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EXISTENCE OF SOLUTIONS OF A NONLINEAR INTEGRAL EQUATION ON AN UNBOUNDED INTERVAL

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ABSTRACT. In this paper we investigate a nonlinear integral equation of Volterra type on an unbounded interval. We show that under some assumptions our equation has solutions belonging to the space of bounded and continuous functions on $\mathbb{R}_+$. The main tool used in our study is the technique associated with the measures of noncompactness.

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1. INTRODUCTION

It is well known that integral equations have many useful applications in describing numerous events and problems of the real world (see, for example, [1, 6, 7, 8, 9]).

The purpose of this paper is to consider the existence of solutions for the following nonlinear integral equation of Volterra type

$$x(t) = (T_1x)(t) + (T_2x)(t) \int_0^t u(t, s, x(s))ds, \quad t \geq 0, \tag{1.1}$$

where T_1, T_2 are given operators on certain space of functions defined on $\mathbb{R}_+$, u is a continuous function while x is an unknown function. We show that under some assumptions Eq. (1.1) has a solution being continuous and bounded on $\mathbb{R}_+$. The result obtained in the paper generalizes several ones obtained earlier [5, 10].

2. PRELIMINARIES

This section is devoted to collect some definitions and auxiliary results which will be used in the sequel.

Assume that $(E, \|\cdot\|)$ is an infinite dimensional Banach space with zero element θ. Denote by $B(x, r)$ the closed ball centered at x and with radius r. The symbol B_r stands for the ball $B(\theta, r)$.

If X is a nonempty subset of E we denote by $\overline{X}$, $ConvX$ the closure and the closed convex closure of X, respectively. The family of all nonempty and bounded

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subsets of E is denoted by $\mathfrak{M}_E$ and by $\mathfrak{N}_E$ its subfamily consisting of all relatively compact sets.

We use the following definition of the concept of measure of noncompactness [4].

Definition 2.1. A mapping $\mu : \mathfrak{M}_E \longrightarrow \mathbb{R}_+$ is said to be a *measure of noncompactness* in the space E if it satisfies the following conditions:

1. The family $\ker \mu = \{X \in \mathfrak{M}_E : \mu(X) = 0\}$ is nonempty and $\ker \mu \subset \mathfrak{N}_E$.
2. $X \subset Y \Rightarrow \mu(X) \leq \mu(Y)$.
3. $\mu(Conv X) = \mu(X)$.
4. $\mu(\overline{X}) = \mu(X)$.
5. $\mu(\lambda X + (1-\lambda)Y) \leq \lambda\mu(X) + (1-\lambda)\mu(Y)$ for $\lambda \in [0, 1]$.
6. If (X_n) is a sequence of closed sets from $\mathfrak{M}_E$ such that $X_{n+1} \subset X_n$ for $n = 1, 2, \ldots$ and if $\lim_{n\to\infty} \mu(X_n) = 0$, then the set $X_\infty = \bigcap_{n=1}^{\infty} X_n$ is nonempty.

The family $\ker \mu$ described in 1 is called *the kernel of the measure of noncompactness* μ.

A measure μ is said to be sublinear if it satisfies the following two conditions:

7. $\mu(\lambda X) = |\lambda|\mu(X)$ for $\lambda \in \mathbb{R}$.
8. $\mu(X + Y) \leq \mu(X) + \mu(Y)$.

Further facts concerning measures of noncompactness and their properties may be found in [4]. For our further purposes we will only need the following fixed point theorem [4].

Theorem 2.2. *Let Ω be a nonempty, bounded, closed and convex subset of the Banach space E and let $F : \Omega \longrightarrow \Omega$ be a continuous operator such that $\mu(FX) \leq k\mu(X)$ for any nonempty subset X of Ω, where $k \in [0, 1)$ is a constant. Then F has a fixed point in the set Ω.*

Remark 2.3. An operator F satisfying the assumptions of Theorem 2.2 is called a Darbo operator with constant k with respect to the measure of noncompactness μ.

Remark 2.4. Under the assumptions of the above theorem it can be shown that the set $\mathrm{Fix}(F)$ of fixed points of F belonging to Ω is a member of the kernel $\ker \mu$.

In the sequel we will work in the Banach space $BC(\mathbb{R}_+)$ consisting of all real functions defined, bounded and continuous on $\mathbb{R}_+$. The space $BC(\mathbb{R}_+)$ is equipped with the standard norm

$$\|x\| = \sup\{|x(t)| : t \geq 0\}.$$

Now we recollect the main facts about some measure of noncompactness in the space $BC(\mathbb{R}_+)$ which will be used in the paper. This measure was introduced in [2].

To do this let us fix a nonempty bounded subset X of $BC(\mathbb{R}_+)$. For $\varepsilon > 0$, $T > 0$ and $x \in X$, denote by $w^T(x,\varepsilon)$ the *modulus of continuity* of the function x on the interval $[0,T]$ defined by the formula

$$w^T(x,\varepsilon) = \sup\{|x(t) - x(s)| : t, s \in [0,T],\ |t-s| \le \varepsilon\}.$$

Further, let us put

$$\begin{aligned} w^T(X,\varepsilon) &= \sup\{w^T(x,\varepsilon) : x \in X\},\\ w_0^T(X) &= \lim_{\varepsilon\to 0} w^T(X,\varepsilon),\\ w_0(X) &= \lim_{T\to\infty} w_0^T(X). \end{aligned}$$

For a fixed number $t \ge 0$ we denote

$$X(t) = \{x(t) : x \in X\}$$

and

$$\operatorname{diam} X(t) = \sup\{|x(t) - y(t)| : x, y \in X\}.$$

Finally, let us define the function μ on the family $\mathfrak{M}_{BC(\mathbb{R}_+)}$ by putting

$$\mu(X) = w_0(X) + \limsup_{t\to\infty} \operatorname{diam} X(t).$$

It can be shown [2] that the function μ is a sublinear measure of noncompactness on the space $BC(\mathbb{R}_+)$. The kernel $\ker\mu$ of this measure contains nonempty and bounded sets X such that functions from X are locally equicontinuous on $\mathbb{R}_+$ and the thickness of the bundle formed by functions from X tends to zero at infinity.

3. MAIN RESULT

In this section, we will investigate the nonlinear integral equation (1.1). Our considerations are placed in the Banach space $BC(\mathbb{R}_+)$ described above.

We will consider Eq. (1.1) under following hypotheses:

(H_1) $T_i : BC(\mathbb{R}_+) \to BC(\mathbb{R}_+)$, $(i = 1,2)$ are continuous and Darbo operators with respect to the measure of noncompactness considered in Section 2 with constants Q_i, $(i = 1,2)$. Moreover, there exist nonnegative constants c_i, d_i, $(i = 1,2)$ such that

$$\|T_1x\| \le c_1 + d_1\|x\| \quad \text{and} \quad \|T_2x\| \le c_2 + d_2\|x\| \quad \text{for} \quad x \in BC(\mathbb{R}_+),$$

(H_2) $u : \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ is a continuous function and there exist continuous functions $a, b, \psi : \mathbb{R}_+ \to \mathbb{R}_+$ with ψ nondecreasing such that

$$|u(t,s,x)| \le a(t)b(s)\psi(|x|), \quad \text{for} \quad t, s \in \mathbb{R}_+ \quad \text{and} \quad x \in \mathbb{R}$$

and

$$\lim_{t\to\infty} a(t)\int_0^t b(s)ds = 0.$$

Remark 3.1. Observe that based on the assumption (H_2) there exists the following finite constant:

$$A = \sup\left\{a(t)\int_0^t b(s)ds : t \geq 0\right\}.$$

(H_3) There exists $r_0 > 0$ such that

$$(c_1 + d_1 r_0) + (c_2 + d_2 r_0)\,\psi(r_0)\,A \leq r_0$$

and, moreover, $Q_1 + Q_2\,\psi(r_0)\,A < 1$.

Now we can formulate our existence result.

Theorem 3.2. *Under assumptions* (H_1)-(H_3) *Eq.* (1.1) *has at least one solution* $x = x(t)$ *which belongs to the space* $BC(\mathbb{R}_+)$.

Proof. Define the operator F on the space $BC(\mathbb{R}_+)$ by putting

$$(Fx)(t) = (T_1x)(t) + (T_2x)(t)\int_0^t u(t,s,x(s))ds, \quad t \geq 0. \tag{3.1}$$

Firstly, we will show that for an arbitrarily fixed $x \in BC(\mathbb{R}_+)$ the function Fx is continuous on $\mathbb{R}_+$.

By (H_1), to do this it is sufficient to show that the function

$$(Bx)(t) = \int_0^t u(t,s,x(s))ds$$

is continuous.

Thus, fix $\varepsilon > 0$, $T > 0$ and $t_1, t_2 \in [0,T]$ such that $|t_1 - t_2| \leq \varepsilon$. Without loss of generality we can assume that $t_1 < t_2$. Then

$$\begin{aligned}
|(Bx)(t_2) - (Bx)(t_1)| &\leq \left|\int_0^{t_2} u(t_2,s,x(s))ds - \int_0^{t_1} u(t_1,s,x(s))ds\right| \\
&\leq \left|\int_0^{t_2} u(t_2,s,x(s))ds - \int_0^{t_1} u(t_2,s,x(s))ds\right| \\
&\quad + \left|\int_0^{t_1} u(t_2,s,x(s))ds - \int_0^{t_1} u(t_1,s,x(s))ds\right| \\
&\leq \int_{t_1}^{t_2} |u(t_2,s,x(s))|ds + \int_0^{t_1} |u(t_2,s,x(s)) - u(t_1,s,x(s))|ds \\
&\leq \int_{t_1}^{t_2} a(t_2)b(s)\psi(|x(s)|)ds + \int_0^{t_1} w_u^T(\varepsilon)ds \\
&\leq \|a_{|[0,T]}\| \cdot \|b_{|[0,T]}\| \cdot \psi(\|x\|)\,(t_2 - t_1) + w_u^T(\varepsilon)\int_0^T ds \\
&\leq \|a_{|[0,T]}\| \cdot \|b_{|[0,T]}\| \cdot \psi(\|x\|)\,\varepsilon + w_u^T(\varepsilon)\,T,
\end{aligned}$$

where

$$\|a_{|[0,T]}\| = \sup\{a(t) : t \in [0,T]\},$$

$$\|b_{|[0,T]}\| = \sup\{b(t) : t \in [0,T]\},$$

$$w_u^T(\varepsilon) = \sup\{|u(t,s,y) - u(t',s,y)| : t,t',s \in [0,T],\ |t-t'| \leq \varepsilon,\ y \in [-\|x\|, \|x\|]\}.$$

Consequently

$$w^T(Bx,\varepsilon) \leq \|a_{|[0,T]}\| \cdot \|b_{|[0,T]}\| \cdot \psi(\|x\|)\,\varepsilon + w_u^T(\varepsilon)\,T.$$

Let us notice that $w_u^T(\varepsilon) \to 0$ as $\varepsilon \to 0$ which is a consequence of the uniform continuity of the function u on the compact $[0,T] \times [0,T] \times [-\|x\|, \|x\|]$. This fact gives us that

$$\lim_{\varepsilon \to 0} w^T(Bx,\varepsilon) = 0$$

and this says us that the function Bx is continuous on the interval $[0,T]$. As T is arbitrary, Bx is continuous on $\mathbb{R}_+$ and this proves that Fx is continuous on $\mathbb{R}_+$.

Moreover, taking into account the assumptions (H_1) and (H_2) we derive the following estimate

$$\begin{aligned}
|(Fx)(t)| &\leq |(T_1x)(t)| + |(T_2x)(t)| \int_0^t |u(t,s,x(s))|ds \\
&\leq \|T_1x\| + \|T_2x\| \int_0^t |u(t,s,x(s))|ds \\
&\leq (c_1 + d_1\|x\|) + (c_2 + d_2\|x\|) \int_0^t a(t)b(s)\psi(|x(s)|)ds \\
&\leq (c_1 + d_1\|x\|) + (c_2 + d_2\|x\|)\,\psi(\|x\|) \int_0^t a(t)b(s)ds \\
&\leq (c_1 + d_1\|x\|) + (c_2 + d_2\|x\|)\,\psi(\|x\|)\,A.
\end{aligned}$$

Thus Fx is bounded on $\mathbb{R}_+$.

Moreover the above estimate gives the following inequality

$$\|Fx\| \leq (c_1 + d_1\|x\|) + (c_2 + d_2\|x\|)\,\psi(\|x\|)\,A.$$

In view of (H_3) we have that $F : B_{r_0} \to B_{r_0}$.

Now, let us take a nonempty subset X of the ball B_{r_0}. Fix $\varepsilon > 0$, $T > 0$ and take $t_1, t_2 \in [0,T]$ such that $|t_1 - t_2| \leq \varepsilon$ with $t_1 < t_2$ and $x \in X$. Then, keeping in mind our assumptions, we can get

$$
\begin{aligned}
&|(Fx)(t_2)-(Fx)(t_1)| \\
&= \Bigg|(T_1x)(t_2)+(T_2x)(t_2)\int_0^{t_2} u(t_2,s,x(s))ds \\
&\quad -(T_1x)(t_1)-(T_2x)(t_1)\int_0^{t_1} u(t_1,s,x(s))ds\Bigg| \\
&\le |(T_1x)(t_2)-(T_1x)(t_1)| \\
&\quad + \left|(T_2x)(t_2)\int_0^{t_2} u(t_2,s,x(s))ds-(T_2x)(t_1)\int_0^{t_2} u(t_2,s,x(s))ds\right| \\
&\quad + \left|(T_2x)(t_1)\int_0^{t_2} u(t_2,s,x(s))ds-(T_2x)(t_1)\int_0^{t_1} u(t_2,s,x(s))ds\right| \\
&\quad + \left|(T_2x)(t_1)\int_0^{t_1} u(t_2,s,x(s))ds-(T_2x)(t_1)\int_0^{t_1} u(t_1,s,x(s))ds\right| \\
&\le w^T(T_1x,\varepsilon)+|(T_2x)(t_2)-(T_2x)(t_1)|\int_0^{t_2}|u(t_2,s,x(s))|ds \\
&\quad +|(T_2x)(t_1)|\int_{t_1}^{t_2}|u(t_2,s,x(s))|ds \\
&\quad +|(T_2x)(t_1)|\int_0^{t_1}|u(t_2,s,x(s))-u(t_1,s,x(s))|ds \\
&\le w^T(T_1x,\varepsilon)+w^T(T_2x,\varepsilon)\,a(t_2)\int_0^{t_2} b(s)\psi(|x(s)|)ds \\
&\quad +(c_2+d_2r_0)\,a(t_2)\int_{t_1}^{t_2} b(s)\psi(|x(s)|)ds+(c_2+d_2r_0)\int_0^{t_1} w^T_{u,r_0}(\varepsilon)ds \\
&\le w^T(T_1x,\varepsilon)+w^T(T_2x,\varepsilon)\,\psi(\|x\|)\,a(t_2)\int_0^{t_2} b(s)ds \\
&\quad +(c_2+d_2r_0)\,\|a_{|[0,T]}\|\cdot\|b_{|[0,T]}\|\cdot\psi(\|x\|)\,(t_2-t_1)+(c_2+d_2r_0)\,w^T_{u,r_0}(\varepsilon)\int_0^{t_1} ds \\
&\le w^T(T_1x,\varepsilon)+w^T(T_2x,\varepsilon)\,\psi(r_0)\,A \\
&\quad +(c_2+d_2r_0)[\|a_{|[0,T]}\|\cdot\|b_{|[0,T]}\|\cdot\psi(r_0)\,\varepsilon+w^T_{u,r_0}(\varepsilon)\,T],
\end{aligned}
$$

where

$$w^T_{u,r_0}(\varepsilon)=\sup\{|u(t,s,y)-u(t',s,y)| : t,t',s\in[0,T],\ y\in[-r_0,r_0],\ |t-t'|\le\varepsilon\}$$

and as u is uniformly continuous on the compact $[0,T]\times[0,T]\times[-r_0,r_0]$ we have that $w^T_{u,r_0}(\varepsilon)\to 0$ as $\varepsilon\to 0$. The last estimate gives us that

$$
\begin{aligned}
w^T(FX,\varepsilon) &\le w^T(T_1X,\varepsilon)+w^T(T_2X,\varepsilon)\,\psi(r_0)\,A \\
&\quad +(c_2+d_2r_0)\,[\|a_{|[0,T]}\|\cdot\|b_{|[0,T]}\|\cdot\psi(r_0)\,\varepsilon+w^T_{u,r_0}(\varepsilon)\,T].
\end{aligned}
$$

Taking limit as $\varepsilon \to 0$ we get

$$w_0^T(FX) \leq w_0^T(T_1X) + w_0^T(T_2X)\,\psi(r_0)\,A$$

and finally passing to limit as $T \to \infty$,

$$w_0(FX) \leq w_0(T_1X) + w_0(T_2X)\,\psi(r_0)\,A. \tag{3.2}$$

Now, for $x, y \in X$ and for a fixed $t \geq 0$, in virtue of our assumptions, we obtain

$$\begin{aligned}
|(Fx)(t) - (Fy)(t)| &\leq |(T_1x)(t) - (T_1y)(t)| \\
&\quad + \left|(T_2x)(t)\int_0^t u(t,s,x(s))ds - (T_2y)(t)\int_0^t u(t,s,y(s))ds\right| \\
&\leq \operatorname{diam}(T_1X)(t) \\
&\quad + \left|(T_2x)(t)\int_0^t u(t,s,x(s))ds - (T_2y)(t)\int_0^t u(t,s,x(s))ds\right| \\
&\quad + \left|(T_2y)(t)\int_0^t u(t,s,x(s))ds - (T_2y)(t)\int_0^t u(t,s,y(s))ds\right| \\
&\leq \operatorname{diam}(T_1X)(t) + |(T_2x)(t) - (T_2y)(t)|\int_0^t |u(t,s,x(s))|ds \\
&\quad + |(T_2y)(t)|\int_0^t |u(t,s,x(s)) - u(t,s,y(s))|ds \\
&\leq \operatorname{diam}(T_1X)(t) + \operatorname{diam}(T_2X)(t)\,\psi(r_0)\,a(t)\int_0^t b(s)ds \\
&\quad + (c_2 + d_2r_0)\left[\int_0^t |u(t,s,x(s))|ds + \int_0^t |u(t,s,y(s))|ds\right] \\
&\leq \operatorname{diam}(T_1X)(t) + \operatorname{diam}(T_2X)(t)\,\psi(r_0)\,a(t)\int_0^t b(s)ds \\
&\quad + (c_2 + d_2r_0)\,2\psi(r_0)\,a(t)\int_0^t b(s)ds.
\end{aligned}$$

Hence we deduce that

$$\begin{aligned}
\operatorname{diam}(FX)(t) &\leq \operatorname{diam}(T_1X)(t) + \operatorname{diam}(T_2X)(t)\,\psi(r_0)\,a(t)\int_0^t b(s)ds \\
&\quad + (c_2 + d_2r_0)\,2\psi(r_0)\,a(t)\int_0^t b(s)ds.
\end{aligned}$$

Now, taking into account our assumptions we get

$$\limsup_{t\to\infty} \operatorname{diam}(FX)(t) \leq \limsup_{t\to\infty} \operatorname{diam}(T_1X)(t). \tag{3.3}$$

Now, linking (3.2) and (3.3) and, keeping in mind the definition of the measure of noncompactness μ in the space $BC(\mathbb{R}_+)$ given in Section 2, we obtain

$$\begin{aligned}\mu(FX) &= w_0(FX) + \limsup_{t\to\infty} \operatorname{diam}(FX)(t)\\ &\le w_0(T_1X) + w_0(T_2X)\,\psi(r_0)\,A + \limsup_{t\to\infty} \operatorname{diam}(T_1X)(t)\\ &= \mu(T_1X) + w_0(T_2X)\,\psi(r_0)\,A.\end{aligned}$$

By (H$_1$), as the operators T_1 and T_2 are Darbo operators with constants Q_1 and Q_2, respectively, we get

$$\begin{aligned}\mu(FX) &\le \mu(T_1X) + w_0(T_2X)\,\psi(r_0)\,A\\ &\le \mu(T_1X) + \mu(T_2X)\,\psi(r_0)\,A\\ &\le Q_1\,\mu(X) + Q_2\,\mu(X)\,\psi(r_0)\,A\\ &= [Q_1 + Q_2\,\psi(r_0)\,A]\,\mu(X).\end{aligned} \tag{3.4}$$

Finally, we will show that F is continuous on the ball B_{r_0}.

In order to do this let us fix $\varepsilon > 0$ and take a sequence $(x_n) \subset B_{r_0}$ and $x \in B_{r_0}$ with $x_n \to x$. Then, for $t \in \mathbb{R}_+$, we get

(3.5)
$$\begin{aligned}|(Fx_n)(t) - (Fx)(t)| &\le |(T_1x_n)(t) - (T_1x)(t)|\\ &\quad + \left|(T_2x_n)(t)\int_0^t u(t,s,x_n(s))ds - (T_2x)(t)\int_0^t u(t,s,x(s))ds\right|\\ &\le |(T_1x_n)(t) - (T_1x)(t)|\\ &\quad + \left|(T_2x_n)(t)\int_0^t u(t,s,x_n(s))ds - (T_2x)(t)\int_0^t u(t,s,x_n(s))ds\right|\\ &\quad + \left|(T_2x)(t)\int_0^t u(t,s,x_n(s)) - (T_2x)(t)\int_0^t u(t,s,x(s))ds\right|\\ &\le |(T_1x_n)(t) - (T_1x)(t)|\\ &\quad + |(T_2x_n)(t) - (T_2x)(t)|\int_0^t |u(t,s,x_n(s))|ds\\ &\quad + |(T_2x)(t)|\int_0^t |u(t,s,x_n(s)) - u(t,s,x(s))|ds.\end{aligned}$$

Next, using assumption (H$_2$), we can choose a number $T > 0$ such that for $t \ge T$, the following inequality holds

$$a(t)\int_0^t b(s)ds < \min\left\{\varepsilon,\ \frac{1}{\psi(r_0)+1},\ \frac{\varepsilon}{3\cdot 2\psi(r_0)\,(c_2 + d_2 r_0)}\right\}.$$

If we put

$$\bar{w}^T_{u,r_0}(\varepsilon) = \sup\{|u(t,s,x) - u(t,s,y)| : t,s \in [0,T],\ x,y \in [-r_0,r_0],\ |x-y| \le \varepsilon\}$$

then the uniform continuity of u on the compact $[0,T]\times[0,T]\times[-r_0,r_0]$ yields that $\lim_{\varepsilon\to 0}\bar{w}^T_{u,r_0}(\varepsilon)=0$ and this means that for $\varepsilon>0$ given there exists $\delta>0$ such that if $0<\delta'<\delta$, then

$$\bar{w}^T_{u,r_0}(\delta')<\frac{\varepsilon}{3(c_2+d_2r_0)\,T}.$$

As $x_n\to x$, for $\delta>0$ there exists $n_1\in\mathbb{N}$ such that $\|x_n-x\|<\delta/2$, for $n\geq n_1$.

Moreover, as T_1 and T_2 are continuous operators there exists $n_2\in\mathbb{N}$ such that for $n\geq n_2$

$$\|T_1x_n-T_1x\|<\frac{\varepsilon}{3},$$
$$\|T_2x_n-T_2x\|<\min\left\{\frac{\varepsilon}{3},\ \frac{\varepsilon}{3\,\|a_{|[0,T]}\|\cdot\|b_{|[0,T]}\|\cdot\psi(r_0)\,T}\right\}.$$

Now we take $n\geq\max\{n_1,n_2\}$ and we consider two cases:

1. If $t\geq T$.
2. If $t<T$.

Case 1. In virtue of (3.5) we get

$$\begin{aligned}|(Fx_n)(t)-(Fx)(t)|&\leq\|T_1x_n-T_1x\|+\|T_2x_n-T_2x\|\cdot\psi(r_0)\,a(t)\int_0^t b(s)ds\\&\quad+(c_2+d_2r_0)\,2\psi(r_0)\,a(t)\int_0^t b(s)ds\\&\leq\frac{\varepsilon}{3}+\frac{\varepsilon}{3}\frac{\psi(r_0)}{\psi(r_0)+1}+\frac{(c_2+d_2r_0)\,2\psi(r_0)\,\varepsilon}{3\cdot 2\,\psi(r_0)\,(c_2+d_2r_0)}\\&\leq\varepsilon.\end{aligned}$$

Case 2. In virtue of (3.5) we get

$$\begin{aligned}|(Fx_n)(t)-(Fx)(t)|&\leq\|T_1x_n-T_1x\|+\|T_2x_n-T_2x\|\cdot\psi(r_0)\cdot\|a_{|[0,T]}\|\cdot\|b_{|[0,T]}\|\,T\\&\quad+(c_2+d_2r_0)\int_0^t w^T_{u,r_0}\left(\frac{\delta}{2}\right)ds\\&\leq\frac{\varepsilon}{3}+\frac{\varepsilon\,\psi(r_0)\cdot\|a_{|[0,T]}\|\cdot\|b_{|[0,T]}\|\,T}{3\,\|a_{|[0,T]}\|\cdot\|b_{|[0,T]}\|\cdot\psi(r_0)\,T}+\frac{\varepsilon\,(c_2+d_2r_0)\,T}{3(c_2+d_2r_0)\,T}\\&\leq\varepsilon.\end{aligned}$$

These facts prove that F is continuous on the ball B_{r_0}.

Finally, taking into account (3.4), assumption (H_3) and Theorem 2.2 we infer that the operator F has at least one fixed point in B_{r_0}. This completes the proof. $\square$

Remark 3.3. Theorem 3.2 can be proved using the Schauder fixed point principle instead Darbo's theorem. In fact, to do this we consider the set B_{r_0} appearing in the proof of Theorem 3.2 and put $B^1_{r_0}=\operatorname{Conv}(F(B_{r_0}))$, $B^2_{r_0}=\operatorname{Conv}(F(B^1_{r_0}))$ and so on. Observe that the sequence of sets $(B^n_{r_0})$ is decreasing i.e. $B^{n+1}_{r_0}\subset B^n_{r_0}$ for $n=1,2,\ldots$.

Moreover the sets of this sequence are closed and convex. From this, in view of (3.4) we get

$$\mu(B^n_{r_0}) \leq q^n \mu(B_{r_0}),$$

where $q = Q_1 + Q_2\,\psi(r_0)\,A$.

Notice that by assumption (H_3) of Theorem 3.2, $q < 1$. From this fact and the last inequality we infer

$$\lim_{n\to\infty} \mu(B^n_{r_0}) = 0.$$

Hence, taking into account the condition 6 of the definition of a measure of noncompactness, we get that the set $Y = \bigcap_{n=1}^{\infty}(B^n_{r_0})$ is nonempty, bounded, closed and convex. Moreover, in view of the condition 2 in the Definition 2.1 and the fact that $\lim_{n\to\infty} \mu(B^n_{r_0}) = 0$, the set Y is member of the $\ker \mu$ and, consequently, Y is relatively compact subset of $BC(\mathbb{R}_+)$. We observe that the operator F transforms the set Y into itself. The proof that F is continuous on Y is the same as that appearing in Theorem 3.2.

Finally, taking into account all facts concerning the set Y and the continuity of the operator $F : Y \to Y$, the Schauder fixed point principle says us that F has at least one fixed point in Y, being a solution of Eq. (1.1).

4. FURTHER DISCUSSIONS, REMARKS AND EXAMPLES

This section is devoted to discuss and to give some examples in connection with our main result proved in the previous section.

Remark 4.1. Let $f : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ be a continuous function and such that $t \mapsto f(t, 0)$ is an element of $BC(\mathbb{R}_+)$ and suppose that there exists a constant k such that

$$|f(t, x) - f(t, y)| \leq k|x - y| \quad \text{for} \quad x, y \in \mathbb{R}.$$

Then the operator $T : BC(\mathbb{R}_+) \longrightarrow BC(\mathbb{R}_+)$ defined by $(Tx)(t) = f(t, x(t))$ is a Darbo operator with respect to the measure of noncompactness μ defined in the Section 1.

In fact, it is easily seen that if $x \in BC(\mathbb{R}_+)$ then, by our assumptions about f, Tx is continuous on $\mathbb{R}_+$. Moreover, if $x \in BC(\mathbb{R}_+)$, then

$$|(Tx)(t)| = |f(t, x(t))| \leq |f(t, x(t)) - f(t, 0)| + |f(t, 0)| \leq k|x(t)| + |f(t, 0)|$$

and this proves that T transforms $BC(\mathbb{R}_+)$ into itself.

In what follows we prove that T is a Darbo operator.

Let X be a nonempty bounded subset of $BC(\mathbb{R}_+)$ and $x, y \in X$, $t \in \mathbb{R}_+$, then we obtain

$$|(Tx)(t) - (Ty)(t)| = |f(t, x(t)) - f(t, y(t))| \leq k|x(t) - y(t)|$$

and this gives us

$$\operatorname{diam}(Tx)(t) \le k \operatorname{diam} X(t)$$

and, consequently

$$\limsup_{t\to\infty} \operatorname{diam}(Tx)(t) \le k \limsup_{t\to\infty} \operatorname{diam} X(t). \tag{4.1}$$

On the other hand, for $T > 0$, $\varepsilon > 0$, $t, p \in [0, T]$ with $|t - p| \le \varepsilon$ and $x \in X$ we have

$$\begin{aligned}|(Tx)(t) - (Tx)(p)| &= |f(t, x(t)) - f(p, x(p))|\\ &\le |f(t, x(t)) - f(t, x(p))| + |f(t, x(p)) - f(p, x(p))|\\ &\le k|x(t) - x(p)| + w_f^T(\varepsilon),\end{aligned}$$

where $w_f^T(\varepsilon) = \sup\{|f(t_1, x) - f(t_2, y)| : t_1, t_2 \in [0, T],\ |t_1 - t_2| \le \varepsilon,\ x, y \in [-\|X\|, \|X\|]\}$ and $\|X\| = \sup\{\|x\| : x \in X\}$.

Notice that as f is uniformly continuous on $[0, T] \times [-\|X\|, \|X\|]$, $\lim_{\varepsilon\to 0} w_f^T(\varepsilon) = 0$.

Taking into account the above mentioned facts and the last inequality, we get

$$w^T(TX, \varepsilon) \le kw^T(X, \varepsilon) + w_f^T(\varepsilon)$$

and taking limit as $\varepsilon \to 0$,

$$w_0^T(TX) \le kw_0^T(X)$$

and, finally, taking limit as $T \to \infty$

$$w_0(TX) \le kw_0(X). \tag{4.2}$$

Now, linking (4.1) and (4.2) we have

$$\mu(TX) \le k\mu(X)$$

and this proves that T is a Darbo operator with constant k.

Functions f satisfying conditions appearing in Remark 4.1 are used in the papers [5, 10] and, consequently, our theorem generalizes the results of these papers in some particular cases (for example, when $m(t)$ is a bounded function, see [5, 10]).

In the paper [3] it is used a function $f : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ continuous and such that there exist positive constants L, M with $M < L$ and

$$|f(t, x) - f(t, y)| \le \frac{M|x - y|}{L + |x - y|} \quad \text{for} \quad x, y \in \mathbb{R}$$

and $t \mapsto f(t, 0)$ is a bounded function.

In this case, this function can be considered as a particular case of the functions appearing in Remark 4.1, in virtue of

$$|f(t, x) - f(t, y)| \le \frac{M|x - y|}{L + |x - y|} \le \frac{M}{L}|x - y|$$

and, consequently, the result proved in [3] is a particular case of our Theorem 3.2.

Now, we present some examples of Darbo operators which can be used in our Theorem 3.2.

Example 4.2. Let $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ be a continuous nondecreasing function with bounded derivative and $\lim_{t\to\infty} \varphi(t) = \infty$. Consider the composition operator:

$$T_\varphi : BC(\mathbb{R}_+) \longrightarrow BC(\mathbb{R}_+),$$

defined by $(T_\varphi x)(t) = x(\varphi(t))$. Obviously, T_φ transforms $BC(\mathbb{R}_+)$ into itself.

In what follows we prove that T_φ is a Darbo operator.

In fact, let X be a nonempty bounded subset of $BC(\mathbb{R}_+)$ and $x, y \in X$, $t \in \mathbb{R}_+$ then we have

$$|(T_\varphi x)(t) - (T_\varphi y)(t)| = |x(\varphi(t)) - y(\varphi(t))|.$$

From this we get

$$\operatorname{diam}(T_\varphi X)(t) \leq \operatorname{diam} X(\varphi(t))$$

and as $\lim_{t\to\infty} \varphi(t) = \infty$, we obtain

$$\limsup_{t\to\infty} \operatorname{diam}(T_\varphi X)(t) \leq \limsup_{t\to\infty} \operatorname{diam} X(\varphi(t)) = \limsup_{t\to\infty} \operatorname{diam} X(t). \tag{4.3}$$

On the other hand, for $T > 0$, $\varepsilon > 0$, $t, p \in [0, T]$ with $|t - p| \leq \varepsilon$ and $x \in X$ we have

$$|(T_\varphi x)(t) - (T_\varphi x)(p)| = |x(\varphi(t)) - x(\varphi(p))|. \tag{4.4}$$

As φ has bounded derivative, we put $h = \sup\{|\varphi'(t)| : t \in \mathbb{R}_+\}$ and the mean value theorem gives us that $|\varphi(t) - \varphi(p)| \leq h|t - p|$. Notice that φ is a nondecreasing and nonnegative function and $t, p \in [0, T]$ then $\varphi(t), \varphi(p) \in [0, \varphi(T)]$ and (4.4) gives us

$$w^T(T_\varphi x, \varepsilon) \leq w^{\varphi(T)}(x, h\varepsilon).$$

Taking supremum in $x \in X$

$$w^T(T_\varphi X, \varepsilon) \leq w^{\varphi(T)}(X, h\varepsilon),$$

and taking limit as $\varepsilon \to 0$

$$w_0^T(T_\varphi X) \leq w_0^{\varphi(T)}(X).$$

Finally, taking limit as $T \to \infty$ and keeping in mind that $\lim_{t\to\infty} \varphi(t) = \infty$ we obtain

$$w_0(T_\varphi X) \leq w_0(X). \tag{4.5}$$

Now, linking (4.3) and (4.5) we get

$$\mu(T_\varphi X) \leq \mu(X)$$

and, consequently, T_φ is a Darbo operator with constant $k = 1$.

An example of function φ is $\varphi(t) = \ln(t + 1)$.

Example 4.3. Consider the composition operator appearing in Example 4.2 with $\varphi(t) = t^2$. Notice that the derivative of φ is not bounded.

In this case, we will prove that T_φ is also a Darbo operator. Obviously, T_φ transforms $BC(\mathbb{R}_+)$ into itself.

Let X be a nonempty bounded subset of $BC(\mathbb{R}_+)$ and $x, y \in X$, $t \in \mathbb{R}_+$, then

$$|(T_\varphi x)(t) - (T_\varphi y)(t)| = |x(t^2) - y(t^2)|.$$

This gives us

$$\operatorname{diam}(T_\varphi X)(t) \leq \operatorname{diam} X(t^2)$$

and, consequently,

$$\limsup_{t\to\infty} \operatorname{diam}(T_\varphi X)(t) \leq \limsup_{t\to\infty} \operatorname{diam} X(t^2) = \limsup_{t\to\infty} \operatorname{diam} X(t). \tag{4.6}$$

On the other hand, for $T > 0$, $\varepsilon > 0$, $t, p \in [0, T]$ with $|t - p| \leq \varepsilon$ and $x \in X$ we have

$$|(T_\varphi x)(t) - (T_\varphi x)(p)| = |x(t^2) - x(p^2)|.$$

As $|t^2 - p^2| = |t + p| \cdot |t - p| \leq 2\,T\,|t - p|$ we obtain

$$w^T(T_\varphi x, \varepsilon) \leq w^{T^2}(x, 2T\varepsilon)$$

and this gives us

$$w^T(T_\varphi X, \varepsilon) \leq w^{T^2}(X, 2T\varepsilon).$$

Taking limit as $\varepsilon \to 0$

$$w_0^T(T_\varphi X) \leq w_0^{T^2}(X)$$

and taking limit as $T \to \infty$

$$w_0(T_\varphi X) \leq w_0(X). \tag{4.7}$$

Finally, linking (4.6) and (4.7)

$$\mu(T_\varphi X) \leq \mu(X)$$

and this proves that T_φ is a Darbo operator with constant $k = 1$.

The condition about the boundedness of derivative of φ can be changed by the following condition:

$$|\varphi(t) - \varphi(p)| \leq \varphi(T)|t - p|, \quad \text{for} \quad t, p \in [0, T]$$

and the Example 4.2 would work for our purposes. (An example of such functions is $\varphi(t) = e^t$).

Example 4.4. Let φ be a continuous and bounded function on $\mathbb{R}_+$ and we consider the multiplication operator $T^\varphi : BC(\mathbb{R}_+) \to BC(\mathbb{R}_+)$ defined by $(T^\varphi x)(t) = x(t)\varphi(t)$. Obviously, T^φ transforms $BC(\mathbb{R}_+)$ into itself.

In order to prove that T^φ is a Darbo operator we take a nonempty bounded subset X of $BC(\mathbb{R}_+)$ and $x, y \in X$, $t \in \mathbb{R}_+$ then

$$|(T^\varphi x)(t) - (T^\varphi y)(t)| = |\varphi(t)| \cdot |x(t) - y(t)| \leq \|\varphi\| \cdot |x(t) - y(t)|$$

and this proves that

$$\limsup_{t\to\infty} \operatorname{diam}(T^\varphi X)(t) \leq \|\varphi\| \limsup_{t\to\infty} \operatorname{diam} X(t). \tag{4.8}$$

On the other hand, for $T > 0$, $\varepsilon > 0$, $t, p \in [0, T]$ with $|t - p| \leq \varepsilon$ and $x \in X$ we have

$$\begin{aligned} |(T^\varphi x)(p) - (T^\varphi x)(t)| &= |\varphi(p)x(p) - \varphi(t)x(t)| \\ &\leq |\varphi(p)x(p) - \varphi(p)x(t)| + |\varphi(p)x(t) - \varphi(t)x(t)| \\ &\leq |\varphi(p)| \cdot |x(p) - x(t)| + |x(t)| \cdot |\varphi(p) - \varphi(t)| \\ &\leq \|\varphi\| \cdot |x(p) - x(t)| + \|x\| \cdot w_\varphi^T(\varepsilon), \end{aligned}$$

where $w_\varphi^T(\varepsilon) = \sup\{|\varphi(t_1) - \varphi(t_2)| : t_1, t_2 \in [0, T],\ |t_1 - t_2| \leq \varepsilon\}$. Obviously, $w_\varphi^T(\varepsilon) \to 0$ as $\varepsilon \to 0$.

The last inequality gives us

$$w^T(T^\varphi x, \varepsilon) \leq \|\varphi\| w^T(x, \varepsilon) + \|x\| w_\varphi^T(\varepsilon)$$

and, consequently,

$$w^T(T^\varphi X, \varepsilon) \leq \|\varphi\| w^T(X, \varepsilon) + \|X\| w_\varphi^T(\varepsilon).$$

Following the same steps that in Example 4.3 we get

$$w_0(T^\varphi X) \leq \|\varphi\| w_0(X). \tag{4.9}$$

Finally, linking (4.8) and (4.9)

$$\mu(T^\varphi X) \leq \|\varphi\| \mu(X)$$

and this proves that T^φ is a Darbo operator with constant $k = \|\varphi\|$.

Example 4.5. Let $\phi : \mathbb{R} \to \mathbb{R}$ be a Lipschitz function. Consider the operator $L_\phi : BC(\mathbb{R}_+) \to BC(\mathbb{R}_+)$ defined by

$$(L_\phi x)(t) = (\phi \circ x)(t).$$

Obviously, if x is continuous on $\mathbb{R}_+$ then $\phi \circ x$ is also continuous. Moreover, if x is bounded on $\mathbb{R}_+$ as ϕ is continuous on $\mathbb{R}_+$, $\phi \circ x$ is bounded.

In what follows we prove that L_ϕ is a Darbo operator.

Let $X \subset BC(\mathbb{R}_+)$ be a nonempty bounded subset and $x, y \in X$, $t \in \mathbb{R}_+$ then

$$\begin{aligned}|(L_\phi x)(t) - (L_\phi y)(t)| &= |(\phi \circ x)(t) - (\phi \circ y)(t)| \\ &= |(\phi(x(t)) - (\phi(y(t))| \\ &\leq M|x(t) - y(t)|,\end{aligned}$$

where M is the Lipschitz constant of ϕ.

Consequently,

$$\operatorname{diam}(L_\phi X)(t) \leq M \operatorname{diam} X(t)$$

and this implies that

$$\limsup_{t\to\infty} \operatorname{diam}(L_\phi X)(t) \leq M \limsup_{t\to\infty} \operatorname{diam} X(t). \tag{4.10}$$

On the other hand, for $T > 0$, $\varepsilon > 0$, $t, p \in [0, T]$ with $|t - p| \leq \varepsilon$ and $x \in X$ we have

$$|(L_\phi x)(p) - (L_\phi x)(t)| = |(\phi(x(p)) - (\phi(x(t))| \leq M|x(p) - x(t)|$$

and this gives us that

$$w^T(L_\phi X, \varepsilon) \leq M w^T(X, \varepsilon).$$

Consequently,

$$w_0(L_\phi X) \leq M w_0(X). \tag{4.11}$$

Finally, linking (4.10) and (4.11)

$$\mu(L_\phi X) \leq M \mu(X)$$

Notice that examples of functions ϕ are: $\phi(t) = \sin t$, $\phi(t) = \arctan t$ and $\phi(t) = \dfrac{|t|}{1 + |t|}$.

Example 4.6. Let $\varphi : \mathbb{R}_+ \to \mathbb{R}$ be a continuous function such that $\int_0^\infty |\varphi(s)| ds < \infty$. Consider the operator $I_\varphi : BC(\mathbb{R}_+) \to BC(\mathbb{R}_+)$ defined by

$$(I_\varphi x)(t) = \int_{t/2}^t x(s)\varphi(s) ds.$$

In view of our assumptions it is easily seen that I_φ transforms $BC(\mathbb{R}_+)$ into itself.

Now, we will show that I_φ is a Darbo operator.

In fact, let $X \subset BC(\mathbb{R}_+)$ be a nonempty bounded subset and $x, y \in X$, $t \in \mathbb{R}_+$ then we have

$$\begin{aligned}|(I_\varphi x)(t) - (I_\varphi y)(t)| &= \left| \int_{t/2}^t x(s)\varphi(s) ds - \int_{t/2}^t y(s)\varphi(s) ds \right| \\ &\leq \int_{t/2}^t |x(s) - y(s)||\varphi(s)| ds \\ &\leq \|x - y\| \int_{t/2}^t |\varphi(s)| ds \leq 2\|X\| \int_{t/2}^t |\varphi(s)| ds,\end{aligned}$$

where $\|X\| = \sup\{\|x\| : x \in X\}$.

This gives

$$\operatorname{diam}(I_\varphi X)(t) \leq 2\|X\| \int_{t/2}^{t} |\varphi(s)|ds$$

and, as $\displaystyle\int_0^\infty |\varphi(s)|ds < \infty$, we get

$$\limsup_{t\to\infty} \operatorname{diam}(I_\varphi X)(t) \leq 2\|X\| \lim_{t\to\infty} \int_{t/2}^{t} |\varphi(s)|ds = 0.$$

On the other hand, for $T > 0$, $\varepsilon > 0$, $t, p \in [0, T]$ with $|t - p| \leq \varepsilon$ and $x \in X$ we have

$$|(I_\varphi x)(p) - (I_\varphi x)(t)| = \left| \int_{p/2}^{p} x(s)\varphi(s)ds - \int_{t/2}^{t} x(s)\varphi(s)ds \right|.$$

Without loss of generality we can suppose that $t < p$. Then

$$\begin{aligned}
|(I_\varphi x)(p) - (I_\varphi x)(t)| &= \left| \int_0^p x(s)\varphi(s)ds - \int_0^{p/2} x(s)\varphi(s)ds \right. \\
&\qquad \left. - \int_0^t x(s)\varphi(s)ds + \int_0^{t/2} x(s)\varphi(s)ds \right| \\
&= \left| \int_t^p x(s)\varphi(s)ds - \int_{t/2}^{p/2} x(s)\varphi(s)ds \right| \\
&\leq \|x\| \left[\int_t^p |\varphi(s)|ds + \int_{t/2}^{p/2} |\varphi(s)|ds \right] \\
&\leq \|x\| \left[\|\varphi_{|[0,T]}\|(p - t) + \|\varphi_{|[0,T]}\|(p/2 - t/2) \right] \\
&\leq \|x\| \|\varphi_{|[0,T]}\| \frac{3}{2}\varepsilon,
\end{aligned}$$

where $\|\varphi_{|[0,T]}\| = \sup\{|\varphi(s)| : s \in [0, T]\}$.

Consequently,

$$w^T(I_\varphi X, \varepsilon) \leq \|x\| \|\varphi_{|[0,T]}\| \frac{3}{2}\varepsilon$$

and this gives

$$w_0^T(I_\varphi X) = 0.$$

This proves that I_φ is a compact operator and thus, a Darbo operator.

Examples of functions φ are: $\varphi(t) = e^{-\lambda t}$ and $\varphi(t) = \dfrac{1}{1 + t^2}$.

Example 4.7. Let $\varphi \in BC(\mathbb{R}_+)$ be a fixed function. Consider the operator $M_\varphi : BC(\mathbb{R}_+) \to BC(\mathbb{R}_+)$ defined by

$$(M_\varphi x)(t) = \max(x(t), \varphi(t)).$$

Obviously, M_φ transforms $BC(\mathbb{R}_+)$ into itself.

Now, we prove that M_φ is a Darbo operator.

In fact, let $X \subset BC(\mathbb{R}_+)$ be a nonempty bounded subset and $x, y \in X$, $t \in \mathbb{R}_+$ then

$$|(M_\varphi x)(t) - (M_\varphi y)(t)| = |\max(x(t), \varphi(t)) - \max(y(t), \varphi(t))|.$$

We can distinguish several cases:

1. $\max(x(t), \varphi(t)) = x(t)$ and $\max(y(t), \varphi(t)) = y(t)$.
2. $\max(x(t), \varphi(t)) = x(t)$ and $\max(y(t), \varphi(t)) = \varphi(t)$.
3. $\max(x(t), \varphi(t)) = \varphi(t)$ and $\max(y(t), \varphi(t)) = y(t)$.
4. $\max(x(t), \varphi(t)) = \varphi(t)$ and $\max(y(t), \varphi(t)) = \varphi(t)$.

Case 1. In this case $|(M_\varphi x)(t) - (M_\varphi y)(t)| = |x(t) - y(t)| \leq \operatorname{diam} X(t)$.

Case 2. In this case $|(M_\varphi x)(t) - (M_\varphi y)(t)| = |x(t) - \varphi(t)|$.

Notice that $\varphi(t) \leq x(t)$ and $y(t) \leq \varphi(t)$.

Consequently, $|x(t) - \varphi(t)| = d(x(t), \varphi(t)) \leq d(x(t), y(t)) \leq |x(t) - y(t)|$, and $|(M_\varphi x)(t) - (M_\varphi y)(t)| \leq |x(t) - y(t)| \leq \operatorname{diam} X(t)$.

Case 3. It is analogous to case 2.

Case 4. In this case $|(M_\varphi x)(t) - (M_\varphi x)(t)| = 0$.

In summary, we have that

$$|(M_\varphi x)(t) - (M_\varphi y)(t)| \leq \operatorname{diam} X(t)$$

and this gives

$$\operatorname{diam}(M_\varphi X)(t) \leq \operatorname{diam} X(t)$$

and, consequently,

$$\limsup_{t\to\infty} \operatorname{diam}(M_\varphi X)(t) \leq \limsup_{t\to\infty} \operatorname{diam} X(t).$$

On the other hand, for $T > 0$, $\varepsilon > 0$, $t, p \in [0, T]$ with $|t - p| \leq \varepsilon$ and $x \in X$ we have

$$|(M_\varphi x)(t) - (M_\varphi x)(p)| = |\max(x(t), \varphi(t)) - \max(x(p), \varphi(p))|.$$

We can distinguish several cases:

1. $\max(x(t), \varphi(t)) = x(t)$ and $\max(x(p), \varphi(p)) = x(p)$.
2. $\max(x(t), \varphi(t)) = x(t)$ and $\max(x(p), \varphi(p)) = \varphi(p)$.
3. $\max(x(t), \varphi(t)) = \varphi(t)$ and $\max(x(p), \varphi(p)) = x(p)$.
4. $\max(x(t), \varphi(t)) = \varphi(t)$ and $\max(x(p), \varphi(p)) = \varphi(p)$.

Following the same argument that we use for diameter it is easily proved that

$$w^T(M_\varphi x, \varepsilon) \leq \max\{w^T(x, \varepsilon), w^T(\varphi, \varepsilon)\}$$

the last inequality gives us that

$$w^T(M_\varphi x, \varepsilon) \leq w^T(x, \varepsilon) + w^T(\varphi, \varepsilon)$$

and this implies that

$$w^T(M_\varphi X, \varepsilon) \leq w^T(X, \varepsilon) + w^T(\varphi, \varepsilon).$$

As φ is continuous we obtain

$$w_0^T(M_\varphi X) \leq w_0^T(X)$$

and, thus,

$$w_0(M_\varphi X) \leq w_0(X).$$

An interesting case appears when $\varphi = 0$ and in this case

$$M_\varphi x = \max(x, 0) = x^+ \quad \text{(positive part of } x\text{)}$$

Remark 4.8. In our Theorem 3.2 we use the condition $\|Tx\| \leq c + d\|x\|$ for $x \in BC(\mathbb{R}_+)$. Notice that this condition is satisfied when $|(Tx)(t)| \leq c + d|x(t)|$ for $t \in \mathbb{R}_+$.

The following example proves that these conditions are not equivalent.

We consider the Darbo operator given in Example 4.3, i.e. $(Tx)(t) = x(t^2)$. Obviously, $\|Tx\| \leq \|x\|$.

Now we suppose that there exist nonnegative constants c and d such that $|(Tx)(t)| \leq c + d|x(t)|$ for every $x \in BC(\mathbb{R}_+)$ and $t \in \mathbb{R}_+$.

We consider the function

$$x(t) = \begin{cases} \frac{1}{2}(\max\{c, d\} + 1)(t - 2), & 0 \leq t \leq 4 \\ \max\{c, d\} + 1, & t > 4. \end{cases}$$

Obviously, $x \in BC(\mathbb{R}_+)$ and $|(Tx)(2)| = |x(4)| = \max\{c, d\} + 1 > c + d|x(2)| = c$.

Remark 4.9. Following the definition in [5] the asymptotic stability of a solution $x = x(t)$ of Eq. (1.1) will be understood in the following sense:

For $\varepsilon > 0$ given there exist $T > 0$ and $r > 0$ such that if $x, y \in B_r$ and $x = x(t)$, $y = y(t)$ are solutions of Eq. (1.1) then $|x(t) - y(t)| \leq \varepsilon$ for $t \geq T$.

Taking into account Remark 2.4 and the description of the kernel of the measure of noncompactness μ given in Section 2, we infer easily from the proof of Theorem 3.2 that any solution of Eq. (1.1) which belongs to the ball B_{r_0} is asymptotically stable.

Remark 4.10. If $(T_1x)(t) = h(t)$, $(T_2x)(t) = 1$ and $u(t, s, x) = \varphi(s)x(s)$ with $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ continuous, our Eq. (1.1) reduces to

$$x(t) = h(t) + \int_0^t \varphi(s)x(s)ds.$$

In this case we can obtain uniqueness of the solution in the ball B_{r_0}. In fact, if $x = x(t)$, $y = y(t)$ are solutions of our Eq. (1.1) then, for $t \in [0, T]$, where $T > 0$ is fixed, we obtain

$$|x(t) - y(t)| \leq \int_0^t \varphi(s)|x(s) - y(s)|ds$$

and Gronwall inequality [11] gives us $|x(t) - y(t)| = 0$ and, consequently, $x(t) = y(t)$ in $[0, T]$ and, as T is arbitrary, this gives us the uniqueness of the solution in B_{r_0}.

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