suppose that for every $k \in \mathbb{Z}_a^\infty$ the following assumptions hold:

1) $\Delta\omega_1(k) < 0$.
2) $\Delta\omega_2(k) > 0$.
3) $\Delta\omega_1(k) + f_1(k+1) > 0$.
4) $f_1(k) + f_2(k)(\Delta\omega_1(k))/\omega_2(k) > 0$.
5) $\Delta f_1(k) > 0$.
6) $\Delta f_2(k) < 0$.

Then the system of difference equations (5.1) *has a solution $u = u^*(k) = (u_1^*(k), u_2^*(k))$ such that*

$$\omega_i(k) < u_i^*(k) < \omega_i(k) + f_i(k) \tag{5.8}$$

for $i = 1, 2$ and $k \in \mathbb{Z}_a^\infty$.

Proof. We will apply Theorem 3.1 with $I_e = \{2\}$ and $I_L = \{1\}$,

$$F_1(k, u_1, u_2) = u_2 \cdot (-\gamma_1(k)u_1 + \beta_1(k)), \quad F_2(k, u_1, u_2) = u_1 \cdot (\gamma_2(k)u_2 - \beta_2(k)), \tag{5.9}$$

$$b_i(k) = \omega_i(k), \quad c_i(k) = \omega_i(k) + f_i(k), \quad i = 1, 2, \tag{5.10}$$

$$\Omega(k) = \{(k, u_1, u_2) \colon \omega_i(k) < u_i < \omega_i(k) + f_i(k), i = 1, 2\}.$$

Remark that due to assumptions 1) and 2), $f_i(k) > 0$ for $k \in \mathbb{Z}_a^\infty$, $i = 1, 2$.

The functions F_1 and F_2 are obviously continuous with respect to their arguments u_1 and u_2.

Let us prove that the set Ω is of Liapunov type with respect to the first variable and to the system (5.1). Suppose that

$$b_1(k) < u_1 < c_1(k) \quad \text{and} \quad b_2(k) < u_2 < c_2(k).$$

As the function

$$G(u_1) := u_1 + F_1(k, u_1, u_2) = u_1 + u_2 \cdot (-\gamma_1(k)u_1 + \beta_1(k))$$

is linear with respect to its argument u_1 for every fixed $k \in \mathbb{Z}_a^\infty$ and every fixed u_2 such that $b_2(k) < u_2 < c_2(k)$, to prove inequalities (2.2), it will be sufficient to show that these inequalities are fulfilled for the end points of the interval $(b_1(k), c_1(k))$, i.e. that

$$b_1(k+1) < b_1(k) + F_1(k, b_1(k), u_2) < c_1(k+1) \tag{5.11}$$

and

$$b_1(k+1) < c_1(k) + F_1(k, c_1(k), u_2) < c_1(k+1) \tag{5.12}$$

for any $k \in \mathbb{Z}_a^\infty$ and $b_2(k) < u_2 < c_2(k)$.

Start with inequalities (5.11). Substituting for $b_1(k)$ and then for $\omega_1(k)$, we get

$$b_1(k) + F_1(k, b_1(k), u_2) = \omega_1(k) + u_2 \cdot (-\gamma_1(k)\omega_1(k) + \beta_1(k)) = \omega_1(k).$$

Thus, inequalities (5.11) reduce to

$$\omega_1(k+1) < \omega_1(k) < \omega_1(k+1) + f_1(k+1).$$

The first inequality ($\omega_1(k+1) < \omega_1(k)$) is fulfilled due to assumption 1). The second inequality is equivalent to the inequality

$$\Delta\omega_1(k) + f_1(k+1) > 0$$

which is supposed to be valid in assumption 3). Thus, inequalities (5.11) hold.

Now let us concentrate on inequalities (5.12). Again, substituting the appropriate values, we get

$$\begin{aligned} c_1(k) + F_1(k, c_1(k), u_2) &= \omega_1(k) + f_1(k) + u_2 \cdot (-\gamma_1(k)(\omega_1(k) + f_1(k)) + \beta_1(k)) \\ &= \omega_1(k) + f_1(k) - u_2\gamma_1(k)f_1(k) = \omega_1(k) + f_1(k) + u_2\frac{\Delta\omega_1(k)}{\omega_2(k)}. \end{aligned}$$

We have to prove that

$$\omega_1(k+1) < \omega_1(k) + f_1(k) + u_2\frac{\Delta\omega_1(k)}{\omega_2(k)} < \omega_1(k+1) + f_1(k+1). \tag{5.13}$$

Find the lower and the upper estimate of $\omega_1(k) + f_1(k) + u_2(\Delta\omega_1(k))/\omega_2(k)$. In the following considerations we will use the fact that $\Delta\omega_1(k) < 0$ (see assumption 1)).

$$\begin{aligned} &\omega_1(k) + f_1(k) + u_2\frac{\Delta\omega_1(k)}{\omega_2(k)} > \omega_1(k) + f_1(k) + (\omega_2(k) + f_2(k))\frac{\Delta\omega_1(k)}{\omega_2(k)} \\ &\quad = \omega_1(k) + f_1(k) + \Delta\omega_1(k) + f_2(k)\frac{\Delta\omega_1(k)}{\omega_2(k)} = \omega_1(k+1) + f_1(k) + f_2(k)\frac{\Delta\omega_1(k)}{\omega_2(k)}, \end{aligned}$$

$$\begin{aligned}\omega_1(k)+f_1(k)+u_2\frac{\Delta\omega_1(k)}{\omega_2(k)} &< \omega_1(k)+f_1(k)+\omega_2(k)\frac{\Delta\omega_1(k)}{\omega_2(k)}\\ &= \omega_1(k)+f_1(k)+\Delta\omega_1(k)=\omega_1(k+1)+f_1(k).\end{aligned}$$

To prove the first inequality of (5.13), it is sufficient to prove that

$$\omega_1(k+1)<\omega_1(k+1)+f_1(k)+f_2(k)\frac{\Delta\omega_1(k)}{\omega_2(k)}$$

which gives

$$0<f_1(k)+f_2(k)\frac{\Delta\omega_1(k)}{\omega_2(k)}.$$

This inequality is fulfilled due to assumption 4) of the Theorem. As for the second inequality from (5.13), it is sufficient to show that

$$\omega_1(k+1)+f_1(k)<\omega_1(k+1)+f_1(k+1).$$

This holds because of assumption 5). That means that inequalities (5.12) hold. Altogether we have shown that the set Ω is of Liapunov type with respect to the first variable and to system (5.1).

Now prove that all the points of the sets

$$\begin{aligned}\Omega_B^2 &= \{(k,u_1,b_2(k))\colon k\in\mathbb{Z}_a^\infty, b_1(k)\le u_1\le c_1(k)\},\\ \Omega_C^2 &= \{(k,u_1,c_2(k))\colon k\in\mathbb{Z}_a^\infty, b_1(k)\le u_1\le c_1(k)\}\end{aligned}$$

are points of strict egress.

Due to (2.3), we have to show that

$$F_2(k,u_1,b_2(k))<b_2(k+1)-b_2(k)$$

which, after substitution for F_2 and b_2 from (5.9) and (5.10), becomes

$$u_1\cdot(\gamma_2(k)\omega_2(k)-\beta_2(k))<\omega_2(k+1)-\omega_2(k). \tag{5.14}$$

From the definition of $\omega_2(k)$ (see (5.4)), we get

$$u_1\cdot(\gamma_2(k)\omega_2(k)-\beta_2(k))=u_1\cdot\left(\gamma_2(k)\frac{\beta_2(k)}{\gamma_2(k)}-\beta_2(k)\right)=0$$

and thus (5.14) reduces to

$$0<\omega_2(k+1)-\omega_2(k)=\Delta\omega_2(k)$$

which holds thanks to assumption 2).

Finally, due to (2.4), prove that

$$F_2(k,u_1,c_2(k))>c_2(k+1)-c_2(k).$$

After substitution, we get

$$u_1\cdot(\gamma_2(k)(\omega_2(k)+f_2(k))-\beta_2(k))>\omega_2(k+1)+f_2(k+1)-(\omega_2(k)+f_2(k)),$$

which can be simplified to

$$u_1\gamma_2(k)f_2(k) > \Delta\omega_2(k) + \Delta f_2(k). \tag{5.15}$$

The minimum possible value of the left-hand side of (5.15) is

$$\omega_1(k)\gamma_2(k)f_2(k) = \omega_1(k)\gamma_2(k)\,\frac{\Delta\omega_2(k)}{\omega_1(k)\gamma_2(k)} = \Delta\omega_2(k).$$

To prove inequality (5.15), it is enough to show that

$$\Delta\omega_2(k) > \Delta\omega_2(k) + \Delta f_2(k)$$

but this certainly holds because of assumption 6).

We have shown that all the assumptions of Theorem 3.1 are fulfilled and thus there exists a solution of system (5.1) satisfying conditions (5.8). □

Example 5.2. Let us consider the system of equations

$$\begin{aligned} \Delta u_1(k) &= u_2(k)\left(-\frac{1}{k^4}\,u_1(k) + \frac{1}{k^5}\right), \\ \Delta u_2(k) &= u_1(k)(k^3u_2(k) - k^4). \end{aligned} \tag{5.16}$$

We will show that for $k \in \mathbb{Z}_2^\infty$, all the assumptions of Theorem 5.1 are fulfilled. In this case

$$\omega_1(k) = \frac{1}{k}, \quad \omega_2(k) = k,$$

$$\Delta\omega_1(k) = \frac{1}{k+1} - \frac{1}{k} = -\frac{1}{k(k+1)}, \quad \Delta\omega_2(k) = (k+1) - k = 1,$$

and

$$f_1(k) = -\frac{-1/(k(k+1))}{k\cdot 1/k^4} = \frac{k^2}{k+1}, \quad f_2(k) = \frac{1}{(1/k)\cdot k^3} = \frac{1}{k^2}.$$

Assumptions 1) and 2) of Theorem 5.1 are obviously fulfilled.

Prove the validity of assumption 3):

$$\Delta\omega_1(k) + f_1(k+1) = -\frac{1}{k(k+1)} + \frac{(k+1)^2}{k+2} = \frac{k^4+3k^3+3k^2-2}{k(k+1)(k+2)}.$$

The last expression is positive for any $k \in \mathbb{Z}_1^\infty$ and thus assumption 3) holds.

The desired inequality in assumption 4) is in our case

$$\begin{aligned} f_1(k) + f_2(k)\frac{\Delta\omega_1(k)}{\omega_2(k)} = \frac{k^2}{k+1} + \frac{1}{k^2}\cdot\frac{-1/(k(k+1))}{k} &= \frac{k^2}{k+1} - \frac{1}{k^4(k+1)} \\ &= \frac{k^6-1}{k^4(k+1)} > 0. \end{aligned}$$

This holds for any $k \in \mathbb{Z}_2^\infty$.

As for assumption 5), we get

$$\Delta f_1(k) = \frac{(k+1)^2}{k+2} - \frac{k^2}{k+1} = \frac{k^2+3k+1}{(k+1)(k+2)} > 0$$

for any $k \in \mathbb{Z}_1^\infty$.

Finally, assumption 6) is fulfilled, too, because the function $f_2(k) = 1/k^2$ is decreasing for $k \in \mathbb{Z}_1^\infty$ and thus $\Delta f_2(k) < 0$.

All the assumptions of Theorem 5.1 are fulfilled and thus there exists a solution $u = u^*(k)$, $k \in \mathbb{Z}_2^\infty$, of system (5.16) that satisfies the conditions

$$\begin{aligned}
\frac{1}{k} &< u_1^*(k) < \frac{1}{k} + \frac{k^2}{k+1}, \\
k &< u_2^*(k) < k + \frac{1}{k^2}.
\end{aligned}$$

6. CONCLUDING REMARKS

The previous results of the first author introduce or apply just one of the in the introduction mentioned techniques (retract or Liapunov). E.g., in papers [6] and [9], the retract type technique is used, meanwhile in [8], the author uses the Liapunov type technique. In [6], the typical assumptions are inequalities of the type (2.3) and (2.4). For [8], inequalities of the type (2.2) are typical, but there they are applied to all the equations of system (1.1).

Combining both methods brings a nontrivial generalization of the cited results. The presented "hybrid" method involves both the single methods as its special cases. If we choose $I_e = \emptyset$ in Theorem 3.1, we get the purely Liapunov case, i.e.the method described in [8], and for the choice $I_e = \{1, \dots, n\}$ we get the purely retract type case, i.e.the method from [9]. The introduced connection of two various methods enables us to apply the result of Theorem 3.1 to discrete systems which could not be treated with help of any of the single methods. As an example let us mention the discrete analogue of the so called Emden-Fowler differential equation which is studied in [11].

Finally, let us underline the flexibility of the new method. In the formulation of the results, the role of the functions b_i and c_i is vital. The suitable choice of them (which often depends upon our resourcefulness – see comments after Theorem 3.1) often allows us to prove a result that could be anticipated by intuition. Sometimes even the best possible and sharp results can be obtained this way. What is meant by the "best possible result" will be illustrated on the following example. Simultaneously we underline that the severity of conditions of our approach is not very restrictive and is adequate to the obtained result due to comparisons with known results given below.

Example 6.1. Let us consider the second order difference equation

$$\Delta v(k+1) = -p(k)v(k). \tag{6.1}$$

In applications, the existence of positive solutions of equations describing various phenomena is discussed very often. The following result on the existence of a positive solution of (6.1) is useful and interesting in its own right.

Theorem 6.2. *If there exists a* $\theta \in [0,1)$ *such that*

$$0 < p(k) \le \frac{1}{4} + \frac{\theta}{16k^2}, \quad k \in \mathbb{Z}_1^\infty, \tag{6.2}$$

then there exists a solution $v = v^*(k)$ *of* (6.1) *such that for* k *sufficiently large,*

$$0 < v^*(k) < \sqrt{k}\left(\frac{1}{2}\right)^k, \tag{6.3}$$

i.e. there exists a solution of equation (6.1) *which is positive for* k *sufficiently large.*

Proof. Using substitutions $u_1(k) := v(k)$, $u_2(k) := v(k+1)$, equation (6.1) can be rewritten as a system of two first order equations

$$\Delta u_1(k) = \quad - u_1(k) + u_2(k), \tag{6.4}$$

$$\Delta u_2(k) = - p(k)u_1(k). \tag{6.5}$$

We will show that if there exists a $\theta \in [0,1)$ such that (6.2) holds then there exists a solution $u = (u_1^*(k), u_2^*(k))$ of (6.4), (6.5) such that

$$0 < u_1^*(k) < \sqrt{k}\left(\frac{1}{2}\right)^k, \tag{6.6}$$

$$0 < u_2^*(k) < \sqrt{k+1}\left(\frac{1}{2}\right)^{k+1} \tag{6.7}$$

for k sufficiently large.

That means that a solution $v = v^*(k)$ of the original equation (6.1) exists such that for k sufficiently large (6.3) holds. It means that a positive solution of equation (6.1) exists for k sufficiently large.

To prove this assertion, we will apply Theorem 3.1 with

$$\begin{aligned}
&I_e = \{2\}, I_L = \{1\},\\
&F_1(k, u_1, u_2) := u_2 - u_1, F_2(k, u_1, u_2) := -p(k)u_1,\\
&b_i(k) := 0, \ i = 1, 2,\\
&c_1(k) := \sqrt{k}\left(\frac{1}{2}\right)^k, \ c_2(k) := \sqrt{k+1}\left(\frac{1}{2}\right)^{k+1}.
\end{aligned}$$

The proof of inequalities (2.2) with $j = 1$ and (2.3) with $j = 2$ is quite obvious and we will omit it. Due to (2.4), we have to show that if

$$u_2(k) = \sqrt{k+1}\left(\frac{1}{2}\right)^{k+1}, \ 0 < u_1(k) < \sqrt{k}\left(\frac{1}{2}\right)^k,$$

then

$$-p(k)u_1 > \sqrt{k+2}\left(\frac{1}{2}\right)^{k+2} - \sqrt{k+1}\left(\frac{1}{2}\right)^{k+1},$$

which is equivalent to the inequality

$$\sqrt{k+1}\left(\frac{1}{2}\right)^{k+1} - \sqrt{k+2}\left(\frac{1}{2}\right)^{k+2} - p(k)u_1 > 0. \tag{6.8}$$

We will estimate the left-hand side of (6.8) with help of the second-degree Maclaurin polynomial of the function $\sqrt{1+x}$, i.e.

$$\sqrt{1+x} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + O(x^3).$$

Then

$$\begin{aligned}
\sqrt{k+1}\left(\frac{1}{2}\right)^{k+1} - \sqrt{k+2}\left(\frac{1}{2}\right)^{k+2} - p(k)u_1 > {} & \sqrt{k+1}\left(\frac{1}{2}\right)^{k+1} \\
& - \sqrt{k+2}\left(\frac{1}{2}\right)^{k+2} - \left(\frac{1}{4} + \frac{\theta}{16k^2}\right)\sqrt{k}\left(\frac{1}{2}\right)^{k} \\
= & \sqrt{k}\left(\frac{1}{2}\right)^{k} \cdot \left(\frac{1}{2}\sqrt{1+\frac{1}{k}} - \frac{1}{4}\sqrt{1+\frac{2}{k}} - \frac{1}{4} - \frac{\theta}{16k^2}\right) \\
= & \sqrt{k}\left(\frac{1}{2}\right)^{k} \cdot \left[\frac{1}{2}\left(1 + \frac{1}{2k} - \frac{1}{8k^2} + O\left(\frac{1}{k^3}\right)\right)\right. \\
& \left. - \frac{1}{4}\left(1 + \frac{2}{2k} - \frac{4}{8k^2} + O\left(\frac{1}{k^3}\right)\right) - \frac{1}{4} - \frac{\theta}{16k^2}\right] \\
= & \sqrt{k}\left(\frac{1}{2}\right)^{k}\left(\frac{1-\theta}{16k^2} + O\left(\frac{1}{k^3}\right)\right) > 0
\end{aligned}$$

for k sufficiently large. Thus, for such k, inequality (6.8) holds and all the assumptions of Theorem 3.1 are fulfilled. Hence, there exists a solution of system (6.4), (6.5) satisfying (6.6), (6.7) which implies the existence of a positive solution of (6.1). □

Now let us investigate the simplest case of equation (6.1), namely $p(k) \equiv p$, where $p > 0$ is a constant. In this case condition (6.2) reduces to

$$0 < p \le 1/4.$$

It can be shown easily (see e.g. [14, Theorem 7.7] and [15, Theorem 7.5.1]) that if $p > 1/4$, then every solution of equation (6.1) oscillates, i.e. there exists no positive solution of (6.1). This means that, as for the bounds b_i and c_i, $i = 1, 2$, we got the best possible result.

Moreover it follows from [13, Corollary 4.7] that all solutions of (6.1) are oscillatory (for $k \to \infty$) if there exists a $\theta^* > 1$ such that

$$p(k) \ge \frac{1}{4} + \frac{\theta^*}{16k^2} \tag{6.9}$$

for all sufficiently large k. Comparing inequalities (6.2) and (6.9) we conclude that they are almost opposite (values $\theta = 1$ or $\theta^* = 1$ are not involved). So, we state again that the choice of bounds b_i and c_i, $i = 1, 2$ led to the best possible and sharp result (in terms of inequalities for the coefficient p of equation (6.1)).

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