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ON POSITIVE SOLUTIONS FOR HIGHER-ORDER BOUNDARY VALUE PROBLEMS WITH IMPULSE

ILKAY YASLAN KARACA

Department of Mathematics Ege University
35100 Bornova, Izmir,Turkey

ABSTRACT. In this paper, we consider a higher-order boundary value problem with impulse. We study the existence of at least one positive solution of an eigenvalue problem. Later, we establish the criteria for the existence of at least two positive solutions of a non-eigenvalue problem. Examples are also included to illustrate our results.

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1. INTRODUCTION

We are concerned with the higher-order boundary value problem with impulse (BVPI)

$$
\begin{cases}
(-1)^n y^{(2n)}(x) = f(x, y(x)), & t \in [a, c) \cup (c, b], \\
y^{(2i)}(c-0) = d_{i+1} y^{(2i)}(c+0), & \\
y^{(2i+1)}(c-0) = \rho_{i+1} y^{(2i+1)}(c+0), & \\
\alpha_{i+1} y^{(2i)}(a) - \beta_{i+1} y^{(2i+1)}(a) = 0, & \\
\gamma_{i+1} y^{(2i)}(b) + \delta_{i+1} y^{(2i+1)}(b) = 0, & 0 \le i \le n-1,
\end{cases}
\tag{1.1}
$$

and the eigenvalue problem $(-1)^n y^{(2n)}(x) = \lambda f(x, y(x))$ with the same boundary conditions where $\lambda > 0$. Here $a < \frac{3a+b}{4} < c < \frac{3b-a}{4} < b$, $y(c-0)$ is the left-hand limit of $y(x)$ at c and $y(c+0)$ is the right-hand limit of $y(x)$ at c.

We assume that the following conditions are satisfied:

(H1) For each $1 \le i \le n$, $d_i > 0$, $\rho_i > 0$, α_i, β_i, γ_i, $\delta_i \ge 0$, $\alpha_i\delta_i + \beta_i\gamma_i + \alpha_i\gamma_i(b-a) > 0$.

(H2) $f(x, \xi)$ is a real-valued function continuous with respect to the collection of its arguments $x \in [a, c) \cup (c, b]$ and $\xi \in \mathbb{R}$, and $f(x, \xi) \ge 0$ for $\xi \in \mathbb{R}^+$, where $\mathbb{R}^+$ denotes the set of nonnegative real numbers. Moreover, for each $\xi \in \mathbb{R}$ there exist finite limits $\lim_{\substack{(x,\xi)\to(c,\xi_0)\\ x<c}} f(x, \xi) = f(c-0, \xi_0)$, $\lim_{\substack{(x,\xi)\to(c,\xi_0)\\ x>c}} f(x, \xi) = f(c+0, \xi_0)$.

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Positive solutions of boundary value problems for differential equations with impulse were earlier studied in [6, 9]. For the basic concepts of impulse differential equations we refer to [2, 16]. Bereketoglu and Huseynov [3] studied nonlinear second-order differential equations subject to seperated linear boundary conditions and to linear impulse conditions by using the Krasnoselskii fixed point theorem. Karaca [12] was interested in proving the existence and multiplicity results for positive solutions to a fourth-order boundary value problem with impulse. For some recent works on the impulsive differential equations we refer the reader to [4, 11, 14, 15, 17].

Higher-order boundary value problems have been studied in recent years [5, 7, 9]. To the author's knowledge, no one has studied of positive solutions for higher-order boundary value problem with impulse.

In this paper, criteria for the existence of at least one positive solution of the eigenvalue problem are first established as a result of the Krasnosel'skii fixed-point theorem. Second, we investigate the existence of at least two positive solutions of BVPI (1.1) by using Avery-Henderson fixed point theorem. Finally, as an application, we also give some examples to demonstrate our results.

2. THE PRELIMINARY LEMMAS

For $1 \leq i \leq n$, denote by θ_i and φ_i the solutions of the homogeneous problem

$$\begin{cases} y''(x) = 0, x \in [a, c) \cup (c, b], \\ y(c-0) = d_i y(c+0), \quad y'(c-0) = \rho_i y'(c+0), \end{cases} \tag{2.1}$$

satisfying the initial conditions

$$\theta_i(a) = \beta_i, \quad \theta_i'(a) = \alpha_i,$$
$$\varphi_i(b) = \delta_i, \quad \varphi_i'(b) = -\gamma_i.$$

Define the number D_i by

$$D_i = \begin{cases} -\beta_i \varphi_i'(a) + \alpha_i \varphi_i(a), & x \in [a, c), \\ \frac{1}{d_i \rho_i}[-\beta_i \varphi_i'(a) + \alpha_i \varphi_i(a)], & x \in (c, b]. \end{cases} \tag{2.2}$$

Lemma 2.1 ([3]). *Let condition* (H1) *hold. For $1 \leq i \leq n$, the number D_i defined by* (2.2) *is positive for $x \in [a, c) \cup (c, b] \cup \{c \pm 0\}$.*

For $1 \leq i \leq n$, let $G_i(x, s)$ be the Green's function for the boundary value problem

$$\begin{cases} y''(x) = 0, & x \in [a, c) \cup (c, b], \\ y(c-0) = d_i y(c+0), & y'(c-0) = \rho_i y'(c+0) \\ \alpha_i y(a) - \beta_i y'(a) = 0, & \gamma_i y(b) + \delta_i y'(b) = 0 \end{cases}$$

which is given by

$$G_i(x,s)=\frac{1}{D_i}\begin{cases}\theta_i(s)\varphi_i(x), & \text{if } a\le s\le x\le b,\\ \theta_i(x)\varphi_i(s), & \text{if } a\le x\le s\le b.\end{cases}\tag{2.3}$$

Lemma 2.2 ([3]). *Let condition* (H1) *hold. For* $1\le i\le n$, *then*

$$G_i(x,s)\ge 0,\qquad \textit{for } x,s\in[a,c)\cup(c,b].$$

Lemma 2.3. *Let condition* (H1) *hold. For* $1\le i\le n$, *then*

$$G_i(x,s)\le G_i(s,s),\quad \textit{for } x,s\in[a,c)\cup(c,b],$$

and

$$G_i(x,s)\ge m_iG_i(s,s),\quad \textit{for } x\in\left[\frac{3a+b}{4},c)\cup(c,\frac{3b-a}{4}\right],\quad s\in[a,c)\cup(c,b],$$

where

$$m_i=\min\left\{\frac{\alpha_i(b-a)+4\beta_i}{4\alpha_i(b-a)+4\beta_i},\frac{\gamma_i(b-a)+4\delta_i}{4\gamma_i(b-a)+4\delta_i}\right\}.\tag{2.4}$$

Proof. After some easy calculations one can see these equalities. □

Lemma 2.4. *Assume that condition* (H1) *is satisfied. For* G *as in* (2.3), *take* $H_1(x,s):=G_1(x,s)$, *and recursively define*

$$H_j(x,s)=\int_a^b H_{j-1}(x,r)G_j(r,s)\Delta r$$

for $2\le j\le n$. *Then* $H_n(x,s)$ *is Green's function for the homogenous problem*

$$\begin{cases}(-1)^n y^{(2n)}(x)=0, & x\in[a,c)\cup(c,b],\\ y^{(2i)}(c-0)=d_{i+1}y^{(2i)}(c+0), & \\ y^{(2i+1)}(c-0)=\rho_{i+1}y^{(2i+1)}(c+0), & \\ \alpha_{i+1}y^{(2i)}(a)-\beta_{i+1}y^{(2i+1)}(a)=0, & \\ \gamma_{i+1}y^{(2i)}(b)+\delta_{i+1}y^{(2i+1)}(b)=0, & 0\le i\le n-1.\end{cases}$$

Lemma 2.5. *Assume* (H1) *holds. If we define*

$$K=\Pi_{j=1}^{n-1}K_j,\qquad L=\Pi_{j=1}^{n-1}m_jL_j$$

then the Green's function $H_n(x,s)$ *in Lemma* 2.4 *satisfies*

$$0\le H_n(x,s)\le KG_n(s,s),\quad (x,s)\in([a,c)\cup(c,b])\times([a,c)\cup(c,b])$$

and

$$H_n(t,s)\ge m_nLG_n(s,s),\quad (x,s)\in\left(\left[\frac{3a+b}{4},c)\cup(c,\frac{3b-a}{4}\right]\right)\times([a,c)\cup(c,b]),$$

where m_n *is given in* (2.4),

$$L_j := \int_{\frac{3a+b}{4}}^{\frac{3b-a}{4}} G_j(s,s)ds > 0, \quad 1 \le j \le n. \tag{2.5}$$

and

$$K_j := \int_a^b G_j(s,s)ds > 0, \quad 1 \le j \le n. \tag{2.6}$$

Proof. Use induction on n and Lemma 2.3. □

3. EXISTENCE OF ONE POSITIVE SOLUTION

In this section we consider the following BVPI with parameter λ,

$$\begin{cases} (-1)^n y^{(2n)}(x) = \lambda f(x, y(x)), & x \in [a,c) \cup (c,b] \\ y^{(2i)}(c-0) = d_{i+1} y^{(2i)}(c+0), \\ y^{(2i+1)}(c-0) = \rho_{i+1} y^{(2i+1)}(c+0), \\ \alpha_{i+1} y^{(2i)}(a) - \beta_{i+1} y^{(2i+1)}(a) = 0, \\ \gamma_{i+1} y^{(2i)}(b) + \delta_{i+1} y^{(2i+1)}(b) = 0, & 0 \le i \le n-1, \end{cases} \tag{3.1}$$

Define the nonnegative extended real numbers f_0, f^0, f_∞ and f^∞ by

$$f_0 := \liminf_{y \to 0^+} \min_{x \in [a,c) \cup (c,b]} \frac{f(x,y)}{y}, \quad f^0 := \limsup_{y \to 0^+} \max_{x \in [a,c) \cup (c,b]} \frac{f(x,y)}{y},$$
$$f_\infty := \liminf_{y \to \infty} \min_{x \in [a,c) \cup (c,b]} \frac{f(x,y)}{y}, \quad f^\infty := \limsup_{y \to \infty} \max_{x \in [a,c) \cup (c,b]} \frac{f(x,y)}{y},$$

respectively. These numbers can be regarded as generalized super or sublinear conditions on the function $f(x,y)$ at $y = 0$ and $y = \infty$. Thus, if $f_0 = f^0 = 0$ $(+\infty)$, then $f(x,y)$ is superlinear (sublinear) at $y = 0$ and if $f_\infty = f^\infty = 0$ $(+\infty)$, then $f(x,y)$ is sublinear (superlinear) at $y = +\infty$. Let

$$M = m_n \prod_{j=1}^{n-1} \frac{m_j L_j}{K_j}. \tag{3.2}$$

We need the following fixed-point theorem (Krasnosel'skii fixed-point theorem) to prove the existence at least one positive solution to BVPI (3.1).

Theorem 3.1 ([10, 13])**.** *Let* $\mathcal{B}$ *be a Banach space, and let* $\mathcal{P} \subset \mathcal{B}$ *be a cone. Assume* Ω_1 *and* Ω_2 *are open bounded subsets of* $\mathcal{B}$ *with* $0 \in \Omega_1$, $\overline{\Omega}_1 \subset \Omega_2$, *and let*

$$A : \mathcal{P} \cap (\overline{\Omega}_2 \setminus \Omega_1) \to \mathcal{P}$$

be a completely continuous operator such that either

(i) $\|Ay\| \le \|y\|$, $y \in \mathcal{P} \cap \partial\Omega_1$, $\|Ay\| \ge \|y\|$, $y \in \mathcal{P} \cap \partial\Omega_2$; *or*
(ii) $\|Ay\| \ge \|y\|$, $y \in \mathcal{P} \cap \partial\Omega_1$, $\|Ay\| \le \|y\|$, $y \in \mathcal{P} \cap \partial\Omega_2$,

holds. Then A *has a fixed point in* $\mathcal{P} \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

Theorem 3.2. *Assume that conditions* (H1) *and* (H2) *are satisfied. Then, for each* λ *satisfying*

(a)

$$\frac{1}{Mm_n\Pi_{j=1}^n L_j f_\infty} < \lambda < \frac{1}{\Pi_{j=1}^n K_j f_0}, \tag{3.3}$$

or

(b)

$$\frac{1}{Mm_n\Pi_{j=1}^n L_j f_0} < \lambda < \frac{1}{\prod_{j=1}^n K_j f_\infty}, \tag{3.4}$$

there exists at least one positive solution of the BVPI (3.1)*, where* m_n, L_j, K_j, M *are as in* (2.4)–(2.6) *and* (3.2)*, respectively. Moreover, in the case* f *is superlinear (sublinear), then equation* (3.3) (*equation* (3.4)) *becomes* $0 < \lambda < \infty$.

Proof. Define $\mathcal{B}$ to be Banach space of all continuous functions on $[a,c) \cup (c,b]$ equipped with the norm $\|.\|$ defined by

$$\|y\| = \max_{x\in[a,c)\cup(c,b]} |y(x)|.$$

Define the cone $\mathcal{P} \subset \mathcal{B}$ by

$$\mathcal{P} = \left\{ y \in \mathcal{B} : y(x) \geq 0, \min_{x\in[\frac{3a+b}{4},c)\cup(c,\frac{3b-a}{4}]} y(x) \geq M\|y\| \right\},$$

where M is as in (3.2). Define an operator A_λ by

$$A_\lambda y(x) = \lambda \int_a^b H_n(x,s) f(s,y(s))ds$$

for $x \in [a,c) \cup (c,b]$. The solutions of the BVPI (3.1) are the fixed points of the operator A_λ.

Firstly, we show that $A_\lambda : \mathcal{P} \to \mathcal{P}$. Note that $y \in \mathcal{P}$ implies that $A_\lambda y(x) \geq 0$ on $[a,c) \cup (c,b]$ and

$$\begin{aligned}\min_{x\in[\frac{3a+b}{4},c)\cup(c,\frac{3b-a}{4}]} A_\lambda y(x) &= \lambda \int_a^b \min_{x\in[\frac{3a+b}{4},c)\cup(c,\frac{3b-a}{4}]} H_n(x,s) f(s,y(s))ds \\ &\geq M\lambda \int_a^b \max_{x\in[a,c)\cup(c,b]} H_n(x,s) f(s,y(s))ds\end{aligned}$$

by Lemma 2.5. It follows that

$$\min_{x\in[\frac{3a+b}{4},c)\cup(c,\frac{3b-a}{4}]} A_\lambda y(x) \geq M\|A_\lambda y\|.$$

Hence $A_\lambda y \in \mathcal{P}$ and so $A_\lambda : \mathcal{P} \to \mathcal{P}$ which is what we want to prove. Moreover since $H_n(x,s)$ and $f(x,y)$ are piece-wise continuous we can prove, in a standard way, that the operator A_λ is completely continuous in $\mathcal{B}$.

Assume that (a) holds. Since $\lambda < \frac{1}{\prod_{j=1}^{n} K_j f_0}$, there exists $\epsilon_1 > 0$ so that

$$0 < \lambda \leq 1/\prod_{j=1}^{n} K_j(f_0 + \epsilon_1).$$

Using the definition of f_0, there is an $r_1 > 0$, sufficiently small, so that

$$f(x,y) \leq (f_0 + \epsilon_1)y \quad \text{for } 0 < y \leq r_1, \quad x \in [a,c) \cup (c,b].$$

If $y \in \mathcal{P}$, with $\|y\| = r_1$, then

$$\begin{aligned} A_\lambda y(x) &= \lambda \int_a^b H_n(x,s) f(s, y(s)) ds \\ &\leq \lambda(f_0 + \epsilon_1) \int_a^b H_n(x,s) y(s) ds \\ &\leq \lambda(f_0 + \epsilon_1) \|y\| K \int_a^b G_n(s,s) ds \\ &\leq \lambda(f_0 + \epsilon_1) \prod_{j=1}^{n} K_j \|y\| \\ &\leq \|y\| \end{aligned}$$

for $x \in [a,c) \cup (c,b]$. So, if we set $\Omega_1 := \{y \in \mathcal{P} : \|y\| < r_1\}$, then $\|A_\lambda y\| \leq \|y\|$ for $y \in \mathcal{P} \cap \partial\Omega_1$.

Now, we use assumption $\frac{1}{Mm_n \prod_{j=1}^{n} L_j f_\infty} < \lambda$.

First, we consider the case when $f_\infty < \infty$. In this case pick an $\epsilon_2 > 0$ so that

$$\lambda M m_n \prod_{j=1}^{n} L_j (f_\infty - \epsilon_2) \geq 1.$$

Using the definition f_∞, there exists $\overline{r}_2 > r_1$, sufficiently large, so that

$$f(x,y) \geq (f_\infty - \epsilon_2) y \quad \text{for } y \geq \overline{r}_2, \quad x \in [a,c) \cup (c,b].$$

We now show that there exists $r_2 \geq \overline{r}_2$ such that if $y \in \partial\mathcal{P}_{r_2}$, then $\|A_\lambda y\| \geq \|y\|$. Let $r_2 = \max\{2r_1, \frac{1}{M}\overline{r}_2\}$ and set $\Omega_2 := \{y \in \mathcal{P} : \|y\| < r_2\}$. If $y \in \mathcal{P} \cap \partial\Omega_2$, then

$$\min_{x \in [a,c) \cup (c,b]} y(x) \geq M\|y\| = M r_2 \geq \overline{r}_2,$$

and so

$$\begin{aligned} A_\lambda y(x) &= \lambda \int_a^b H_n(x,s) f(s, y(s)) ds \\ &\geq \lambda(f_\infty - \epsilon_2) \int_a^b H_n(x,s) y(s) ds \\ &\geq \lambda(f_\infty - \epsilon_2) \int_{\frac{3a+b}{4}}^{\frac{3b-a}{4}} H_n(x,s) y(s) ds \end{aligned}$$

$$\geq \lambda(f_\infty - \epsilon_2)M\|y\|m_n L \int_{\frac{3a+b}{4}}^{\frac{3b-a}{4}} G_n(s,s)ds$$

$$\geq \lambda(f_\infty - \epsilon_2)Mm_n \prod_{j=1}^{n} L_j\|y\|$$

$$\geq \|y\| = r_2.$$

Consequently, $\|A_\lambda y(x)\| \geq \|y(x)\|$, for $x \in [a,c) \cup (c,b]x$.

Finally, we consider the case $f_\infty = \infty$. In this case the hypothesis becomes $\lambda > 0$. Choose $N > 0$ sufficiently large so that

$$\lambda N M m_n \prod_{j=1}^{n} L_j \geq 1.$$

Hence there exists $\overline{r}_2 > r_1$ so that $f(x,y) \geq Ny$ for $y \geq \overline{r}_2$ and for all $x \in [a,c) \cup (c,b]$. Now define r_2 as before and assume $y \in \partial\mathcal{P}_{r_2}$. Then

$$A_\lambda y(x) \geq \lambda N \int_a^b H_n(x,s)y(s)ds$$

$$\geq \lambda N M\|y\|m_n L \int_{\frac{3a+b}{4}}^{\frac{3b-a}{4}} G_n(s,s)ds$$

$$= \lambda N M m_n \prod_{j=1}^{n} L_j\|y\|$$

$$\geq \|y\| = r_2$$

for $x \in [a,c) \cup (c,b]$. Hence $\|A_\lambda y\| \leq \|y\|$ for $y \in \mathcal{P} \cap \partial\Omega_1$ and $\|A_\lambda y\| \geq \|y\|$ for $y \in \mathcal{P} \cap \partial\Omega_2$ hold. Then A_λ has a fixed point in $\mathcal{P} \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

Now we show (b). Since $\frac{1}{Mm_n\Pi_{j=1}^{n}L_j f_0} < \lambda$, there exists $\epsilon_3 > 0$ so that

$$\lambda M m_n \prod_{j=1}^{n} L_j(f_0 - \epsilon_3) \geq 1.$$

From the definition of f_0, there exists an $r_3 > 0$ such that $f(x,y) \geq (f_0 - \epsilon_3)y$ for $0 < y \leq r_3$. If $y \in \mathcal{P}$ with $\|y\| = r_3$, then

$$A_\lambda y(x) = \lambda \int_a^b H_n(x,s)f(s,y(s))ds$$

$$\geq \lambda(f_0 - \epsilon_3) \int_{\frac{3a+b}{4}}^{\frac{3b-a}{4}} H_n(x,s)y(s)ds$$

$$\geq \lambda M(f_0 - \epsilon_3)\|y\|m_n L \int_{\frac{3a+b}{4}}^{\frac{3b-a}{4}} G_n(s,s)ds$$

$$= \lambda(f_0 - \epsilon_3)Mm_n \prod_{j=1}^{n} L_j\|y\|$$

$$\geq \|y\| = r_3.$$

Hence $\|A_\lambda y\| \geq \|y\|$. So, if we set $\Omega_3 := \{y \in \mathcal{P} : \|y\| < r_3\}$, then $\|Ay\| \geq \|y\|$ for $y \in \mathcal{P} \cap \partial\Omega_3$.

Now, we use assumption $\frac{1}{\prod_{j=1}^n K_j f_\infty} > \lambda$. Pick an $\epsilon_4 > 0$ so that

$$\lambda \prod_{j=1}^n K_j(f_\infty + \epsilon_4) \leq 1.$$

Using definition of f_∞, there exists an $\overline{r}_4 > 0$ such that $f(x, y) \leq (f_\infty + \epsilon_4)y$ for all $y \geq \overline{r}_4$. We consider the two cases.

Case I. Suppose $f(x, y)$ is bounded on $([a, c) \cup (c, b]) \times (0, \infty)$. In this case, there is $N > 0$ such that $f(x, y) \leq N$ for $x \in [a, c) \cup (c, b]$, $y \in (0, \infty)$. Let $r_4 = \max\{2r_3, \lambda N \prod_{j=1}^n K_j\}$. Then for $y \in \mathcal{P}$ with $\|y\| = r_4$,

$$\begin{aligned} A_\lambda y(x) &= \lambda \int_a^b H_n(x, s) f(s, y(s)) ds \\ &\leq \lambda N K \int_a^b G_n(s, s) ds \\ &= \lambda N \prod_{j=1}^n K_j \\ &\leq \|y\| = r_4, \end{aligned}$$

so that $\|A_\lambda y\| \leq \|y\|$.

Case II. Suppose $f(x, y)$ is unbounded on $[a, c) \cup (c, b] \times (0, \infty)$. In this case,

$$g(r) := \max\{f(x, y) : x \in [a, c) \cup (c, b], 0 \leq y \leq r\}$$

satisfies

$$\lim_{r\to\infty} g(r) = \infty.$$

We can therefore choose

$$r_4 = \max\{2r_3, \overline{r}_4\}$$

such that

$$g(r_4) \geq g(r)$$

for $0 < r \leq r_4$ and hence for $y \in \mathcal{P}$ and $\|y\| = r_4$, we have

$$\begin{aligned} A_\lambda y(x) &= \lambda \int_a^b H_n(x, s) f(s, y(s)) ds \\ &\leq \lambda \int_a^b H_n(x, s) g(r_4) ds \\ &\leq \lambda (f_\infty + \epsilon_4) r_4 K \int_a^b G_n(s, s) ds \end{aligned}$$

$$= \lambda(f_\infty + \epsilon_4) \prod_{j=1}^{n} K_j r_4$$
$$\leq r_4 = \|y\|,$$

and again we hence have $\|A_\lambda y\| \leq \|y\|$ for $y \in \mathcal{P} \cup \partial\Omega_4$, where $\Omega_4 = \{y \in \mathcal{B} : \|y\| < H_4\}$ in both cases. It follows from part (ii) of Theorem 3.1 that A has a fixed point in $\mathcal{P} \cap (\overline{\Omega}_4 \setminus \Omega_3)$, such that $r_3 \leq \|y\| \leq r_4$. The proof of part (b) of this theorem is completed. Therefore, the BVPI (3.1) has at least one positive solution. □

4. EXISTENCE OF TWO POSITIVE SOLUTIONS

In this section, using Theorem 4.1 (Avery-Henderson fixed-point theorem) we prove the existence of at least two positive solutions of the BVPI (1.1).

Theorem 4.1 ([1]). *Let $\mathcal{P}$ be a cone in a real Banach space S. If φ and ψ are increasing, nonnegative continuous functionals on $\mathcal{P}$, let θ be a nonnegative continuous functional on $\mathcal{P}$ with $\theta(0) = 0$ such that, for some positive constants r and M,*

$$\psi(u) \leq \theta(u) \leq \varphi(u) \quad \textit{and } \|u\| \leq M\psi(u)$$

for all $u \in \overline{\mathcal{P}(\psi, r)}$. Suppose that there exist positive numbers $p < q < r$ such that

$$\theta(\lambda u) \leq \lambda\theta(u), \quad \textit{for all } 0 \leq \lambda \leq 1 \quad \textit{and } u \in \partial P(\theta, q).$$

If $A : \overline{\mathcal{P}(\psi, r)} \to \mathcal{P}$ is a completely continuous operator satisfying

(i) *$\psi(Au) > r$ for all $u \in \partial\mathcal{P}(\psi, r)$,*
(ii) *$\theta(Au) < q$ for all $u \in \partial\mathcal{P}(\theta, q)$,*
(iii) *$\mathcal{P}(\varphi, p) \neq \{\}$ and $\varphi(Au) > p$ for all $u \in \partial\mathcal{P}(\varphi, p)$,*

then A has at least two fixed points u_1 and u_2 such that

$$p < \varphi(u_1) \quad \textit{with } \theta(u_1) < q \quad \textit{and } q < \theta(u_2) \quad \textit{with } \psi(u_2) < r.$$

Let the Banach space $\mathcal{B} = \mathcal{C}([a, c) \cup (c, b])$ with the norm $\|\cdot\|$ defined by $\|y\| = \max_{x\in[a,c)\cup(c,b]} |y(x)|$. Again define the cone $\mathcal{P} \subset \mathcal{B}$ by

$$\mathcal{P} = \left\{ y \in \mathcal{B} : y(x) \geq 0, \quad \min_{x\in[\frac{3a+b}{4},c)\cup(c,\frac{3b-a}{4}]} y(x) \geq M\|y\| \right\}$$

where M is as in (3.2), and the operator $A : \mathcal{P} \to \mathcal{B}$ by

$$Ay(x) = \int_a^b H_n(x, s) f(s, y(s)) ds.$$

Let the nonnegative, increasing, continuous functionals ψ, θ, and φ be defined on the cone $\mathcal{P}$ by

$$\begin{cases} \psi(y) := \min_{x\in[\frac{3a+b}{4},c)\cup(c,\frac{3b-a}{4}]} y(x), \\ \theta(y) := \max_{x\in[\frac{3a+b}{4},c)\cup(c,\frac{3b-a}{4}]} y(x), \\ \varphi(y) := \max_{x\in[a,c)\cup(c,b]} y(x) \end{cases} \tag{4.1}$$

and let $\mathcal{P}(\psi, r) := \{y \in \mathcal{P} : \psi(y) < r\}$.

In the next theorem, we will assume

(H3) $f \in \mathcal{C}([a, c) \cup (c, b] \times [0, \infty), [0, \infty))$.

Theorem 4.2. *Assume* (H1) *and* (H3) *hold. Suppose there exist positive numbers* $0 < p < q < r$ *such that the function* f *satisfies the following conditions:*

(D1) $f(x, y) > p/(m_n \prod_{j=1}^{n} L_j)$ *for* $x \in [\frac{3a+b}{4}, c) \cup (c, \frac{3b-a}{4}]$ *and* $y \in [Mp, p]$,
(D2) $f(x, y) < q/\prod_{j=1}^{n} K_j$ *for* $x \in [a, c) \cup (c, b]$ *and* $y \in [0, q/M]$,
(D3) $f(x, y) > r/(Mm_n \prod_{j=1}^{n} L_j)$ *for* $x \in [\frac{3a+b}{4}, c) \cup (c, \frac{3b-a}{4}]$ *and* $y \in [r, r/M]$,

where m_n, L_j, K_j, M *are as defined in* (2.4)–(2.6) *and* (3.2), *respectively. Then the BVPI* (1.1) *has at least two positive solutions* y_1 *and* y_2 *such that*

$$p < \max_{x\in[a,c)\cup(c,b]} y_1(x) \quad \textit{with} \quad \max_{x\in[\frac{3a+b}{4},c)\cup(c,\frac{3b-a}{4}]} y_1(x) < q,$$

$$q < \max_{x\in[\frac{3a+b}{4},c)\cup(c,\frac{3b-a}{4}]} y_2(x) \quad \textit{with} \quad \min_{x\in[\frac{3a+b}{4},c)\cup(c,\frac{3b-a}{4}]} y_2(x) < r.$$

Proof. From (H3), Lemma 2.2 and Lemma 2.5, $A\mathcal{P} \subset \mathcal{P}$. Moreover, A is completely continuous. From (4.1), for each $y \in \mathcal{P}$ we have

$$\psi(y) \le \theta(y) \le \varphi(y), \tag{4.2}$$

$$\|y\| \le \frac{1}{M} \min_{x\in[\frac{3a+b}{4},c)\cup(c,\frac{3b-a}{4}]} y(x) = \frac{1}{M}\psi(y) \le \frac{1}{M}\theta(y) \le \frac{1}{M}\varphi(y). \tag{4.3}$$

For any $y \in \mathcal{P}$, (4.2) and (4.3) imply

$$\psi(y) \le \theta(y) \le \varphi(y), \quad \|y\| \le \frac{1}{M}\psi(y).$$

For all $y \in \mathcal{P}$, $\lambda \in [0, 1]$ we have

$$\theta(\lambda y) = \max_{x\in[\frac{3a+b}{4},c)\cup(c,\frac{3b-a}{4}]} (\lambda y)(x) = \lambda \max_{x\in[\frac{3a+b}{4},c)\cup(c,\frac{3b-a}{4}]} y(x) = \lambda\theta(y).$$

It is clear that $\theta(0) = 0$.

We now show that the remaining conditions of Theorem 4.1 are satisfied.

Firstly, we shall verify that the condition (iii) of Theorem 4.1 is satisfied. Since $0 \in \mathcal{P}$ and $p > 0$, $\mathcal{P}(\varphi, p) \neq \{\}$. Since $y \in \partial\mathcal{P}(\varphi, p)$, $Mp \leq y(x) \leq \|y\| = p$ for $x \in [\frac{3a+b}{4}, c) \cup (c, \frac{3b-a}{4}]$. Therefore,

$$\begin{aligned}\varphi(Ay) &= \max_{x\in[a,c)\cup(c,b]} Ay(x)\\ &\geq Ay(x)\\ &= \int_a^b H_n(x,s)f(s,y(s))ds\\ &> \frac{p}{m_n \prod_{j=1}^n L_j} m_n L \int_{\frac{3a+b}{4}}^{\frac{3b-a}{4}} G_n(s,s)ds\\ &= p\end{aligned}$$

using hypothesis (D1).

Now we shall show that the condition (ii) of Theorem 4.1 is satisfied. Since $y \in \partial\mathcal{P}(\theta, q)$, from (4.3) we have that $0 \leq y(x) \leq \|y\| \leq q/M$ for $x \in [a, c) \cup (c, b]$. Thus

$$\begin{aligned}\theta(Ay) &= \max_{x\in[\frac{3a+b}{4},c)\cup(c,\frac{3b-a}{4}]} Ay(x)\\ &= \max_{x\in[\frac{3a+b}{4},c)\cup(c,\frac{3b-a}{4}]} \int_a^b H_n(x,s)f(s,y(s))ds\\ &< \frac{q}{\prod_{j=1}^n K_j} K \int_a^b G_n(s,s)ds = q\end{aligned}$$

by hypothesis (D2).

Finally using hypothesis (D3), we shall show that the condition (i) of Theorem 4.1 is satisfied. Since $y \in \partial\mathcal{P}(\psi, r)$, from (4.3) we have that $\min_{x\in[\frac{3a+b}{4},c)\cup(c,\frac{3b-a}{4}]} y(x) = r$ and $r \leq \|y\| \leq r/M$. Then

$$\begin{aligned}\psi(Ay(x)) &= \min_{[\frac{3a+b}{4},c)\cup(c,\frac{3b-a}{4}]} \int_a^b H_n(x,s)f(s,y(s))ds\\ &= \int_a^b \min_{[\frac{3a+b}{4},c)\cup(c,\frac{3b-a}{4}]} H_n(x,s)f(s,y(s))ds\\ &\geq M \int_{\frac{3a+b}{4}}^{\frac{3b-a}{4}} H_n(s,s)f(s,y(s))ds\\ &> M \frac{r}{M m_n \prod_{j=1}^n L_j} m_n L \int_{\frac{3a+b}{4}}^{\frac{3b-a}{4}} G_n(s,s)ds = r.\end{aligned}$$

This completes the proof. □

5. EXAMPLES

Example 5.1. Consider the following boundary value problem:

$$(5.1)\qquad \begin{cases} (-1)^n y^{(2n)}(x) = e^{-y^2}, & x \in [2,3) \cup (3,5], \\ y^{(2i)}(3-0) = d_{i+1} y^{(2i)}(3+0), & \\ y^{(2i+1)}(3-0) = \rho_{i+1} y^{(2i+1)}(3+0), & \\ \alpha_{i+1} y^{(2i)}(2) - \beta_{i+1} y^{(2i+1)}(2) = 0, & \\ \gamma_{i+1} y^{(2i)}(5) + \delta_{i+1} y^{(2i+1)}(5) = 0, & 0 \le i \le n-1. \end{cases}$$

Then $a = 2$, $b = 5$, $c = 3$, and

$$f(x, y) = f(x) = e^{-y^2},\ y \in [0, \infty).$$

Since $\lim_{y \to 0^+} (f(y)/y) = +\infty$, $\lim_{y \to +\infty} (f(y)/y) = 0$.

We assume that the constants α_i, β_i, γ_i, δ_i, $(0 \le i \le n-1)$ satisfy the condition (H1). Thus the BVPI (5.1) has at least one positive solution by Theorem 3.2.

Example 5.2. Let us introduce an example to illustrate the usage of Theorem 4.2. Consider the BVPI:

$$(5.2)\qquad \begin{cases} (-1)^n y^{(vi)}(x) = f(x, y(x)), & x \in [0,2) \cup (2,3], \\ y^{(2i)}(2-0) = d_{i+1} y^{(2i)}(2+0), & \\ y^{(2i+1)}(2-0) = \rho_{i+1} y^{(2i+1)}(2+0), & \\ \alpha_{i+1} y^{(2i)}(0) - \beta_{i+1} y^{(2i+1)}(0) = 0, & \\ \gamma_{i+1} y^{(2i)}(3) + \delta_{i+1} y^{(2i+1)}(3) = 0, & 0 \le i \le 2, \end{cases}$$

Here $n = 3$, $f(x, y) = f(y) = \frac{900(y+9)^3}{(y+9)^2 + 64.10^8}$, $a = 0$, $b = 3$, $c = 2$, $\alpha_1 = 1$, $\alpha_2 = 2$, $\alpha_3 = 4$, $\beta_1 = 1/6$, $\beta_2 = 3$, $\beta_3 = 4$, $\gamma_1 = 3/2$, $\gamma_2 = 7$, $\gamma_3 = 1/7$, $\delta_1 = 2/5$,$\delta_2 = 2$, $\delta_3 = 5$. Then the conditions (H1) and (H2) are satisfied. For $1 \le i \le n$, the Green's function $G_i(x, s)$ in Lemma 2.1 is

$$G_i(x, s) = \frac{1}{D_i} \begin{cases} \theta_i(s)\varphi_i(x), & 0 \le s \le x \le 3, \\ \theta_i(x)\varphi_i(s), & 0 \le x \le s \le 3, \end{cases}$$

where

$$D_1 = \begin{cases} 111/40, & x \in [0,2), \\ 111/20, & x \in (2,3], \end{cases} \quad D_2 = \begin{cases} 59/4, & x \in [0,2), \\ 177, & x \in (2,3], \end{cases} \quad D_3 = \begin{cases} 328/7, & x \in [0,2), \\ 328/70, & x \in (2,3], \end{cases}$$

$$\theta_1(x) = \begin{cases} x + 1/6, & x \in [0,2), \\ 2x - 11/6, & x \in (2,3], \end{cases} \quad \theta_2(x) = \begin{cases} 2x + 3, & x \in [0,2), \\ 8x + 5, & x \in (2,3], \end{cases}$$

$$\theta_3(x) = \begin{cases} 4(x+1), & x \in [0,2), \\ (4x+22)/5, & x \in (2,3], \end{cases}$$

$$\varphi_1(x) = \begin{cases} -3x/4 + 17/5, & x \in [0,2), \\ -3x/2 + 49/10, & x \in (2,3], \end{cases} \quad \varphi_2(x) = \begin{cases} -7x/4 + 13/2, & x \in [0,2), \\ -7x + 23, & x \in (2,3], \end{cases}$$

and

$$\varphi_3(x) = \frac{1}{7}\begin{cases} 82 - 5x, & x \in [0,2), \\ 38 - x, & x \in (2,3]. \end{cases}$$

From (2.4)–(2.6), we get

$$\begin{aligned} m_1 &= 11/38, \quad K_1 = 1771/666, \quad L_1 = 19039/10656 \\ m_2 &= 29/92, \quad K_2 = 4049/1062, \quad L_2 = 37523/16992 \\ m_3 &= 7/16, \quad K_3 = 1745/164, \quad L_3 = 22975/5248 \end{aligned}$$

Clearly f is continuous and increasing on $(-\infty, \infty)$. If we take $p = 10^{-4}$, $q = 1/64$ and $r = 10^5$ then

$$0 < p < q < r.$$

It is clear that (D1), (D2), and (D3) of Theorem 4.2 are satisfied. Thus the BVPI (5.2) has at least two positive solutions y_1, y_2 satisfying

$$\begin{aligned} 10^{-4} &< \max_{t \in [0,2)\cup(2,3]} y_1(t) \quad \text{with} \quad \max_{t \in [3/4,2)\cup(2,9/4]} y_1(t) < 1/64 \\ \frac{1}{64} &< \max_{t \in [3/4,2)\cup(2,9/4]} y_2(t) \quad \text{with} \quad \min_{t \in [3/4,2)\cup(2,9/4]} y_2(t) < 10^5. \end{aligned}$$

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