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OSCILLATION RESULTS RELATED TO INTEGRAL AVERAGING TECHNIQUE FOR LINEAR HAMILTONIAN SYSTEMS

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ABSTRACT. For linear Hamiltonian systems, even for self-adjoint second order differential systems, we obtain new oscillation results without the assumptions which have been required for related results given before. The main tool used is a generalized Riccati transformation and the standard integral averaging technique.

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1. PRELIMINARIES

Consider the linear Hamiltonian system

$$\begin{cases} x' = A(t)x + B(t)y, \\ y' = C(t)x - A^*(t)y, \end{cases} \quad t \geq t_0, \tag{1.1}$$

where $A(t)$, $B(t)$, $C(t)$ are real $n \times n$ matrix-valued functions, B, C are Hermitian, B is positive definite and $x, y \in \mathbf{R}^n$. By M^* we mean the conjugate transpose of the matrix M.

A Hermitian matrix $M \in C^{n\times n}$ is positive semi-definite (positive definite) if for all $u \in C^n, u \neq 0,\ u^*Mu \geq 0 (> 0)$. A positive semi–definite (positive definite) Hermitian matrix M will be denoted by $M \geq 0 (M > 0)$, with the usual ordering of the eigenvalues of M given by $\lambda_1(M) \geq \lambda_2(M) \geq \cdots \geq \lambda_n(M)$, and as usual $\mathrm{tr}M = \sum_{i=1}^n \lambda_i(M)$.

We also consider the corresponding matrix system

$$\begin{cases} X' = A(t)X + B(t)Y, \\ Y' = C(t)X - A^*(t)Y, \end{cases} \quad t \geq t_0. \tag{1.2}$$

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A solution $(X(t), Y(t))$ of system (1.2) is said to be nontrivial, if $\det X(t) \neq 0$ is fulfilled for at least one $t \geq t_0$. A nontrivial solution $(X(t), Y(t))$ of system (1.2) is said to be conjoined (prepared) if $X^*(t)Y(t) - Y^*(t)X(t) \equiv 0$, $t \geq t_0$. A conjoined solution $(X(t), Y(t))$ of (1.2) is said to be a conjoined basis of (1.1) (or (1.2)) if the rank of the $2n \times n$ matrix $\binom{X(t)}{Y(t)}$ is n.

Two distinct points a, b in $[t_0, \infty)$ are said to be (mutually) conjugate with respect to (1.1) if there exists a solution $(x(t), y(t))$ of (1.1) with $x(a) = x(b) = 0$ and $x(t) \neq 0$ (i.e., not equal to the zero vector in $\mathbf{R}^n$) on the subinterval with end-points a and b. The system (1.1) is said to be disconjugate on a subinterval J of $[t_0, \infty)$ if no two distinct points are conjugate. If (1.1) is disconjugate on J and $(X(t), Y(t))$ is the conjoined basis of (1.2) satisfying $X(a) = 0, U(a) = I$, $a \in J$, where by I we men the $n \times n$ identity matrix, then $\det X(t) \neq 0$ for $t \in J \backslash \{a\}$. A conjoined basis $(X(t), Y(t))$ of system (1.2) is said to be oscillatory in case the determinant of $X(t)$ vanishes on $[T, \infty)$ for each $T \geq t_0$.

Let $\Phi(t)$ be a fundamental matrix for the linear system $v' = A(t)v$. The pair $(A(t), B(t))$ is called controllable if the rows of $\Phi^{-1}(t)B(t)$ are linearly independent over any subinterval of $[t_0, \infty)$ (see [5, p. 36-37] and [9, p. 107]). This definition coincides with the following fact: if for any solution $(x(t), y(t))$ of (1.1), one has that $x(t) \equiv 0$ on any non-degenerate subinterval $J \subseteq [t_0, \infty)$ implies $x = y \equiv 0$ on $[t_0, \infty)$. Observe that since $B(t) > 0$, we have the pair $(A(t), B(t))$ is controllable. Suppose there exists an oscillatory conjoined basis of system (1.2), then by Sturm's separation theorem [5, Theorem 16, p. 71] and [9, Theorem 7.3.5], we know that each conjoined basis of system (1.2) is oscillatory, so system (1.1) (or (1.2)) is called oscillatory. Now the definition of oscillation agrees with the non-disconjugacy of system (1.1) (or (1.2)) on any neighborhood of $+\infty$.

In the case when $A(t) \equiv 0$, system (1.2) reduces to the second order self-adjoint matrix differential system

$$(P(t)X')' + Q(t)X = 0 \tag{1.3}$$

with $P(t) = B^{-1}(t)$, $Q(t) = -C(t)$. Oscillation and non-oscillation of system (1.3) have been extensively studied by many authors [1–8, 10–13, 18, 19], it is studied in [16] when the system with damping. A discrete version of (1.3) is studied in [14]. Particularly, for the case when $P(t) \equiv I$, i.e., for the system

$$X'' + Q(t)X = 0, \tag{1.4}$$

it was conjectured by Hinton and Lewis [8] that (1.4) is oscillatory if

$$\lim_{t\to\infty} \lambda_1 \left[\int_{t_0}^{t} Q(s)ds \right] = \infty.$$

This conjecture was settled with some additional assumptions on the rate of growth of the trace of $\int_0^t Q(s)ds$ by Mingarelli [13], Kwong et al. [11], Butler and Erbe [1, 2], and Butler et al. [3]. This conjecture was settled by Kwong and Kaper [10] for the two-demensional case, and by Byers et al. [4] for arbitrary $n-$dimensional cases.

Oscillation properties for Hamiltonian system (1.2) are widely studied , too (see [5, 9, 15, 17, 20, 21]). In paper [15], Meng studied the Hamiltonian systems (1.1), and obtained some new oscillation criteria, here we list the main results of [15] as follows:

We say that a function $H = H(t,s)$ belongs to a function class $\mathcal{W}$, denotes by $H \in \mathcal{W}$, if $H \in C(D, \mathbf{R}_+)$, where $D = \{(t,s) : t \geq s \geq t_0\}$ which satisfies

(i) $H(t,t) = 0$ and $H(t,s) > 0$ for $t_0 < s < t < +\infty$;

(ii) H has a continuous non-positive partial derivative $\partial H/\partial s$ satisfying the condition $\partial[H(t,s)k(s)]/\partial s = -h(t,s)\sqrt{H(t,s)k(s)}$, for some $h \in L_{loc}(D, \mathbb{R})$, $k \in C^1([t_0,\infty),(0,\infty))$.

Theorem 1.1 ([15, Theorem 1])**.** *Suppose that there exist two positive and real-valued functions* ϕ, $\theta \in C^1[t_0,\infty)$, *such that, for some* $H \in \mathcal{W}$ *with* $k(t) \equiv 1$,

$$(C_1) \qquad \limsup_{t\to\infty} \frac{1}{H(t,t_0)} \lambda_1 \left[\int_{t_0}^t [H(t,s)C_2(s) - \frac{1}{4} h^2(t,s) B_2^{-1}(s)] ds \right] = \infty,$$

where $B_2(t)$, $C_2(t)$ *are given by* (C_7) *and* (C_8). *Then system* (1.2) *is oscillatory.*

Theorem 1.2 ([15, Theorem 3])**.** *Let* H, h, ϕ *and* θ *be as in theorem A, suppose that*

$$(C_2) \qquad 0 < \inf_{s\geq t_0} \left\{ \liminf_{t\to\infty} \frac{H(t,s)}{H(t,t_0)} \right\} \leq \infty$$

and

$$(C_3) \qquad \liminf_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^t H(t,s) tr(C_2(s)) ds > -\infty,$$

$$(C_4) \qquad \limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^t \frac{h^2(t,s)}{\lambda_n(B_2(s))} ds < \infty.$$

If there exists a function $m \in C[t_0,\infty)$ *such that*

$$(C_5) \quad \limsup_{t\to\infty} \frac{1}{H(t,T)} \lambda_1 \left[\int_T^t [H(t,s)C_2(s) - \frac{1}{4} h^2(t,s) B_2^{-1}(s)] ds \right] \geq m(T),\ T \geq t_0,$$

and

$$(C_6) \qquad int_{t_0}^{\infty} \lambda_n(B_2(t)) m_+^2(t) dt = \infty,$$

where $m_+(t) = \max\{m(t), 0\}$, $B_2(t)$, $C_2(t)$ *are the same as in Theorem A. Then system* (1.2) *is oscillatory.*

The purpose of this paper is to establish some new oscillation criteria for the Hamiltonian system (1.2) using a generalized Riccati transformation and the standard integral averaging technique, which allow us to remove conditions (C_3) and (C_4) in Theorem B.

2. MAIN RESULTS

In the sequel, we need the following lemmas:

Lemma 2.1 ([3]). *If A is an $n \times n$ Hermitian matrix, then*
(i) $[\lambda_1(A)]^2 \leq \lambda_1(A^2) \leq tr(A^2)$;
(ii) $(trA)^2 \leq n trA^2$.

Lemma 2.2 ([15]). *If A is an $n \times n$ Hermitian matrix and R is an $n \times n$ positive definite Hermitian matrix, then* $tr(ARA) \geq \lambda_n(R) trA^2$.

Let $\phi(t)$ and $\theta(t)$ be positive, smooth and real-valued functions on $[0, +\infty)$. Since $B(t) > 0$, this allows us to make the transformation:

$$U = \phi X, \quad V = \theta Y + \alpha B^{-1} X,$$

where

$$\alpha = \frac{\theta}{2}(\frac{\phi'}{\phi} - \frac{\theta'}{\theta}).$$

Then U and V satisfy the following differential system:

$$\begin{cases} U' = A(t)U + B_1(t)V + \frac{1}{2}(\frac{\phi'}{\phi} + \frac{\theta'}{\theta})U, \\ V' = C_1(t)U - A^*(t)V + \frac{1}{2}(\frac{\phi'}{\phi} + \frac{\theta'}{\theta})V, \end{cases} \tag{2.1}$$

where

$$B_1(t) = \frac{\phi(t)}{\theta(t)} B(t),$$

$$C_1(t) = \frac{\theta}{\phi}\left\{C(t) + \frac{\alpha}{\theta}(B^{-1}(t)A(t) + A^*(t)B^{-1}(t)) + (\frac{\alpha}{\theta}B^{-1}(t))' - \frac{\alpha^2}{\theta^2}B^{-1}(t)\right\}.$$

Recalling that $\Phi(t)$ is a fundamental matrix of the linear system $v' = A(t)v$, set

$$B_2(t) = \Phi^{-1}(t)B_1(t)\Phi^{*-1}(t) = \frac{\phi(t)}{\theta(t)}\Phi^{-1}(t)B(t)\Phi^{*-1}(t), \tag{C_7}$$

$$C_2(t) = -\Phi^*(t)C_1(t)\Phi(t). \tag{C_8}$$

Then we have the following results.

Theorem 2.3. *Suppose that there exist three positive and real-valued functions ϕ, θ, $k \in C^1[t_0, \infty)$, such that, for some $\beta \geq 1$, and for some $H \in \mathcal{W}$,*

$$\limsup_{t\to\infty} \frac{1}{H(t, t_0)} \lambda_1 \left\{ \int_{t_0}^t \left[H(t, s)k(s)C_2(s) - \frac{\beta}{4} h^2(t, s) B_2^{-1}(s) \right] ds \right\} = \infty, \tag{2.2}$$

where $B_2(t)$, $C_2(t)$ are given by (C_7) and (C_8). Then system (1.2) *is oscillatory.*

Proof. Suppose to the contrary that there exists a conjoined basis $(X(t), Y(t))$ of (1.2) which is not oscillatory. Without loss of generality, we may suppose that $\det X(t) \neq 0$ for $t \geq t_0$. Define

$$W(t) = V(t)U^{-1}(t), \quad t \geq t_0.$$

From (2.1) we have

$$W'(t) = C_1(t) - A^*(t)W(t) - W(t)A(t) - W(t)B_1(t)W(t).$$

Let $R(t) = \Phi^*(t)W(t)\Phi(t)$, by calculation, it follows that $R(t)$ satisfies the following Riccati equation:

$$R'(t) = -C_2(t) - R(t)B_2(t)R(t).$$

Since $B_2(t) > 0$, let $E(t) = [B_2(t)]^{1/2}$. Multiplying the Riccati equation, with t replaced by s, by $H(t,s)k(s)$ and integrating it from T to t, for all $t \geq T \geq t_0$, we obtain

$$\begin{aligned}
\int_T^t H(t,s)k(s)C_2(s)ds &= -\int_T^t H(t,s)k(s)R'(s)ds \\
&\quad - \int_T^t H(t,s)k(s)[R(s)B_2(s)R(s)]ds \\
&= H(t,T)k(T)R(T) - \int_T^t H(t,s)k(s)[R(s)B_2(s)R(s)]ds \\
&\quad - \int_T^t h(t,s)\sqrt{H(t,s)k(s)}R(s)ds \\
&= H(t,T)k(T)R(T) - \int_T^t H(t,s)k(s)E^{-1}(s)[E(s)R(s)E(s)]^2E^{-1}(s)ds \\
&\quad - \int_T^t h(t,s)\sqrt{H(t,s)k(s)}E^{-1}(s)[E(s)R(s)E(s)]E^{-1}(s)ds \\
&= H(t,T)k(T)R(T) + \frac{\beta}{4}\int_T^t h^2(t,s)B_2^{-1}(s)ds \\
&\quad - \frac{\beta-1}{\beta}\int_T^t H(t,s)k(s)R(s)B_2(s)R(s)ds \\
&\quad - \int_T^t E^{-1}(s)\left\{\sqrt{\frac{H(t,s)k(s)}{\beta}}E(s)R(s)E(s) + \frac{\sqrt{\beta}}{2}h(t,s)I\right\}^2 E^{-1}(s)ds
\end{aligned}$$

for some $\beta \geq 1$. Hence, we have

$$\int_T^t \left[H(t,s)k(s)C_2(s) - \frac{\beta}{4}h^2(t,s)B_2^{-1}(s)\right]ds \leq H(t,T)k(T)R(T), \quad t > T \geq t_0.$$

This implies that for all $t \geq t_0$,

$$\int_{t_0}^t \left[H(t,s)k(s)C_2(s) - \frac{\beta}{4}h^2(t,s)B_2^{-1}(s)\right]ds \leq H(t,t_0)k(t_0)R(t_0).$$

It follows that

$$\begin{aligned}
\lambda_1\left[\int_{t_0}^t (H(t,s)k(s)C_2(s) - \frac{\beta}{4}h^2(t,s)B_2^{-1}(s))ds\right] &\leq \lambda_1[H(t,t_0)k(t_0)R(t_0)] \\
&= H(t,t_0)k(t_0)\lambda_1[R(t_0)].
\end{aligned}$$

This gives

$$\limsup_{t\to\infty}\frac{1}{H(t,t_0)}\lambda_1\left[\int_{t_0}^{t}(H(t,s)k(s)C_2(s)-\frac{\beta}{4}h^2(t,s)B_2^{-1}(s))ds\right]$$
$$\leq k(t_0)\lambda_1[R(t_0)]<\infty,$$

which contradicts (2.2). This completes the proof of the Theorem. □

Under modifications of the hypotheses of Theorem 2.3, we obtain the following results (Corollary 2.4; Theorem 2.5 and Corollary 2.6).

Corollary 2.4. *In Theorem* 2.3, *if the condition* (2.2) *is replaced by the conditions*

$$\limsup_{t\to\infty}\frac{1}{H(t,t_0)}\lambda_1\left[\int_{t_0}^{t}h^2(t,s)B_2^{-1}(s)ds\right]<\infty$$

and

$$\limsup_{t\to\infty}\frac{1}{H(t,t_0)}\lambda_1\left[\int_{t_0}^{t}H(t,s)k(s)C_2(s)ds\right]=\infty,$$

where $B_2(t)$, $C_2(t)$ *are the same as in Theorem* 2.3. *Then system* (1.2) *is oscillatory.*

Theorem 2.5. *Suppose that there exist three positive and real-valued functions* ϕ, θ, $k\in C^1[t_0,\infty)$, *such that, for some* $\beta\geq 1$, *and for some* $H\in\mathcal{W}$,

$$\limsup_{t\to\infty}\frac{1}{H(t,t_0)}\int_{t_0}^{t}\left[H(t,s)k(s)tr[C_2(s)]-\frac{\beta}{4}h^2(t,s)tr[B_2^{-1}(s)]\right]ds=\infty, \tag{2.3}$$

where $B_2(t)$, $C_2(t)$ *are the same as in Theorem* 2.3. *Then system* (1.2) *is oscillatory.*

Corollary 2.6. *In Theorem* 2.5 *if the condition* (2.3) *is replaced by the conditions*

$$\limsup_{t\to\infty}\frac{1}{H(t,t_0)}\int_{t_0}^{t}h^2(t,s)tr[B_2^{-1}(s)]ds<\infty$$

and

$$\limsup_{t\to\infty}\frac{1}{H(t,t_0)}\int_{t_0}^{t}H(t,s)k(s)tr[C_2(s)]ds=\infty,$$

where $B_2(t)$, $C_2(t)$ *are the same as in Theorem* 2.3. *Then system* (1.2) *is oscillatory.*

Theorem 2.7. *Let the functions* H, h, ϕ, θ *and* k *be as in Theorem* 2.3, *and suppose*

$$0<\inf_{s\geq t_0}\left\{\liminf_{t\to\infty}\frac{H(t,s)}{H(t,t_0)}\right\}\leq\infty. \tag{2.4}$$

If there exists a function $m\in C[t_0,\infty)$ *such that*

$$\limsup_{t\to\infty}\frac{1}{H(t,T)}\int_{T}^{t}\left[H(t,s)k(s)tr[C_2(s)]-\frac{\beta}{4}h^2(t,s)tr[B_2^{-1}(s)]\right]ds\geq m(T) \tag{2.5}$$

for all $t\geq T\geq t_0$, *and for some* $\beta>1$. *Moreover,*

$$\int_{t_0}^{\infty}\frac{\lambda_n(B_2(t))m_+^2(t)}{k(t)}dt=\infty, \tag{2.6}$$

where $m_+(t)=\max\{m(t),0\}$, $B_2(t)$, $C_2(t)$ *are the same as in Theorem* 2.3. *Then system* (1.2) *is oscillatory.*

Proof. Assume to the contrary that (1.2) is non-oscillatory. Followed the proof of Theorem 2.3, for some $\beta > 1$, we obtain

$$\int_T^t H(t,s)k(s)C_2(s)ds \le H(t,T)k(T)R(T) + \frac{\beta}{4}\int_T^t h^2(t,s)B_2^{-1}(s)ds$$
$$- \frac{\beta-1}{\beta}\int_T^t H(t,s)k(s)R(s)B_2(s)R(s)ds.$$

So, for all $t > T \ge t_0$, we have

$$\limsup_{t\to\infty}\frac{1}{H(t,T)}\int_T^t \left[H(t,s)k(s)C_2(s) - \frac{\beta}{4}h^2(t,s)B_2^{-1}(s)\right]ds$$
$$\le k(T)R(T) - \frac{\beta-1}{\beta}\liminf_{t\to\infty}\frac{1}{H(t,T)}\int_T^t H(t,s)k(s)R(s)B_2(s)R(s)ds.$$

So we get

$$\limsup_{t\to\infty}\frac{1}{H(t,T)}\int_T^t \left[H(t,s)k(s)\mathrm{tr}[C_2(s)] - \frac{\beta}{4}h^2(t,s)\mathrm{tr}[B_2^{-1}(s)]\right]ds$$
$$\le k(T)\mathrm{tr}[R(T)] - \frac{\beta-1}{\beta}\liminf_{t\to\infty}\frac{1}{H(t,T)}\int_T^t H(t,s)k(s)\mathrm{tr}\left[R(s)B_2(s)R(s)\right]ds.$$

For all $T \ge t_0$ and for any $\beta > 1$, by (2.5) we have

$$k(T)\mathrm{tr}[R(T)] \ge m(T) + \frac{\beta-1}{\beta}\liminf_{t\to\infty}\frac{1}{H(t,T)}\int_T^t H(t,s)k(s)\mathrm{tr}\left[R(s)B_2(s)R(s)\right]ds.$$

So

$$k(T)\mathrm{tr}[R(T)] \ge m(T) \quad \text{or} \quad nk(T)\mathrm{tr}[R^2(T)] \ge \frac{m_+^2(T)}{k(T)}, \tag{2.7}$$

and

$$\liminf_{t\to\infty}\frac{1}{H(t,t_0)}\int_{t_0}^t H(t,s)k(s)\mathrm{tr}\left[R(s)B_2(s)R(s)\right]ds$$
$$\le \frac{\beta}{\beta-1}\left[k(t_0)\mathrm{tr}(R(t_0)) - m(t_0)\right] < \infty. \tag{2.8}$$

Now, we claim that

$$\int_{t_0}^\infty k(s)\lambda_n[B_2(s)]\mathrm{tr}[R^2(s)]ds < \infty. \tag{2.9}$$

Suppose to the contrary that

$$\int_{t_0}^\infty k(s)\lambda_n[B_2(s)]\mathrm{tr}[R^2(s)]ds = \infty.$$

By (2.4), there is a positive constant ξ satisfying

$$\inf_{s\ge t_0}\left\{\liminf_{t\to\infty}\frac{H(t,s)}{H(t,t_0)}\right\} > \xi > 0. \tag{2.10}$$

Let η be any arbitrary positive number, then there exists a $t_1 > t_0$ such that, for all $t \geq t_1$,

$$\int_{t_0}^{t} k(s)\lambda_n[B_2(s)]\mathrm{tr}[R^2(s)]ds \geq \frac{\eta}{\xi}.$$

For the convenience, let

$$p(t) = \frac{1}{H(t,t_0)} \int_{t_0}^{t} H(t,s)k(s)\lambda_n[B_2(s)]\mathrm{tr}[R^2(s)]ds,$$

then, for $t \geq t_1$, we have

$$\begin{aligned}
p(t) &= \frac{1}{H(t,t_0)} \int_{t_0}^{t} H(t,s)d\left\{\int_{t_0}^{s} k(\tau)\lambda_n[B_2(\tau)]\mathrm{tr}[R^2(\tau)]d\tau\right\} \\
&= \frac{1}{H(t,t_0)} \int_{t_0}^{t} -\frac{\partial H(t,s)}{\partial s}\left\{\int_{t_0}^{s} k(\tau)\lambda_n[B_2(\tau)]\mathrm{tr}[R^2(\tau)]d\tau\right\} ds \\
&\geq \frac{1}{H(t,t_0)} \int_{t_1}^{t} -\frac{\partial H(t,s)}{\partial s}\left\{\int_{t_0}^{s} k(\tau)\lambda_n[B_2(\tau)]\mathrm{tr}[R^2(\tau)]d\tau\right\} ds \\
&\geq \frac{\eta}{\xi}\frac{1}{H(t,t_0)} \int_{t_1}^{t} -\frac{\partial H(t,s)k(s)}{\partial s} ds \\
&= \frac{\eta}{\xi}\frac{H(t,t_1)}{H(t,t_0)}.
\end{aligned}$$

By (2.10), there exists a $t_2 \geq t_1$ such that, for all $t \geq t_2$,

$$\frac{H(t,t_1)}{H(t,t_0)} \geq \xi,$$

which implies $p(t) \geq \eta$ for all $t \geq t_2$. Since η is arbitrary, we have

$$\lim_{t\to\infty} p(t) = \infty,$$

which implies

$$\liminf_{t\to\infty} p(t) = \lim_{t\to\infty} p(t) = \infty.$$

Then we obtain

$$\begin{aligned}
\liminf_{t\to\infty} \frac{1}{H(t,t_0)} &\int_{t_0}^{t} H(t,s)k(s)\mathrm{tr}[R(s)B_2(s)R(s)]ds \\
&\geq \liminf_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} H(t,s)k(s)\lambda_n[B_2(s)]\mathrm{tr}[R^2(s)]ds \\
&= \liminf_{t\to\infty} p(t) = \infty,
\end{aligned}$$

which contradicts (2.8), thus (2.9) holds. Then by (2.7) we get

$$\int_{t_0}^{\infty} \frac{m_+^2(t)}{k(t)}\lambda_n[B_2(t)]dt \leq n\int_{t_0}^{\infty} k(t)\mathrm{tr}[R^2(t)]\lambda_n[B_2(t)]dt < \infty,$$

which contradicts (2.6). This completes the proof. $\square$

Remark 2.8. Let $\beta = 1$, $k(t) \equiv 1$, $t \in [t_0, \infty)$ in Theorem 2.3, Theorem 2.3 reduces to Theorem A; we obtain the same result in Theorem 2.7 without the two assumptions (C_3) and (C_4) in Theorem B. Therefore, Theorem 2.3 (2.5) and Theorem 2.7 are generalizations and improvements of [15, Theorem 1, 3], respectively.

Remark 2.9. Different choices of H, h, k, ϕ and θ give many new criteria for the oscillation of system (1.2).

We observe that only the ratio α/θ and ϕ/θ are involved in the coefficients of the formulaes, and $\alpha(t) = \theta\{\frac{1}{2}\log[\phi(t)/\theta(t)]\}'$. Therefore, choose

$$a(t) = \frac{\theta(t)}{\phi(t)} = \exp\left(-2\int_{t_0}^{t} f(s)ds\right),$$

where $f \in C^1[t_0, \infty)$, then

$$\frac{\alpha(t)}{\theta(t)} = f(t), \frac{\alpha(t)}{\phi(t)} = a(t)f(t),$$

and

$$B_1(t) = \frac{1}{a(t)}B(t), \qquad B_2(t) = \frac{1}{a(t)}\Phi^{-1}(t)B(t)\Phi^{*-1}(t),$$

$$C_1(t) = a(t)\left\{C(t) + f(t)\left[B^{-1}(t)A(t) + A^*(t)B^{-1}(t)\right] + [f(t)B^{-1}(t)]' - f^2(t)B^{-1}(t)\right\}.$$

In other words, to carry out the transformation, we need only to choose one appropriate smooth function $f(t)$.

Let $k(t) \equiv 1$, $H(t,s) = (t-s)^\lambda$, $t \geq s \geq t_0$, then we have $h(t,s) = \lambda(t-s)^{\frac{\lambda-2}{2}}$, where $\lambda > 1$ is a constant, and for any $s \geq t_0$, we have

$$\lim_{t\to\infty} \frac{H(t,s)}{H(t,t_0)} = \lim_{t\to\infty} \frac{(t-s)^\lambda}{(t-t_0)^\lambda} = 1.$$

Consequently, using Theorem 2.7, we have the following corollary:

Corollary 2.10. *Let $\lambda > 1$ be a constant and suppose that there exists a function $m \in C[t_0, \infty)$ such that, for some $\beta > 1$,*

$$\limsup_{t\to\infty} \frac{1}{t^\lambda}\int_T^t \left[(t-s)^\lambda tr[C_2(s)] - \frac{\beta\lambda^2}{4}(t-s)^{\lambda-2} tr[B_2^{-1}(s)]\right] ds \geq m(T), \quad t \geq T \geq t_0,$$

and (2.10) hold, where $B_2(t)$, $C_2(t)$ are the same as in Theorem 2.3. Then system (1.2) is oscillatory.

Example 2.11. Let $t \in (0, \infty)$, consider the linear Hamiltonian system

$$\begin{cases} X' = \frac{1}{2t(3+\sin t)^2}Y, \\ Y' = \left[(1+\sin t)^2(1+t-\frac{10}{t}) + \frac{1}{2t}(3+\sin t)^2 + 2\cos t(3+\sin t) - \frac{3t}{2} - \frac{3}{2} - \frac{27}{2t}\right]X, \end{cases} \tag{2.11}$$

where

$$A(t) \equiv 0, \quad B(t) = \frac{1}{2t(3+\sin t)^2}I,$$

$$C(t) = \left[(1+\sin t)^2(1+t-\frac{10}{t}) + \frac{1}{2t}(3+\sin t)^2 + 2\cos t(3+\sin t) - \frac{3t}{2} - \frac{3}{2} - \frac{27}{2t}\right] I$$

are 2×2-matrices, and $B(t)$, $C(t)$ are Hermitian. In this case $\Phi(t) \equiv I$, choose $\lambda = 2$, $f(t) = -1/2t$, then $f'(t) = 1/2t^2$, $a(t) = t$, by direct calculation, we get

$$B_2(t) = \frac{1}{2t^2(3+\sin t)^2} I,$$
$$C_2(t) = \frac{1}{2}\left[20(1+\sin t)^2 + 3t^2 + 3t + 27 - 2(t+t^2)(1+\sin t)^2\right] I.$$

Let $\beta = 6$, these yield

$$\begin{aligned}
&\limsup_{t\to\infty} \frac{1}{t^2}\mathrm{tr}\left[\int_T^t ((t-s)^2 C_2(s) - 6B_2^{-1}(s))ds\right] \\
&= \limsup_{t\to\infty} \frac{1}{t^2}\int_T^t (t-s)^2\left[20(1+\sin s)^2 + 3s^2 + 3s + 27 - 2(s+s^2)(1+\sin s)^2\right] \\
&\quad - (t-s)^2 6s^2(3+\sin s)^2 ds \\
&\triangleq m(T) = \frac{123}{4} - \frac{113}{2}T - T\cos T\sin T - T^2\cos T\sin T - 4T\cos T + \frac{21}{2}\cos T\sin T \\
&\quad - T\cos^2 T - 4T^2\cos T + 8T\sin T + 48\cos T + 4\sin T - \frac{1}{2}\cos^2 T.
\end{aligned}$$

It is easy to verify that (2.10) holds. Therefore, system (2.11) is oscillatory by Corollary 2.10. However, we can easily find that

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)}\int_{t_0}^t \frac{h^2(t,s)}{\lambda_n(B_2(s))}ds = \limsup_{t\to\infty} \frac{1}{t^2}\int_{t_0}^t \frac{4}{\lambda_n(B_2(s))}ds = \infty,$$

so condition (C_4) in Theorem B is not satisfied.

Let $k(t) = \frac{1}{t^2}$, $H(t,s) = (t-s)^2$, $t \ge s \ge t_0$, then we have $h(t,s) = \frac{2}{s^2}$, and for any $s \ge t_0$, we have

$$\lim_{t\to\infty} \frac{H(t,s)}{H(t,t_0)} = \lim_{t\to\infty} \frac{(t-s)^2}{(t-t_0)^2} = 1.$$

Consequently, using Theorem 2.7, we have the following corollary:

Corollary 2.12. *Suppose that there exists a function $m \in C[t_0,\infty)$ such that, for some $\beta > 1$,*

$$\limsup_{t\to\infty} \frac{1}{t^2}\int_T^t \left[(t-s)^2\frac{1}{s^2}tr[C_2(s)] - \frac{\beta}{s^4}tr[B_2^{-1}(s)]\right] ds \ge m(T), \quad t \ge T \ge t_0,$$

and (2.10) *hold, where $B_2(t)$, $C_2(t)$ are the same as in Theorem* 2.3. *Then system* (1.2) *is oscillatory.*

Example 2.13. Let $t \in (0,\infty)$, consider the linear Hamiltonian system

(2.12)
$$\begin{cases} X' = \frac{1}{t^5(3+\sin t)^2}Y, \\ Y' = \left[(1+\sin t)^2(t^3+t^2-10t) + \frac{9}{4}t^3(3+\sin t)^2 + t^4\cos t(3+\sin t) - \frac{3}{2}t^3 - \frac{3}{2}t^2 - \frac{27}{2}t\right]X, \end{cases}$$

where

$$A(t) \equiv 0, \quad B(t) = \frac{1}{t^5(3+\sin t)^2} I,$$

$$C(t) = \Bigg[(1+\sin t)^2(t^3+t^2-10t) + \frac{9}{4}t^3(3+\sin t)^2 + t^4\cos t(3+\sin t) - \frac{3}{2}t^3 - \frac{3}{2}t^2 - \frac{27}{2}t\Bigg] I$$

are 2×2-matrices, and $B(t), C(t)$ are Hermitian. In this case $\Phi(t) \equiv I$, choose $f(t) = -1/2t$, then $f'(t) = 1/2t^2$, $a(t) = t$, by direct calculation, we get

$$B_2(t) = \frac{1}{t^6(3+\sin t)^2} I,$$
$$C_2(t) = \frac{1}{2}\left[20t^2(1+\sin t)^2 + 3t^4 + 3t^3 + 27t^2 - 2(t^3+t^4)(1+\sin t)^2\right] I.$$

Let $\beta = 3$, these yield

$$\begin{aligned}
&\limsup_{t\to\infty} \frac{1}{t^2}\int_T^t \left[(t-s)^2\frac{1}{s^2}\mathrm{tr}[C_2(s)] - \frac{\beta}{s^4}\mathrm{tr}[B_2^{-1}(s)]\right] ds \\
&\quad = \limsup_{t\to\infty} \frac{1}{t^2}\int_T^t (t-s)^2\left[20(1+\sin s)^2 + 3s^2 + 3s + 27 - 2(s+s^2)(1+\sin s)^2\right] \\
&\quad\quad - 6s^2(t-s)^2(3+\sin s)^2 ds \\
&\quad \triangleq m(T) = \frac{123}{4} - \frac{113}{2}T - T\cos T\sin T - T^2\cos T\sin T - 4T\cos T + \frac{21}{2}\cos T\sin T \\
&\quad\quad - T\cos^2 T - 4T^2\cos T + 8T\sin T + 48\cos T + 4\sin T - \frac{1}{2}\cos^2 T.
\end{aligned}$$

It is easy to verify that (2.10) holds. Therefore, system (2.12) is oscillatory by Corollary 2.12. However, we note that

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)}\int_{t_0}^t \frac{h^2(t,s)}{\lambda_n(B_2(s))} ds = \infty,$$

so condition (C_4) in Theorem B is not satisfied.

Examples 2.11 and 2.13 show that Theorem B cannot be applied to system (2.11) or system (2.12), obviously our results are superior to the results obtained before.

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