Dynamic Systems and Applications 18 (2009) 621-636

PERIODIC SOLUTION FOR NON-AUTONOMOUS SECOND ORDER HAMILTONIAN SYSTEMS ON TIME SCALES

YOU-HUI SU AND WAN-TONG LI

School of Mathematics and Physical Sciences, XuZhou Institute of Technology
Xuzhou, Jiangsu, 221008, People's Republic of China
suyouhui@xzit.edu.cn suyh05@lzu.cn
School of Mathematics and Statistics, Lanzhou University
Lanzhou, Gansu, 730000, People's Republic of China

We consider the following non-autonomous second order Hamiltonian system on time scales $\mathbb{T}$ of the form

$$\begin{cases} u^{\Delta\Delta}(\rho(t)) = \overline{\bigtriangledown} H(t,u(t)) & \Delta\text{-a.e. } t \in [0,T]_{\mathbb{T}}, \\ u(0) - u(T) = u^{\Delta}(\rho(0)) - u^{\Delta}(\rho(T)) = 0. \end{cases}$$

As is well known, it is very difficult to use the Hilger's integral to consider the existence of periodic solutions of some second order Hamiltonian systems on time scales since it is only concerned with antiderivatives. Therefore, in this paper, we use a new integral on time scales $\mathbb{T}$ defined by Rynne (*J. Math. Anal. Appl.* **328** (2007) 1217–1236), and establish a new existence result for periodic solutions in $H_T^1(\mathbb{T},\mathbb{R}^n)$ space of the above-mentioned second order Hamiltonian system on time scales $\mathbb{T}$ by applying variational methods and critical theory. As an application, an example is given to illustrate the result.

Key words: Time scales; second order Hamiltonian system; periodic solution; critical point

2000 AMS Subject Classification: 39A10, 39A11, 34C25, 37J45

1. INTRODUCTION

In this paper, motivated by the references on continuous and discrete Hamiltonian systems [16, 28, 30], we consider the following non-autonomous second order Hamiltonian system on time scales $\mathbb{T}$ of the form

$$(1.1) \qquad \begin{cases} u^{\Delta\Delta}(\rho(t)) = \overline{\bigtriangledown} H(t,u(t)) & \Delta\text{-a.e. } t \in [0,T]_{\mathbb{T}}, \\ u(0) - u(T) = u^{\Delta}(\rho(0)) - u^{\Delta}(\rho(T)) = 0, \end{cases}$$

where $0, T \in \mathbb{T}$, $\overline{\bigtriangledown} H(t,u) = D_u H(t,u)$ and $H : [0,T]_{\mathbb{T}} \times \mathbb{R}^n \to \mathbb{R}$, $(t,x) \to H(t,x)$ satisfies the following assumption:

Supported by the NSFC under grant 10571078 and XZIT under grant XKY2008311.
Received October 1, 2008

(H_0): $H(t,x)$ is measurable in t for every $x \in \mathbb{R}^n$ and continuously differentiable in x for Δ-a.e. $t \in [0,T]_{\mathbb{T}}$, and there exist $a \in C(\mathbb{R}^+, \mathbb{R}^+)$, $b \in L^1([0,T]_{\mathbb{T}}, \mathbb{R}^+)$ satisfying $|H(t,x)| \le a(|x|)b(t)$, $|\overline{\bigtriangledown} H(t,x)| \le a(|x|)b(t)$ for all $x \in \mathbb{R}^n$ and Δ-a.e. $t \in [0,T]_{\mathbb{T}}$.

We say that a property holds for Δ-a.e. $t \in A \subset \mathbb{T}$ or Δ-a.e. on $A \subset \mathbb{T}$, whenever there exists a set $E \subset A$ with null Lebesgue Δ-measure such that this property holds for every $t \in A \backslash E$. We refer the reader to [6, 10] for a broad introduction on Lebesgue Δ-measure.

Recently, some authors have obtained many results on the existence of positive solutions of dynamic equations on time scales, for details, see [4, 14, 15, 19, 21, 22, 23, 24, 25, 26, 27] and the references therein. On the one hand, the above-mentioned problems on time scale [4, 14, 15, 19, 21, 22, 23, 24, 25, 26, 27] have often been considered in a set involving Banach spaces of continuous (rd-continuous) functions on $\mathbb{T}$, which motivate us to consider the existence of solutions for dynamic equations on time scales in Hilbert spaces rather than Banach spaces. On the other hand, to the best of our knowledge, there is only one paper [20] concerned with the existence of periodic solutions of second order Hamiltonian systems on time scales. Moreover, there is very little work [1, 2] on the existence of solutions of dynamic equations on time scales by using the variational methods and critical theory. Now, it is natural to consider the existence of periodic solutions of second order Hamiltonian systems on time scales by using variational methods and critical theory.

We make the blanket assumption that $0, T$ are points in $\mathbb{T}$, for an interval $[0,T]_{\mathbb{T}}$, we always mean $[0,T] \cap \mathbb{T}$. Other type of intervals are defined similarly.

If the function $u : \mathbb{T}_\kappa \to \mathbb{R}^n$ is delta differential, u^Δ and $u^{\Delta\Delta}$ are both continuous on Δ-a.e. $\mathbb{T}^\kappa \cap \mathbb{T}_\kappa$, and u satisfies Hamiltonian system (1.1), then we say u is a solution of Hamiltonian system (1.1).

Recently, Su and Li [20] considered the existence of periodic solutions for second order Hamiltonian system on time scales

$$\begin{cases} u^{\Delta\Delta}(\rho(t)) + \mu b(t)|u(t)|^{\mu-2}u(t) + \overline{\bigtriangledown} H(t,u(t)) = 0 \quad \Delta\text{-a.e. } t \in [0,T]_{\mathbb{T}}, \\ u(0) - u(T) = u^\Delta(\rho(0)) - u^\Delta(\rho(T)) = 0. \end{cases} \tag{1.2}$$

Under $\mu > 2$ and certain conditions, we obtain that the problem (1.2) has at least one nonzero periodic solution. However, in [20], by using the generalized mountain pass theorem [17], we only obtain the existence results of Hamiltonian system (1.2) for $\mu > 2$. That is, if continue to use the tools to solve the problems for (1.2) when $\mu \le 2$, the existence of periodic solutions of Hamiltonian system is not satisfied. However, when $\mu = 0$, the Hamiltonian system (1.2) reduces to the Hamiltonian system (1.1). The existence of periodic solutions for Hamiltonian system (1.1) not only has it's theoretical value but also has its practical value. Naturally, it is quite necessary to consider the existence of periodic solutions of Hamiltonian system (1.1). Hence, in this paper, by using the tools completely different from that of [20], we consider the existence for periodic solutions of the non-autonomous second order Hamiltonian system (1.1).

However, just as Ahlbrandt (MR1962542) reviewed for the reference [6], Hilger's integral is based solely on antiderivatives. The so-called "delta integral" and "nabla integral" are defined by a sort of Darboux integral and as a limit of a modified Riemann sum, respectively. The absence of these integrals has hindered the development of time scales integral operator proofs of existence and uniqueness theorems, unified variational theory and a possible Hilbert space spectral theorem for Jacobi operators on time scales. Hence, it is very difficult in considering the existence of periodic solutions of some second order Hamiltonian systems on time scales using the Hilger's integral. In this paper, we attempt to use a new integral on time scales $\mathbb{T}$ defined by Rynne [18] to consider the existence of periodic solutions of non-autonomous second order Hamiltonian system on time scales. By using variational methods and critical theory, we establish a new existence theorem for periodic solutions of the non-autonomous second order Hamiltonian system (1.1). Moreover, we prove some lemmas, which will be very important in proving the existence of periodic solutions in $W_T^{1,p}(\mathbb{T}, \mathbb{R}^n)$ spaces for Hamiltonian systems on time scales. As an application, an example is given to illustrate our result.

Firstly, we present some basic definitions which can be found in [5, 6, 7, 11, 13].

A time scale $\mathbb{T}$ is a nonempty closed subset of $\mathbb{R}$. If $\mathbb{T}$ has a right-scattered minimum m, define $\mathbb{T}_\kappa = \mathbb{T} - \{m\}$; otherwise, set $\mathbb{T}_\kappa = \mathbb{T}$. If $\mathbb{T}$ has a left-scattered maximum M, define $\mathbb{T}^\kappa = \mathbb{T} - \{M\}$; otherwise, set $\mathbb{T}^\kappa = \mathbb{T}$. The forward graininess is $\mu(t) := \sigma(t) - t$. Similarly, the backward graininess is $\nu(t) := t - \rho(t)$.

If $f : \mathbb{T} \to \mathbb{R}$ is a function and $t \in \mathbb{T}^\kappa$, then the delta derivative [7] of f at the point t is defined by the number $f^\Delta(t)$ (provided it exists) with the property that for any $\epsilon > 0$, there is a neighborhood $U \subset \mathbb{T}$ of t such that $\left|f(\sigma(t)) - f(s) - f^\Delta(t)(\sigma(t) - s)\right| \leq \epsilon\left|\sigma(t) - s\right|$ for all $s \in U$.

If $f : \mathbb{T} \to \mathbb{R}$ and $t \in \mathbb{T}_\kappa$, then the nabla derivative of f at the point t is defined by the number $f^\nabla(t)$ (provided it exists) with the property that for any $\epsilon > 0$, there is a neighborhood $U \subset \mathbb{T}$ of t such that $\left|f(\rho(t)) - f(s) - f^\nabla(t)(\rho(t) - s)\right| \leq \epsilon\left|\rho(t) - s\right|$ for all $s \in U$.

Assume that $f : \mathbb{T} \to R$. Continuity of f is defined in the usual manner, while f is said to be rd-continuous provided it is continuous at all right dense points in $\mathbb{T}$ and has finite left-sided limits at all left dense points in $\mathbb{T}$. Let $C_{rd}(\mathbb{T})$ denote the set of rd-continuous functions $f : \mathbb{T} \to \mathbb{R}$, and let

$$|f|_{0,\mathbb{T}} := \sup\{|f| : t \in \mathbb{T}\} \quad \text{for } f \in C_{rd}(\mathbb{T}).$$

With this norm these spaces are Banach spaces. If f is differentiable at every $t \in \mathbb{T}^\kappa$, then f is said to be differentiable. Let $C^1_{rd}(\mathbb{T})$ denote the set of functions $f \in C(\mathbb{T})$ which are differentiable and $f^\Delta \in C_{rd}(\mathbb{T}^\kappa)$. The norm of the functions f are defined by

$$|f|_{1,\mathbb{T}} := |f|_{0,\mathbb{T}} + |f^\Delta|_{0,\mathbb{T}^\kappa} \quad \text{for } f \in C^1_{rd}(\mathbb{T}).$$

These spaces are also Banach spaces.

Secondly, we refer the reader to [10] for measure on time scales. The definition of absolutely continuous on time scales can be founded in [8].

Thirdly, we provide the definition in [18] and simply summarize the main points, which are also described in [9].

Let $a := \inf\{s : s \in \mathbb{T}\}$ and $b := \sup\{s : s \in \mathbb{T}\}$, define a function $E : [a, b] \to \mathbb{R}$ by

$$E(t) := \sup\{s \in \mathbb{T} : s \le t\} \quad \text{for } t \in [a, b].$$

Now, suppose that $f : \mathbb{T}^\kappa \to \mathbb{R}$ is arbitrary function on $\mathbb{T}^\kappa$, if $f \circ E$ is measurable and integrable on the real interval $[a, b)$ in the usual Lebesgue sense, respectively, then we say f is measurable and integrable, respectively. Let $L^1(\mathbb{T})$ denote the set of such integrable functions on $\mathbb{T}$. Furthermore, for any $f \in L^1(\mathbb{T})$, we define the integral of f as

$$\int_s^t f\Delta := \int_s^t f \circ E d\tau \quad \text{for } s, t \in \mathbb{T}, \tag{1.3}$$

with the norm defined by $\|f\|_{L^1(\mathbb{T})} = \int_a^b |f|\Delta$ for $f \in L^1(\mathbb{T})$.

We use the notation $\int_s^t f\Delta$ to denote the Lebesgue integral of a function f between $s, t \in \mathbb{T}$ (when it is defined). That is, we use the same notation for the Lebesgue-type integral defined in [18] as it is also used in the time scale literature for a Riemann-type integral defined in terms of anti-derivative. A detailed discussion of the Lebesgue-type integral and its relationship with the usual time scale integral is given in [9, 18]. With the Lebesgue integral defined, let

$$L^p(\mathbb{T}) := \{f \in L^1(\mathbb{T}) : |f|^p \in L^1(\mathbb{T})\},$$

and the corresponding norm $\|f\|_{L^p(\mathbb{T})} = \left(\int_a^b |f|^p\Delta\right)^{\frac{1}{p}}$. It is shown that in [9] $L^p(\mathbb{T})$ is completed with respect to the norm $\|f\|_{L^p(\mathbb{T})}$.

In terms of $L^p(\mathbb{T})$ spaces, one can easily define Sobolev-type spaces as,

$$W^{1,p}(\mathbb{T}) = \{u : \mathbb{T} \to \mathbb{R} \mid u \text{ is absolutely continuous, } u^\Delta \in L^p(\mathbb{T}^\kappa)\},$$

with the norm defined by

$$\|u\|_{W^{1,p}(\mathbb{T})} = \left(\int_a^b |u|^p \Delta + \int_a^b |u^\Delta|^p \Delta\right)^{\frac{1}{p}} \quad \text{for } u \in W^{1,p}(\mathbb{T}).$$

The Sobolev-type spaces $W^{1,p}(\mathbb{T})$ is a time scale analogue of the usual Sobolev-type spaces $W^{1,p}(I)$ on a real interval I [9].

Finally, we refer the reader to [3] for another introduction on basic properties of Sobolev's spaces on bounded time scales. In addition, we list two Lemmas in [3] which will be used in our proof. Let $a, b \in \mathbb{T}$, $J = [a, b]_{\mathbb{T}}$ and $J^0 = [a, b)_{\mathbb{T}}$.

Lemma 1.1 ([3])**.** *Let $p \in \overline{\mathbb{R}} \equiv [-\infty, +\infty]$ be such that $p \geq 1$. Then, for every $q \in [1, +\infty)$, the immersion $W_\Delta^{1,p}(J) \hookrightarrow L_\Delta^q(J^0)$ is compact.*

Lemma 1.2 ([3])**.** *Let $p \in \overline{\mathbb{R}}$ be such that $p \geq 1$, let $\{u_m\}_{m\in\mathbb{N}} \subset W_\Delta^{1,p}(J)$, and let $u \in W_\Delta^{1,p}(J)$. If $\{u_m\}_{m\in\mathbb{N}}$ converges weakly in $W_\Delta^{1,p}(J)$ to u, then $\{u_m\}_{m\in\mathbb{N}}$ converges strongly in $C(J)$ to u.*

From [3], we also have the following relation.

$$V_\Delta^{1,p}(J) = \{u : J \to \mathbb{R} \mid u \text{ is absolutely continuous, } u^\Delta \in L_\Delta^p(J^0)\} \subset W_\Delta^{1,p}(J). \tag{1.4}$$

The rest of the paper is organized as follows. In Section 2, we list some lemmas, which are important in proving the existence of periodic solutions. By applying these lemmas, we establish a new existence result for periodic solutions of problem (1.1) in Section 3. In final Section, an example is given to illustrate our main result.

2. SOME LEMMAS

In this section, to represent Hamiltonian systems on time scales in a functional-analytic set, we introduce some spaces and some lemmas, which will be used in the rest of the paper and be very important in proving the existence of periodic solutions in $W_T^{1,p}(\mathbb{T}, \mathbb{R}^n)$ spaces on second order Hamiltonian systems on time scales.

Similar to [18], let $a_1 := \inf\{s : s \in \mathbb{T}\}$ and $b_1 := \sup\{s : s \in \mathbb{T}\}$, define a function $E : [a_1, b_1] \to \mathbb{R}^n$ by

$$E(t) := \sup\{s \in \mathbb{T} : s \leq t\} \quad \text{for } t \in [a_1, b_1].$$

Then, suppose that $u : \mathbb{T}^\kappa \to \mathbb{R}^n$ is arbitrary function on $\mathbb{T}^\kappa$, if $u \circ E$ is measurable and integrable on the real interval $[a_1, b_1)$ in the usual Lebesgue senses, respectively, then we say u is measurable and integrable, respectively. We let $L^1(\mathbb{T}, \mathbb{R}^n)$ denotes the set of such integrable functions on $\mathbb{T}$. Furthermore, for any $u \in L^1(\mathbb{T}, \mathbb{R}^n)$, we define the integral of u as

$$\int_s^t u\Delta := \int_s^t u \circ E d\tau \quad \text{for } s, t \in \mathbb{T},$$

with the norm defined by $\|u\|_{L^1(\mathbb{T},\mathbb{R}^n)} = \int_{a_1}^{b_1} |u|\Delta$ for $u \in L^1(\mathbb{T},\mathbb{R}^n)$. Moreover, let

$$L^p(\mathbb{T},\mathbb{R}^n) = \{u : \mathbb{T} \to \mathbb{R}^n, u \in L^1(\mathbb{T},\mathbb{R}^n), |u|^p \in L^1(\mathbb{T},\mathbb{R}^n)\}$$

with the norm defined by

$$\|u\|_{L^p(\mathbb{T},\mathbb{R}^n)} = \left(\int_{a_1}^{b_1} |u|^p\Delta\right)^{\frac{1}{p}} \text{ and } \|u\|_{\infty,\mathbb{R}^n} = \sup_{t\in[a_1,b_1]_{\mathbb{T}}} |u|.$$

One can easily define Sobolev-type spaces as,

$$W^{1,p}(\mathbb{T},\mathbb{R}^n) = \{u : \mathbb{T} \to \mathbb{R}^n \mid u \text{ is absolutely continuous, } u^\Delta \in L^p(\mathbb{T}^\kappa,\mathbb{R}^n)\},$$

with the norm defined by

$$\|u\|_{W^{1,p}(\mathbb{T},\mathbb{R}^n)} = \left(\int_{a_1}^{b_1} |u|^p\Delta + \int_{a_1}^{b_1} |u^\Delta|^p\Delta\right)^{\frac{1}{p}} \quad \text{for } u \in W^{1,p}(\mathbb{T},\mathbb{R}^n).$$

If we replace $u : \mathbb{T} \to \mathbb{R}$ with $u : \mathbb{T} \to \mathbb{R}^n$, then, by using the similar way to Lemma 1.1 and Lemma 1.2, respectively, and then using the relation 1.4 for $u : \mathbb{T} \to \mathbb{R}^n$, we can obtain the following two Lemmas, respectively. Let $a_1, b_1 \in \mathbb{T}$, $J_1 = [a_1, b_1]_{\mathbb{T}}$ and $J_1^0 = [a_1, b_1)_{\mathbb{T}}$.

Lemma 2.1. *Let $p \in \overline{\mathbb{R}}$ be such that $p \geq 1$. Then, for every $q \in [1,+\infty)$, the immersion $W^{1,p}(J_1,\mathbb{R}^n) \hookrightarrow L^q(J_1^0,\mathbb{R}^n)$ is compact.*

Lemma 2.2. *Let $p \in \overline{\mathbb{R}}$ be such that $p \geq 1$, let $\{u_m\}_{m\in\mathbb{N}} \subset W^{1,p}(J_1,\mathbb{R}^n)$, and let $u \in W^{1,p}(J_1,\mathbb{R}^n)$. If $\{u_m\}_{m\in\mathbb{N}}$ converges weakly in $W^{1,p}(J_1,\mathbb{R}^n)$ to u, then $\{u_m\}_{m\in\mathbb{N}}$ converges strongly in $[a_1,b_1]_{\mathbb{T}}$ to u.*

Now, let $W_T^{1,p}(\mathbb{T},\mathbb{R}^n)$ be the Sobolev space given by

$$\begin{aligned} W_T^{1,p}(\mathbb{T},\mathbb{R}^n) = \{&u : [0,T]_{\mathbb{T}} \to \mathbb{R}^n \mid u \text{ is absolutely continuous,} \\ &u(0) = u(T), i \quad u^\Delta \in L_T^p([0,T]_{\mathbb{T}^\kappa},\mathbb{R}^n)\}, \end{aligned}$$

and the norm is defined by

$$\|u\|_{W_T^{1,p}(\mathbb{T},\mathbb{R}^n)} = \left(\int_0^T |u|^p\Delta + \int_0^T |u^\Delta|^p\Delta\right)^{\frac{1}{p}} \quad \text{for } u \in W_T^{1,p}(\mathbb{T},\mathbb{R}^n),$$

where

$$L_T^p([0,T]_{\mathbb{T}^\kappa},\mathbb{R}^n) = \left\{u : [0,T]_{\mathbb{T}^\kappa} \to \mathbb{R}^n, u \in L^1([0,T]_{\mathbb{T}^\kappa},\mathbb{R}^n), |u|^p \in L^1([0,T]_{\mathbb{T}^\kappa},\mathbb{R}^n)\right\}.$$

It is obvious that

$$L_T^p([0,T]_{\mathbb{T}},\mathbb{R}^n) = \left\{u : [0,T]_{\mathbb{T}} \to \mathbb{R}^n,\ u \in L^1([0,T]_{\mathbb{T}},\mathbb{R}^n), |u|^p \in L^1([0,T]_{\mathbb{T}},\mathbb{R}^n)\right\},$$

with the norm defined by

$$\|u\|_{L_T^p(\mathbb{T},\mathbb{R}^n)} = \left(\int_0^T |u|^p\Delta\right)^{\frac{1}{p}} \quad \text{and} \quad \|u\|_\infty = \sup_{t\in[0,T]_{\mathbb{T}}} |u|.$$

We also denote the Hilbert space $H^1_T(\mathbb{T},\mathbb{R}^n) = W^{1,2}_T(\mathbb{T},\mathbb{R}^n)$ with the inner product

$$(u,v) = \int_0^T [u\cdot v + u^\Delta \cdot v^\Delta]\Delta,$$

here u, v are vector-valued and the dot indicates a dot product. Hence, the corresponding norm $\|u\| = \|u\|_{W^{1,2}_T(\mathbb{T},\mathbb{R}^n)} = \|u\|_{H^1_T(\mathbb{T},\mathbb{R}^n)}$.

Motivated by reference [16], we obtain the following two Lemmas which are very important in proving our result.

Lemma 2.3. *There exist $c_1, c_2 > 0$ such that, if $u(t) \in W^{1,p}_T(\mathbb{T},\mathbb{R}^n)$, then*

$$\|u(t)\|_\infty \le c_1\|u(t)\|_{W^{1,p}_T(\mathbb{T},\mathbb{R}^n)}. \tag{2.1}$$

Moreover, if $\int_0^T u(t)\Delta = 0$, then

$$\|u(t)\|_\infty \le c_2\|u^\Delta(t)\|_{L^p_T(\mathbb{T},\mathbb{R}^n)}. \tag{2.2}$$

Proof. Going to the components of $u(t)$, we can assume that $n = 1$. If $u(t) \in W^{1,p}_T(\mathbb{T},\mathbb{R}^n)$, then there exists a $\tau \in [0,T]_\mathbb{T}$ such that $u(\tau) = \inf_{t\in[0,T]_\mathbb{T}} u(t)$, it follows that

$$\frac{1}{T}\int_0^T u(s)\Delta \ge \frac{1}{T}\int_0^T u(\tau)\Delta = u(\tau),$$

thus, there exists constant $c_3 > 0$ such that $|u(\tau)| \le c_3\left|\int_0^T u(s)\Delta\right|$. Hence, for $t \in [0,T]_\mathbb{T}$, according to the Hölder inequality on time scales [7], one can get

$$\begin{aligned}|u(t)| &= \left|u(\tau) + \int_\tau^t u^\Delta(s)\Delta\right| \le |u(\tau)| + \left|\int_\tau^t u^\Delta(s)\Delta\right| \\ &\le c_3\left|\int_0^T u(s)\Delta\right| + T^{\frac{1}{q}}\left(\int_0^T |u^\Delta(s)|^p\Delta\right)^{\frac{1}{p}} = c_3\left|\int_0^T u(s)\Delta\right| + T^{\frac{1}{q}}\|u^\Delta(t)\|_{L^p(\mathbb{T})}.\end{aligned}$$

If $\int_0^T u(s)\Delta = 0$, then (2.2) holds. In the general case, according to Hölder inequality on time scales again, one has

$$\begin{aligned}|u(t)| &\le c_3\left|\int_0^T u(s)\Delta\right| + T^{\frac{1}{q}}\left(\int_0^T |u^\Delta(s)|^p\Delta\right)^{\frac{1}{p}} \\ &\le c_3T^{\frac{1}{q}}\|u\|_{L^p_T(\mathbb{T},\mathbb{R}^n)} + T^{\frac{1}{q}}\|u^\Delta(s)\|_{L^p_T(\mathbb{T},\mathbb{R}^n)} \le \left(c_3T^{\frac{1}{q}} + T^{\frac{1}{q}}\right)\|u\|_{W^{1,p}_T(\mathbb{T},\mathbb{R}^n)},\end{aligned}$$

which implies (2.1) holds. □

Lemma 2.4. *Let $L : [0,T]_\mathbb{T}\times\mathbb{R}^n\times\mathbb{R}^n \to \mathbb{R}$, $(t,x,y) \to L(t,x,y)$ be measurable in t for each $[x,y] \in \mathbb{R}^n\times\mathbb{R}^n$ and continuously differentiable in $[x,y]$ for Δ-a.e. $t \in [0,T]_\mathbb{T}$. If there exists $a \in C(\mathbb{R}^+,\mathbb{R}^+)$, $b \in L^1_T([0,T]_\mathbb{T},\mathbb{R}^+)$ and $c \in L^q_T([0,T]_\mathbb{T},\mathbb{R}^+)$, $1 < q < \infty$,* *such that, for Δ-a.e. $t \in [0,T]_{\mathbb{T}}$ and each $[x,y] \in \mathbb{R}^n \times \mathbb{R}^n$, one has*

$$\begin{cases} |L(t,x,y)| \le a(|x|)(b(t)+|y|^p), \\ |D_x L(t,x,y)| \le a(|x|)(b(t)+|y|^p), \\ |D_y L(t,x,y)| \le a(|x|)(c(t)+|y|^{p-1}), \end{cases} \tag{2.3}$$

where $\frac{1}{p}+\frac{1}{q}=1$, then the functional φ defined by $\varphi(u)=\int_0^T L(t,u(t),u^\Delta(t))\Delta$ is a continuously differential on $W_T^{1,p}(\mathbb{T},\mathbb{R}^n)$ and

$$\langle \varphi'(u), v\rangle = \int_0^T \left[D_x L(t,u(t),u^\Delta(t))\cdot v(t) + D_y L(t,u(t),u^\Delta(t))\cdot v^\Delta(t)\right]\Delta. \tag{2.4}$$

Proof. It suffices to prove that φ has a directional derivative $\varphi'(u) \in \left(W_T^{1,p}(\mathbb{T},\mathbb{R}^n)\right)^*$ given by (2.4) at every point u and that the mapping

$$\varphi' : W_T^{1,p}(\mathbb{T},\mathbb{R}^n) \to \left(W_T^{1,p}(\mathbb{T},\mathbb{R}^n)\right)^*, \quad u \to \varphi'(u)$$

is continuous.

(i) It follows easily from (2.3) that φ is everywhere finite on $W_T^{1,p}(\mathbb{T},\mathbb{R}^n)$. For u and v fixed in $W_T^{1,p}(\mathbb{T},\mathbb{R}^n)$, we define

$$F(\lambda,t) = L(t,u(t)+\lambda v(t), u^\Delta(t)+\lambda v^\Delta(t)) \quad \text{for } t\in[0,T]_{\mathbb{T}} \text{ and } \lambda\in[-1,1],$$

$$\psi(\lambda)=\varphi(u+\lambda v)=\int_0^T F(\lambda,t)\Delta.$$

In the following, we shall apply the Leibniz formula of differentiation under integral sign to ψ. By assumption (2.3), one obtains

$$\begin{aligned} |D_\lambda F(\lambda,E)| &= \left|D_x L\left(E,u+\lambda v,u^\Delta+\lambda v^\Delta\right)\cdot v + D_y L\left(E,u+\lambda v,u^\Delta+\lambda v^\Delta\right)\cdot v^\Delta\right| \\ &\le a(|u+\lambda v|)\left[\left(b+|u^\Delta+\lambda v^\Delta|^p\right)|v| + \left(c+|u^\Delta+\lambda v^\Delta|^{p-1}\right)|v^\Delta|\right] \\ &\le a_0\left[\left(b+(|u^\Delta|+|v^\Delta|)^p]\right)|v| + \left(c+\left(|u^\Delta|+|v^\Delta|\right)^{p-1}\right)|v^\Delta|\right], \end{aligned}$$

where $a_0=\max_{(\lambda,t)\in[-1,1]\times[0,T]} a\left(|u(E)+\lambda v(E)|\right)$.

Since $b\circ E \in L^1([0,T],\mathbb{R}^+)$, $(|u^\Delta|+|v^\Delta|)^p\circ E \in L^1([0,T],\mathbb{R}^+)$, $c\circ E \in L^q([0,T],\mathbb{R}^+)$, $v^\Delta\circ E \in L^p([0,T],\mathbb{R}^+)$ and $v\circ E\in L^1([0,T],\mathbb{R}^+)$, we have

$$|D_\lambda F(\lambda,E(t))| \le d(E(t)) \in L^1([0,T],\mathbb{R}^+).$$

Hence, it follows from the Leibniz formula that

$$\begin{aligned} \psi'(0) = \langle\varphi'(u),v\rangle &= \int_0^T D_\lambda F(0,E(t))dt \\ &= \int_0^T D_x L\left(E(t),u(E(t)),u^\Delta(E(t))\right)\cdot v(E(t))dt \\ &\quad + \int_0^T D_y L\left(E(t),u(E(t)),u^\Delta(E(t))\right)\cdot v^\Delta(E(t))dt \end{aligned}$$

$$= \int_0^T \left[D_x L\left(t, u(t), u^\Delta(t)\right) \cdot v(t) + D_y L\left(t, u(t), u^\Delta(t)\right) \cdot v^\Delta(t)\right] \Delta.$$

Moreover

$$|D_x L(t, u, u^\Delta)| \le a(|u|)(b(t) + |u^\Delta|^p) \in L^1_T([0,T]_{\mathbb{T}}, \mathbb{R}^+)$$
$$|D_y L(t, u, u^\Delta)| \le a(|u|)(c(t) + |u^\Delta|^{p-1}) \in L^q_T([0,T]_{\mathbb{T}}, \mathbb{R}^+).$$

Thus, by Lemma 2.3,

$$\int_0^T D_\lambda F[\lambda, t]\Delta = \int_0^T \left[D_x L(t, u, u^\Delta) \cdot v + D_y L\left(t, u, u^\Delta\right) \cdot v^\Delta\right] \Delta$$
$$\le c_1\|v\|_\infty + c_2\|v^\Delta\|_{L^p(\mathbb{T})} \le c_3\|v\|_{W^{1,p}_T(\mathbb{T})},$$

and φ has a directional derivative $\varphi'(u) \in \left(W^{1,p}_T(\mathbb{T}, \mathbb{R}^n)\right)^*$ at u given by (2.4).

(ii) By Krasnosel'skii's theorem[12], (2.3) implies that the mapping from $W^{1,p}_T(\mathbb{T}, \mathbb{R}^n)$ into $L^1_T([0,T]_{\mathbb{T}}, \mathbb{R}^+) \times L^q_T([0,T]_{\mathbb{T}}, \mathbb{R}^+)$ defined by $u \to \left(D_x L\left(t, u, u^\Delta\right), D_y L\left(t, u, u^\Delta\right)\right)$ is continuous, so that φ' is continuous from $W^{1,p}_T(\mathbb{T}, \mathbb{R}^n)$ into $\left(W^{1,p}_T(\mathbb{T}, \mathbb{R}^n)\right)^*$, and the proof is complete. □

We also need the following lemma, which is presented by Rabinowitz [17].

Lemma 2.5 ([17])**.** *Let $E = V \oplus X$, where X is a real Banach space and $V \ne \{0\}$ and is finite dimensional. Suppose $I \in C^1(E, \mathbb{R})$, satisfies* (P.S.) *condition, and*

(I_1) *there is a constant α and a bounded neighborhood D of 0 in V such that $I|_{\partial D} \le \alpha$;*
(I_2) *there is a constant $\beta > \alpha$ such that $I|_X \ge \beta$.*

Then I possesses a critical value $c^* \ge \beta$. Moreover c^* can be characterized as

$$c^* = \inf_{h \in \Gamma} \max_{u \in \overline{D}} I(h(u)),$$

where $\Gamma = \{h \in C(\overline{D}, E) | h = id \text{ on } \partial D\}$.

3. EXISTENCE RESULTS

In this section, by using the variational methods and critical theory, we obtain a new existence theorem for periodic solutions of the second order Hamiltonian system (1.1) on time scales.

If we let

$$L(t, x, y) = L(t, u(t), u^\Delta(t)) = \frac{1}{2}|u^\Delta(t)|^2 + H(t, u(t)),$$

then by Lemma 2.4, the functional φ given by

$$\varphi(u) = \frac{1}{2}\int_0^T |u^\Delta(t)|^2 \Delta + \int_0^T H(t, u(t))\Delta \tag{3.1}$$

is continuously differentiable on $H_T^1(\mathbb{T},\mathbb{R}^n)$. Hence, we obtain

(3.2)
$$\langle\varphi'(u),v\rangle=\int_0^T u^{\Delta}(t)\cdot v^{\Delta}(t)\Delta+\int_0^T \overline{\bigtriangledown}H(t,u(t))\cdot v(t)\Delta \quad \text{for all } u,v\in H_T^1(\mathbb{T},\mathbb{R}^n).$$

That is, for all $u,v\in H_T^1(\mathbb{T},\mathbb{R}^n)$, one has

$$\begin{aligned}\langle\varphi'(u),v\rangle&=-\int_0^T u^{\Delta\Delta}(\rho(t))\cdot v(t)\Delta+\int_0^T \overline{\bigtriangledown}H(t,u(t))\cdot v(t)\Delta\\&=\int_0^T\big(-u^{\Delta\Delta}(\rho(t))+\overline{\bigtriangledown}H(t,u(t)\big)\cdot v(t)\Delta.\end{aligned}$$

Consequently, $u\in H_T^1(\mathbb{T},\mathbb{R}^n)$ is a solution of problem (1.1) if and only if u is a critical point of φ.

If $u_n\rightharpoonup u_0$ in $H_T^1(\mathbb{T},\mathbb{R}^n)$, then by Lemma 2.2, $u_n\to u_0$ on $[0,T]_{\mathbb{T}}$, this implies that u_n is bounded in $[0,T]_{\mathbb{T}}$. The mean value theorem on time scales [6] implies that there exist $\tau_1^*,\tau_2^*\in[0,T)_{\mathbb{T}}$ satisfying

$$u_n^{\Delta}(\tau_1^*)\le\frac{u_n(T)-u_n(0)}{T}\le u_n^{\Delta}(\tau_2^*). \tag{3.3}$$

From (1.1) and (3.3), we have

$$\begin{aligned}|u_n^{\Delta}(t)|&\le|u_n^{\Delta}(\tau_1^*))|+\left|\int_{\sigma(\tau_1^*)}^{\sigma(t)}\overline{\bigtriangledown}H(t,u(t))\Delta\right|\\&\le\left|\frac{u_n(T)-u_n(0)}{T}\right|+\int_{\sigma(\tau_1^*)}^{T}a(x)b(t)\Delta \text{ for } t\in[0,T]_{\mathbb{T}},\end{aligned}$$

and

$$\begin{aligned}|u_n^{\Delta}(t)|&\ge|u_n^{\Delta}(\tau_2^*)|-\left|\int_{\sigma(\tau_2^*)}^{\sigma(t)}\overline{\bigtriangledown}H(t,u(t))\Delta\right|\\&\ge\left|\frac{u_n(T)-u_n(0)}{T}\right|-\left|\int_{\sigma(\tau_2^*)}^{T}a(x)b(t)\Delta\right| \text{ for } t\in[0,T]_{\mathbb{T}}.\end{aligned}$$

It follows from Lebesgue dominated convergence theorem on time scales [5] that

$$\liminf_{n\to\infty}\varphi(u_n)=\liminf_{n\to\infty}\left(\frac{1}{2}\int_0^T|u_n^{\Delta}(t)|^2\Delta+\int_0^T H(t,u_n(t))\Delta\right)=\varphi(u_0).$$

Hence, φ is weakly lower semi-continuous on $H_T^1(\mathbb{T},\mathbb{R}^n)$.

In order to prove the main result, for $u\in H_T^1(\mathbb{T},\mathbb{R}^n)$, we let

$$\overline{u}(t)=\frac{1}{T}\int_0^T u(t)\Delta,\quad \widetilde{u}(t)=u(t)-\overline{u}(t),$$

and $\widetilde{H}_T^1(\mathbb{T},\mathbb{R}^n)$ is the subspace of $H_T^1(\mathbb{T})$ given by

$$\widetilde{H}_T^1(\mathbb{T},\mathbb{R}^n)=\{u\in H_T^1(\mathbb{T},\mathbb{R}^n)|\overline{u}(t)=0\}.$$

In the following, we shall prove a lemma, which will be used in the proof of the main result.

Lemma 3.1. *If a sequence* $\{u_n(t)\} \subset H_T^1(\mathbb{T}, \mathbb{R}^n)$ *be such that* $\varphi'(u_n(t)) \to 0$ *and* $\{u_n(t)\}$ *is bounded in* $H_T^1(\mathbb{T}, \mathbb{R}^n)$*, then* $\{u_n(t)\}$ *has a convergent subsequence in* $H_T^1(\mathbb{T}, \mathbb{R}^n)$*.*

Proof. From Lemma 2.1, we have

$$H_T^1(\mathbb{T}, \mathbb{R}^n) \hookrightarrow L_T^2([0,T]_{\mathbb{T}}, \mathbb{R}^n)$$

is compact. Hence, there exists a subsequence (still denoted by $\{u_n(t)\}$) and we assume that there is a point $u_0 \in H_T^1(\mathbb{T}, \mathbb{R}^n)$ such that $u_n \rightharpoonup u_0$ in $H_T^1(\mathbb{T}, \mathbb{R}^n)$ and $u_n \to u_0$ in $L_T^2([0,T]_{\mathbb{T}}, \mathbb{R}^n)$. By Lemma 2.2, we have u_n converges uniformly to u_0 on $[0,T]_{\mathbb{T}}$ also.

Hence, there is a $M > 0$ such that $\max_{0 \le t \le T} |u_n(t)| \le M$, $n = 1, 2, \ldots$.

Since

$$\begin{aligned}
&\langle \varphi'(u_n(t)) - \varphi'(u_m(t)), u_n(t) - u_m(t) \rangle \\
&\qquad = \int_0^T \left(u_n^\Delta(t) - u_m^\Delta(t)\right) . \left(u_n^\Delta(t) - u_m^\Delta(t)\right) \Delta \\
&\qquad - \int_0^T \left(\overline{\nabla} H(t, u_n(t)) - \overline{\nabla} H(t, u_m(t))\right) . \left(u_n(t) - u_m(t)\right) \Delta,
\end{aligned}$$

It follows from (H_0) that

$$\begin{aligned}
\int_0^T |u_n^\Delta(t) - u_m^\Delta(t)|^2 \Delta &\le \|\varphi'(u_n(t)) - \varphi'(u_m(t))\| \|u_n(t) - u_m(t)\| \\
&\quad + 2a_M \|u_n(t) - u_m(t)\|_\infty \int_0^T b(t) \Delta \to 0 \text{ as } n, m \to \infty.
\end{aligned}$$

Consequently

$$\|u_n(t) - u_m(t)\|^2 = \int_0^T |u_n^\Delta(t) - u_m^\Delta(t)|^2 \Delta + \int_0^T |u_n(t) - u_m(t)|^2 \Delta \to 0 \text{ as } n, m \to \infty,$$

which implies that $\{u_n\}$ is a Cauchy sequence in $H_T^1(\mathbb{T}, \mathbb{R}^n)$. By the completeness of $H_T^1(\mathbb{T}, \mathbb{R}^n)$, we know $\{u_n\}$ is a convergent sequence in $H_T^1(\mathbb{T}, \mathbb{R}^n)$. □

Now, we list and prove our main result.

Theorem 3.2. *Assume that the following conditions hold:*

(H_1) *there exists a bounded measurable function* $g : [0,T]_{\mathbb{T}} \to \mathbb{R}$ *such that*

$$g(t) \le \lim \inf_{|x| \to \infty} \frac{\overline{\nabla} H(t,x).x}{|x|} \quad \textit{for all } x \in \mathbb{R}^n \quad \textit{and } \Delta\textit{-a.e. } t \in [0,T]_{\mathbb{T}};$$

(H_2) *If* $\{u_n\} \subset H_T^1(\mathbb{T}, \mathbb{R}^n)$ *is such that* $\|u_n\| \to \infty$ *and* $\frac{|\overline{u_n}|\sqrt{T}}{\|u_n\|} \to 1$, *then*

$$\liminf_{n\to\infty} \int_0^T \overline{\nabla} H(t, u_n(t)). \frac{\overline{u_n}}{\|u_n\|} \Delta < 0.$$

Then the problem (1.1) *has at least one solution in* $H_T^1(\mathbb{T}, \mathbb{R}^n)$.

Proof. Firstly, we show that φ satisfies the (P.S.) condition.

By (H_1), there exist $\lambda < 0$ and $M > 0$ such that

$$\overline{\nabla} H(t, x).x \geq \lambda |x| \quad \text{for all } x \in \mathbb{R}^n \text{ with } |x| > M \text{ and } \Delta\text{-a.e. } t \in [0, T]_{\mathbb{T}}.$$

Moreover, let $a_M = \max_{|x| \leq M} a(|x|)$. Then, it follows from (H_0) that

$$\overline{\nabla} H(t, x) \cdot x \geq -a_M b(t)|x| \quad \text{for all } x \in \mathbb{R}^n \text{ with } |x| \leq M \text{ and } \Delta\text{-a.e. } t \in [0, T]_{\mathbb{T}}.$$

Hence

$$\overline{\nabla} H(t, x).x \geq -a_M b(t)|x| + \lambda |x| \quad \text{for all } x \in \mathbb{R}^n \text{ and } \Delta\text{-a.e. } t \in [0, T]_{\mathbb{T}}.$$

Consequently

$$\begin{aligned} H(t, x) &= H(t, x) - H(t, 0) + H(t, 0) \\ &= \int_0^1 \overline{\nabla} H(t, sx).xds + H(t, 0) \geq -a_M b(t)|x| + \lambda |x| + H(t, 0). \end{aligned} \tag{3.4}$$

If a sequence $\{u_n\} \subset H_T^1(\mathbb{T}, \mathbb{R}^n)$ is such that $\varphi'(u_n) \to 0$ and there exists a constant c such that $\varphi(u_n) \leq c$, $n = 1, 2, \ldots$, then $\{u_n\}$ is bounded in $H_T^1(\mathbb{T}, \mathbb{R}^n)$.

Otherwise, there exists a subsequence (still denoted by $\{u_n(t)\}$), we may assume that $\|u_n\| \to \infty$. Let $v_n = \frac{u_n}{\|u_n\|}$. Since $H_T^1(\mathbb{T}, \mathbb{R}^n)$ is Hilbert space and $H_T^1(\mathbb{T}) \hookrightarrow L^2([0, T]_{\mathbb{T}}, \mathbb{R}^n)$ is compact, there exist a point $v_0 \in H_T^1(\mathbb{T}, \mathbb{R}^n)$ and a subsequence of $\{v_n\}$ (we still note by $\{v_n\}$) such that

$$\begin{aligned} &v_n(t) \rightharpoonup v_0(t) \text{ on } H_T^1(\mathbb{T}, \mathbb{R}^n), \\ &v_n(t) \to v_0(t) \text{ on } L_T^2\left([0, T]_{\mathbb{T}}, \mathbb{R}^n\right), \\ &v_n(t) \to v_0(t) \text{on } t \in [0, T]_{\mathbb{T}}, \end{aligned}$$

and there exists a function $\omega \in L_T^2([0, T]_{\mathbb{T}}, \mathbb{R}^n)$ such that $|v_n(t)| \leq \omega(t)$ for Δ-a.e. $t \in [0, T]_{\mathbb{T}}$.

In view of Lemma 2.3, (3.4) and Lebesgue dominated convergence theorem on time scales [5], one has

$$\begin{aligned} \frac{c}{\|u_n\|^2} \geq \frac{\varphi(u_n)}{\|u_n\|^2} &= \frac{1}{2} \int_0^T |v_n^{\Delta}(t)|^2 \Delta + \frac{1}{\|u_n\|^2} \int_0^T H(t, u_n(t)) \Delta \\ &\geq \frac{1}{2} \int_0^T |v_n^{\Delta}(t)|^2 \Delta + \frac{1}{\|u_n\|^2} \int_0^T \left[-a_M b(t)|u_n(t)| + \lambda |u_n(t)| + H(t, 0)\right] \Delta \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2}\int_0^T |v_n^\Delta(t)|^2\Delta - \frac{a_M}{\|u_n\|}\int_0^T b(t)|v_n(t)|\Delta \\
&\quad + \frac{\lambda}{\|u_n\|}\int_0^T |v_n(t)|\Delta + \frac{1}{\|u_n\|^2}\int_0^T H(t,0)\Delta \\
&= \frac{1}{2}\left[1 - \int_0^T |v_n(t)|^2\Delta\right] - \frac{a_M}{\|u_n\|}\int_0^T b(t)|v_n(t)|\Delta \\
&\quad + \frac{\lambda}{\|u_n\|}\int_0^T |v_n(t)|\Delta + \frac{1}{\|u_n\|^2}\int_0^T H(t,0)\Delta,
\end{aligned}
$$

which implies $\int_0^T |v_0(t)|^2\Delta \geq 1$.

On the one hand, in terms of the weak lower semi-continuity of the norm, we obtain

$$
\begin{aligned}
\left[\int_0^T |v_0(t)|^2\Delta + \int_0^T |v_0^\Delta(t)|^2\Delta\right]^{\frac{1}{2}} &= \|v_0(t)\| \\
&\leq \liminf \|v_n(t)\| = 1 \quad \text{for } \Delta\text{-a.e } t \in [0,T]_{\mathbb{T}},
\end{aligned}
$$

which means $v_0^\Delta(t) = 0$ for Δ-a.e $t \in [0,T]_{\mathbb{T}}$. Hence

$$
|v_0(t)| = \text{ constant for } t \in [0,T]_{\mathbb{T}},
$$

which implies $|v_0(t)|^2 = \frac{1}{T}$. Consequently, $\frac{|\overline{u_n}|^2}{\|u_n\|^2} \to \frac{1}{T}$, then

$$
\lim \inf_{n\to\infty} \int_0^T \overline{\triangledown} H(t,u_n(t)).\frac{\overline{u_n}}{\|u_n\|}\Delta < 0.
$$

On the other hand, it follows from (3.2) that

$$
\int_0^T \overline{\triangledown} H(t,u_n(t)) \cdot \frac{\overline{u_n(t)}}{\|u_n(t)\|}\Delta = \left\langle \varphi'(u_n), \frac{\overline{u_n(t)}}{\|u_n(t)\|}\right\rangle \to 0 \qquad \text{as } n \to \infty.
$$

This is a contradiction.

Hence $\{u_n\}$ is bounded in $H_T^1(\mathbb{T},\mathbb{R}^n)$. By Lemma 3.1, $\{u_n\}$ has a convergent subsequence in $H_T^1(\mathbb{T},\mathbb{R}^n)$. Thus, φ satisfies the (P.S.) condition.

Secondly, we shall show that φ is anti-coercive on $\mathbb{R}^n$. That is ,

$$
\varphi(x) \to -\infty \text{ as } |x| \to \infty \qquad \text{for } x \in \mathbb{R}^n, \tag{3.5}
$$

which implies that the condition (I_1) of Lemma 2.5 holds.

In order to obtain (3.5), we first verify that there exist $\delta_1 > 0, \rho_1 > 0$ such that

$$
\int_0^T \overline{\triangledown} H(t,x).x\Delta \leq -\delta_1|x| \qquad \text{for all } x \in \mathbb{R}^n \text{ with } |x| \geq \rho_1. \tag{3.6}
$$

If not, there is a sequence $\{x_n\} \subset \mathbb{R}^n$ with $|x_n| \to \infty$ and

$$
\int_0^T \overline{\triangledown} H(t,x_n) \cdot \frac{\overline{x_n}}{|\overline{x_n}|}\Delta > -\frac{1}{n} \qquad \text{for } n \geq 1,
$$

this contradicts (H_2). Hence, (3.6) is true.

For all $x \in \mathbb{R}^n$ with $|x| > \rho_1$, one has

$$\begin{aligned}\varphi(x) &= \frac{1}{2}\int_0^T |x^{\Delta}|^2\Delta + \int_0^T H(t,x)\Delta \\ &= \int_0^T H(t,x)\Delta = \int_0^T \int_0^1 \overline{\bigtriangledown} H(t,sx).xds\Delta + \int_0^T H(t,0)\Delta \\ &= \int_0^T \left(\int_0^{\frac{\rho_1}{|x|}} \overline{\bigtriangledown} H(t,sx).xds + \int_{\frac{\rho_1}{|x|}}^1 \overline{\bigtriangledown} H(t,sx)\cdot xds\right)\Delta + c_6,\end{aligned} \tag{3.7}$$

where $c_6 = \int_0^T H(t,0)\Delta$.

According to (H_0), we get

$$|\int_0^T \int_0^{\frac{\rho_1}{|x|}} \overline{\bigtriangledown} H(t,sx).xds\Delta| \le \int_0^T \int_0^{\frac{\rho_1}{|x|}} |\overline{\bigtriangledown} H(t,sx)||x|ds\Delta \le \int_0^T \rho_1 a_{\rho_1} b(t)\Delta, \tag{3.8}$$

where $a_{\rho_1} = \max_{|x|\le\rho_1} a(|x|)$.

It follows from (3.6) that

$$\begin{aligned}&\int_0^T \int_{\frac{\rho_1}{|x|}}^1 \overline{\bigtriangledown} H(t,sx)\cdot x\,ds\,\Delta = \int_0^T \int_{\frac{\rho_1}{|x|}}^1 \overline{\bigtriangledown} H(E(t),sx)\cdot x\,ds\,dt \\ &\quad= \int_{\frac{\rho_1}{|x|}}^1 \left(\int_0^T \overline{\bigtriangledown} H(E(t),sx).x\right) dt\,ds \\ &= \int_{\frac{\rho_1}{|x|}}^1 \left(\int_0^T \overline{\bigtriangledown} H(t,sx)\cdot x\right)\Delta\,ds \le -\delta_1|x|\left(1-\frac{\rho_1}{|x|}\right) = -\delta_1|x| + \delta_1\rho_1.\end{aligned} \tag{3.9}$$

(3.7), (3.8) and (3.9) imply that (3.5) is satisfied. Consequently, φ is anti-coercive on $\mathbb{R}^n$.

Finally, we shall prove that φ is coercive on $\widetilde{H}_T^1(\mathbb{T},\mathbb{R}^n)$, which implies that the condition (I_2) of Lemma 2.5 holds.

If there are a constant c^* and a sequence $\{u_n\} \subset \widetilde{H}_T^1(\mathbb{T},\mathbb{R}^n)$ such that $\|u_n\| \to \infty$ and $\varphi(u_n) \le c^*, n = 1,2,\ldots$, then, according to Lemma 2.3, (3.4), $\lambda < 0$ and the Hölder inequality on time scales [7], we have

$$\begin{aligned}c^* \ge \varphi(u_n) &= \frac{1}{2}\int_0^T |u_n^{\Delta}(t)|^2\Delta + \int_0^T H(t,u_n(t))\Delta \\ &\ge \frac{1}{2}\int_0^T |u_n^{\Delta}(t)|^2\Delta + \int_0^T [-a_M b(t)|u_n(t)| + \lambda|u_n(t)| + H(t,0)]\,\Delta \\ &= \frac{1}{2}\|u_n\|^2 - \frac{1}{2}\int_0^T |u_n(t)|^2\Delta + \int_0^T [-a_M b(t)|u_n(t)| + \lambda|u_n(t)| + H(t,0)]\,\Delta \\ &\ge \frac{1}{2}\|u_n\|^2 - \frac{1}{2}\|u_n(t)\|_\infty \int_0^T |u_n(t)|\Delta + \int_0^T [-a_M b(t)|u_n(t)| + \lambda|u_n(t)| + H(t,0)]\,\Delta \\ &= \frac{1}{2}\|u_n\|^2 - \left(\frac{1}{2}\|u_n(t)\|_\infty + a_M B - \lambda\right)\int_0^T |u_n(t)|\Delta + \int_0^T H(t,0)\Delta\end{aligned}$$

$$\geq \frac{1}{2}\|u_n\|^2 - T^{\frac{1}{2}}\left(\frac{1}{2}\|u_n(t)\|_\infty + a_M B - \lambda\right)\left(\int_0^T |u_n(t)|^2\Delta\right)^{\frac{1}{2}} + \int_0^T H(t,0)\Delta$$
$$> c_3\|u_n\|^2 - c_4\|u_n\| - c_5,$$

where c_3, c_4, c_5 are positive constants and $B = \sup_{t\in[0,T]_{\mathbb{T}}} u(t)$. This contradicts to $\|u_n\| \to \infty$. Consequently, φ is coercive on $\widetilde{H}_T^1(\mathbb{T}, \mathbb{R}^n)$.

Now, it is easy to see that all conditions of Lemma 2.5 are satisfied. Hence, the problem (1.1) has at least one solution in $H_T^1(\mathbb{T}, \mathbb{R}^n)$. □

4. AN EXAMPLE

In this section, we present a simple example to illustrate our result.

Let

$$\mathbb{T} = \left\{\frac{1}{4^n}\right\} \cup \{0,\ 0.35,\ 0.4,\ 0.45,\ 0.5,\ 0.55,\ 0.6,\ 0.65\} \cup [0.7, 1] \text{ and } T = 1,$$

here $n \in \mathbb{N}$.

Consider the following second order Hamiltonian systems on time scales $\mathbb{T}$ of the form

$$\begin{cases} u^{\Delta\Delta}(\rho(t)) = \overline{\bigtriangledown} H(t, u(t))\ \Delta\text{-a.e. } t \in [0,1)_{\mathbb{T}}, \\ u(0) - u(1) = u^{\Delta}(0) - u^{\Delta}(1) = 0, \end{cases} \tag{4.1}$$

where $H(t,u) = -t^2 u(t)$. It is easy to verify that (H_0) and all conditions of Theorem 3.2 are satisfied. By Theorem 3.2 we see that the problem (4.1) has at least *one* solution.

REFERENCES

[1] R. P. Agarwal, V. O. Espinar, K. Perera and D. R. Vivero, Multiple positive solutions of singular Dirichlet problems on time scales via variational methods, *Nonlinear Anal.*, 67: 368–381, 2007.

[2] R. P. Agarwal, V. O. Espinar, K. Perera and D. R. Vivero, Existence of multiple positive solutions for second order nonlinear dynamic BVPs by variational methods, *J. Math. Anal. Appl.*, 331: 1263–1274, 2007 .

[3] R. P. Agarwal, V. O. Espinar, K. Perera and D. R. Vivero, Basic properties of Sobolev's spaces on bounded time scales, *Adv. Difference Equ.*, 67: 368–381, 2006.

[4] F. M. Atici and S. G. Topal, Nonlinear three point boundary value problems on time scales, *Dynam. Systems Appl.*, 13: 327–337, 2004.

[5] B. Aulbach and L. Neidhart, *Integration on measure chains, in: Proceedings of the Sixth International Conference on Difference Equations*, CRC, Boca Raton, FL, 239–252, 2004.

[6] M. Bohner and A. Peterson, *Advances in Dynamic Equation on Time Scales*, Birkhäuser, Boston, 2003.

[7] M. Bohner and A. Peterson, *Dynamic Equation on Time Scales: An Introduction with Applications*, Birkhäuser, Boston, MA, 2001.

[8] A. Cabada and D. R. Vivero, Criterions for absolutely continuity on time scales, *J. Difference Equ. Appl.*, 11: 1013–1028, 2005.

[9] F. A. Davidson and B. P. Rynne, Eigenfunction expansions in L^2 spaces for boundary value problems on time-scales, *J. Math. Anal. Appl.*, 335: 1038–1051, 2007.

[10] G. Sh. Guseinov, Integration on time scales, *J. Math. Anal. Appl.*, 285: 107–127, 2003.

[11] S. Hilger, Analysis on measure chains–a unified approach to continuous and discrete calculus, *Results Math.*, 18: 18–56, 1990.

[12] M. A. Krasnosel'skii, *Topological Methods in the Theory of Nonlinear Integral Equations,* A Pergamon Press Book, The Macmillan Co., New York, 1964.

[13] V. Lakshmikantham, S. Sivasundaram and B. Kaymakcalan, *Dynamic Systems on Measure Chains*, Kluwer Academic Publishers, Boston, 1996.

[14] W. T. Li and H. R. Sun, Multiple positive solutions for nonlinear dynamic systems on a measure chain, *J. Comput. Appl. Math.*, 162: 421–430, 2004 .

[15] R. Ma, Positive solutions of a nonlinear m-point boundary value problem, *Comput. Math. Appl.*, 42: 755–765, 2001.

[16] J. Mawhin and M. Willen, *Critical Point Theory and Hamiltonian Systems*, *Appl. Math. Sci.* Vol 74, Springer-Verlag, New York, 1989.

[17] P. H. Rabinowitz, *Minimax Method in Critical Point Theory with Applications to Differential Equations, in: CBMS Regional Conference Series in Mathematics*, *Amer. Math. Soc.* Vol 65, Providence, RI, 1986.

[18] B. P. Rynne, L^2 spaces and boundary value problems on time-scales, *J. Math. Anal. Appl.*, 328: 1217–1236, 2007.

[19] Y. H. Su, Multiple positive pseudo-symmetric solutions of p-Laplacian dynamic equations on time scales, *Math. Comput. Modelling,* in press.

[20] Y. H. Su and W. T. Li, Periodic solutions for second order Hamiltonian systems with a change sign potential on time scales, submitted.

[21] Y. H. Su and W. T. Li, Triple positive solutions of m-point BVPs for p-Laplacian dynamic equations on time scales, *Nonlinear Anal.*, 69: 3811–3820, 2008.

[22] Y. H. Su, W. T. Li and H. R. Sun, Triple positive pseudo-symmetric solutions of three-point BVPs for p-Laplacian dynamic equations on time scales, *Nonlinear Anal.*, 68: 1442–1452, 2008.

[23] Y. H. Su, W. T. Li and H. R. Sun, Positive solutions of singular p-Laplacian BVPs with sign changing nonlinearity on time scales, *Math. Comput. Modelling*, 48: 845–858, 2008.

[24] Y. H. Su, W. T. Li and H. R. Sun, Positive solutions of singular p-Laplacian dynamic equations with sign changing nonlinearity, *Appl. Math. Comput.*, 200: 352–368, 2008.

[25] H. R. Sun and W. T. Li, Existence theory for positive solutions to one-dimensional p-Laplacian boundary value problems on time scales, *J. Differential Equations*, 240: 217–248, 2007.

[26] J. P. Sun, Existence of solution and positive solution of BVP for nonlinear third-order dynamic equation, *Nonlinear Anal.*, 64: 629–636, 2006.

[27] J. P. Sun and W. T. Li, Existence of positive solutions to semipositone Dirichlet BVPs on time scales, *Dynam. Systems Appl.*, 16: 571–578, 2007.

[28] X. Wu, S. H. Chen and F. Zhao, New existence multiplicity theorems of periodic solutions for non-autonomous second order Hamiltonian systems, *Math. Comput. Modelling*, 46: 550–556, 2007.

[29] X. Wu and C. Tang, Periodic solutions of nonautonomous second-order Hamiltonian systems with even-typed potentials, *Nonlinear Anal.*, 55: 759–769, 2003.

[30] J. Yu, H. Bin and Z. Guo, Multiple periodic solutions for discrete Hamiltonian systems, *Nonlinear Anal.*, 66: 1498–1512, 2007.