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NONLINEAR INITIAL VALUE PROBLEMS WITH p-LAPLACIAN

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ABSTRACT. We study the nonlinear initial value problem consisting of the equation $-[p(t)\phi(y')]'+q(t)\phi(y) = w(t)f(y)$ with $\phi(y) = |y|^{r-1}y$ for $r > 0$ and the initial conditions $y(t_0) = y_0$, $(p^{1/r}y')(t_0) = z_0$. By establishing nonlinear integral inequalities and applying a generalized energy function and a generalized Prüfer transformation, we prove that the solution of this initial value problem exists on the whole domain and is unique. This paper provides a foundation for a forthcoming paper on the existence of nodal solutioins of second order nonlinear boundary value problems with p-Laplacian.

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1. INTRODUCTION AND MAIN RESULT

Research on differential equations with p-Laplacian has been active in recent years. In this paper, we study the global existence and uniqueness of solutions of the initial value problems (IVPs) consisting of the equation with p-Laplacian

$$-[p(t)\phi(y')]' + q(t)\phi(y) = w(t)f(y) \quad \text{on } [a,b], \tag{1.1}$$

where $\phi(y) = |y|^{r-1}y$ for $r > 0$, and the initial conditions

$$y(t_0) = y_0, \quad (p^{\frac{1}{r}}y')(t_0) = z_0, \tag{1.2}$$

where $t_0 \in [a,b]$, and $y_0, z_0 \in \mathbb{R}$.

When $r > 0$ and $r \neq 1$, if $f(y) = \phi(y)$, then Eq. (1.1) is a half-linear equation in the sense that $y(t)$ is a solution implies that $cy(t)$ is also a solution for any $c \in \mathbb{R}$. However, it is impossible for Eq. (1.1) to be linear or to be transformed to a linear equation in any situation. Therefore, many tools used for the case when $r = 1$ can not be applied to the general problems with p-Laplacian. For instance, the fundamental solution set, the classical Prüfer transformation and the Gronwall inequality are among such tools.

It is well-known that even for the case when $r = 1$, the solution of IVP (1.1), (1.2), if exists, may not be unique and may not be extended to the whole interval $[a,b]$, see Wong [4] for details. Kong [2] proved that under some smoothness assumptions, IVP

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(1.1), (1.2) with $r = 1$ has a unique solution which exists on the whole interval $[a, b]$. Due to the p-Laplacian form, the establishment of the global existence and uniqueness of the general IVP (1.1), (1.2) is much harder than the case with $r = 1$. This is because the first order system transformed from Eq. (1.1) is of more complicated structure. Recently, Naito and Tanaka [3] proved an existence and uniqueness theorem for the special case of IVP (1.1), (1.2) where $p(t) \equiv 1$ and $q(t) \equiv 0$. However, the work in [3] cannot be simply extended to the IVP (1.1), (1.2). In fact, the energy function and the Gronwall-inequality used in their proofs fail to work especially for the case when $q(t) \not\equiv 0$. Here, by using a generalized energy function and a generalized Prüfer transformation and by deriving some nonlinear inequalities, we are able to prove that the solution of IVP (1.1), (1.2) exists on the whole interval $[a, b]$ and is unique.

Initial value problems are closely related to boundary value problems. For instance, shooting method is one of the fundamental methods which can be used to investigate the existence of solutions of the boundary value problems consisting of Eq. (1.1) and the boundary condition

$$\begin{aligned} a_{11}y(a) - a_{12}(p^{\frac{1}{r}}y')(a) = 0, \\ a_{21}y(b) - a_{22}(p^{\frac{1}{r}}y')(b) = 0, \end{aligned} \tag{1.3}$$

where $a_{ij} \in \mathbb{R}$ and $a_{i1}^2 + a_{i2}^2 \neq 0$ for $i, j = 1, 2$.

To implement the shooting method, we may begin with a solution of Eq. (1.1) satisfying an initial condition generated from the first boundary condition of (1.3) and containing a parameter. Then adjust the value of the parameter to meet the second boundary condition of (1.3). Such an approach requires that solutions to IVP associated with Eq. (1.1) exist uniquely on the whole interval $[a, b]$ and hence depend on the parameter in a continous way.

This paper provides a foundation for a forthcoming paper on the existence of nodal solutions of boundary value problem (1.1), (1.3).

Throughout this papter, we make the following assumptions:

(H1) $p, q, w \in C^1[a, b]$ with $p, w > 0$ on $[a, b]$;
(H2) $f \in C(\mathbb{R})$ such that $yf(y) > 0$ for $y \neq 0$ and f is locally Lipschitz continuous on $\mathbb{R}\backslash\{0\}$;
(H3) There exist limits f_0 and f_∞ such that $0 \le f_0, f_\infty \le \infty$, where

$$f_0 = \lim_{y \to 0} \frac{f(y)}{\phi(y)} \quad \text{and } f_\infty = \lim_{y \to \pm\infty} \frac{f(y)}{\phi(y)}.$$

The following is our main result.

Theorem 1.1. *For any $t_0 \in [a, b]$ and $y_0, z_0 \in \mathbb{R}$, IVP* (1.1), (1.2) *has a solution which exists uniquely on the whole interval $[a, b]$.*

2. PRELIMINARIES FOR THE PROOF

2.1. **Generalized Trignometric Functions.** We will use the generalized Prüfer transformation introduced by Elbert [1] in the proof of the main theorem, so we first introduce some basic knowledge on the generalized trigonometric functions.

Let $S = S(\theta)$ be the unique solution of the half-linear differential equation

$$\frac{d}{d\theta}(\phi(\frac{dS}{d\theta})) + r\phi(S) = 0$$

satisfying the initial condition

$$S(0) = 0, \frac{dS(\theta)}{d\theta}|_{\theta=0} = 1.$$

Then $S = S(\theta)$ is called the generalized sine function. It is easy to see that $S = S(\theta)$ is periodic with period $2\pi_r$, where

$$\pi_r = \frac{2\pi}{(r+1)} / \sin\frac{\pi}{r+1}.$$

For $k \in \mathbb{Z}$, $S(k\pi_r) = 0$, $S(\theta) > 0$ for $\theta \in (2k\pi_r, (2k+1)\pi_r)$ and $S(\theta) < 0$ for $\theta \in ((2k+1)\pi_r, (2k+2)\pi_r)$. The generalized cosine function $C(\theta)$ is defined by $C(\theta) = dS(\theta)/d\theta$. $C(\theta)$ is even and periodic with period $2\pi_r$. For $k \in \mathbb{Z}$, $C((k+1/2)\pi_r) = 0$, $C(\theta) > 0$ for $\theta \in ((2k-1/2)\pi_r, (2k+1/2)\pi_r)$ and $C(\theta) < 0$ for $\theta \in ((2k+1/2)\pi_r, (2k+3/2)\pi_r)$. The functions $S(\theta)$ and $C(\theta)$ satisfy the relation that

$$|S(\theta)|^{r+1} + |C(\theta)|^{r+1} = 1 \quad \text{for } \theta \in \mathbb{R}.$$

The generalized tangent function $T(\theta)$ is defined by

$$T(\theta) = \frac{S(\theta)}{C(\theta)} \quad \text{for } \theta \neq (k+1/2)\pi_r, \quad k \in \mathbb{Z}.$$

Hence it is a periodic function of period π_r and satisfies

$$T'(\theta) = 1 + |T(\theta)|^{r+1} \quad \text{for } \theta \neq (k+1/2)\pi_r, \quad k \in \mathbb{Z}.$$

For $k \in \mathbb{Z}$, $T(\theta)$ is strictly increasing for $\theta \in ((k-1/2)\pi_r, (k+1/2)\pi_r)$. The inverse of $T(\cdot)$ on $(-\pi_r/2, \pi_r/2)$ is denoted by $T^{-1}(\cdot)$.

2.2. **Integral Inequalities.** We need the following lemma to prove our main theorem.

Lemma 2.1. *Let $x(t) \geq 0$ be a continuous solution of the inequality*

$$x(t) \leq C + \int_{t_0}^{t} \phi^{-1}\left(D + K\int_{t_0}^{s} \phi(x(\tau))d\tau\right) ds, \quad t \in [t_0, t_0+T),$$

where $0 < T < \infty$, $C, D, K \in \mathbb{R}$ such that $C, D \geq 0$ and $K > 0$. Then

(a) $x(t)$ is bounded on $[t_0, t_0+T)$ if $D > 0$;
(b) $x(t) \equiv 0$ on $[t_0, t_0+T)$ if $C = D = 0$.

Proof. Let

$$r(t)=C+\int_{t_0}^{t}\phi^{-1}\left(D+K\int_{t_0}^{s}\phi(x(\tau))d\tau\right)ds.$$

Then $x(t)\le r(t)$ for $t\in[t_0,t_0+T)$, and

$$\begin{aligned} r'(t)&=\phi^{-1}\left(D+K\int_{t_0}^{t}\phi(x(s))ds\right)\le\phi^{-1}\left(D+K\int_{t_0}^{t}\phi(r(s))ds\right)\\ &\le\phi^{-1}\left(D+KT\phi(r(t))\right). \end{aligned}\tag{2.1}$$

(i) Assume $D>0$. Then for $t\in[t_0,t_0+T)$

$$\frac{dr(t)}{\phi^{-1}(D+KT\phi(r(t))}\le dt.$$

Integrating both sides from t_0 to t and making the substitution $u=r(t)$, we have that for $t\in[t_0,t_0+T)$

$$\int_{r(t_0)}^{r(t)}\frac{du}{\phi^{-1}(D+KT\phi(u))}\le t-t_0<T.\tag{2.2}$$

Assume $x(t)$ is unbounded on $[t_0,t_0+T)$. Then $\limsup_{t\to(t_0+T)-}x(t)=\infty$, so $\lim_{t\to(t_0+T)-}r(t)=\infty$. Therefore

$$\begin{aligned} \lim_{t\to(t_0+T)-}\int_{r(t_0)}^{r(t)}\frac{du}{\phi^{-1}(D+KT\phi(u))}&=\int_{r(t_0)}^{\infty}\frac{du}{\phi^{-1}(D+KT\phi(u))}\\ &\ge\int_{r^*(t_0)}^{\infty}\frac{du}{\phi^{-1}\left((D+KT)\phi(u)\right)}=\int_{r^*(t_0)}^{\infty}\frac{du}{\phi^{-1}(D+KT)u}=\infty, \end{aligned}$$

where $r^*(t_0)=\max\{r(t_0),1\}$. This contradicts (2.2). Therefore, $x(t)$ is bounded on $[t_0,t_0+T)$.

(ii) Assume $C=D=0$. From (2.1) we have that for $t\in[t_0,t_0+T)$

$$r'(t)\le\phi^{-1}\left(KT\phi(r(t))\right)=\phi^{-1}(KT)r(t).$$

Hence

$$r'(t)-\phi^{-1}(KT)r(t)\le 0.$$

By multiplying both sides of the inequality by $e^{-\phi^{-1}(KT)t}$, we have that for $t\in[t_0,t_0+T)$

$$[e^{-\phi^{-1}(KT)t}r(t)]'\le 0,$$

so

$$e^{-\phi^{-1}(KT)t}r(t)\le e^{-\phi^{-1}(KT)t_0}r(t_0)=0.$$

This implies $r(t)\le 0$, and hence $x(t)\le r(t)\le 0$. Note that $x(t)\ge 0$. Therefore, $x(t)\equiv 0$. ∎

3. PROOF OF THE MAIN RESULT

Notice that, if we let $\tau(t) = \int_a^t 1/p^{1/r}(s)ds$, $t = t(\tau)$ the inverse function, and $u(\tau) = y(t(\tau))$, then IVP (1.1), (1.2) can be transformed to an IVP for the unknown function $u(\tau)$ consisting of the equation

$$-\frac{d}{d\tau}(\phi(\frac{du}{d\tau})) + Q(\tau)\phi(u) = W(\tau)f(u) \quad \text{on } \left[0, \int_a^b 1/p^{1/r}(s)ds\right]$$

and the initial condition

$$u(\tau_0) = y_0, \quad \frac{du}{d\tau}|_{\tau=\tau_0} = z_0,$$

where $Q(\tau) = p^{1/r}(t(\tau))q(t(\tau))$, $W(\tau) = p^{1/r}(t(\tau))w(t(\tau))$, and $\tau_0 = \tau(t_0)$. Thus for simplicity, we will only prove the case when $p(t) \equiv 1$, i.e., the IVP consisting of the equation

$$-[\phi(y')]' + q(t)\phi(y) = w(t)f(y) \quad \text{on } [a, b] \tag{3.1}$$

and the initial conditions

$$y(t_0) = y_0, \quad y'(t_0) = z_0. \tag{3.2}$$

Let $z = \phi(y')$. Then IVP (3.1), (3.2) can be written as the IVP consisting of the equation

$$\begin{pmatrix} y \\ z \end{pmatrix}' = \begin{pmatrix} \phi^{-1}(z) \\ -w(t)f(y) + q(t)\phi(y) \end{pmatrix} \tag{3.3}$$

and the initial condition

$$y(t_0) = y_0, \quad z(t_0) = \phi(z_0). \tag{3.4}$$

Let $\theta(t)$ and $\rho(t)$ be the generalized Prüfer angle and the Prüfer distance of $y(t)$, i.e., they are continuous functions satisfying

$$\begin{cases} y(t) = \rho(t)S(\theta(t)) \\ y'(t) = \rho(t)C(\theta(t)), \end{cases}$$

and equivalently,

$$\rho(t) = (|y(t)|^{r+1} + |y'(t)|^{r+1})^{\frac{1}{r+1}}, \quad \theta(t) = T^{-1}\left(\frac{y(t)}{y'(t)}\right). \tag{3.5}$$

It is easy to see that IVP (3.1), (3.2) is equivalent to the IVP consisting of the system

$$\begin{aligned} \theta'(t) &= |C(\theta(t))|^{r+1} + \frac{w(t)f(\rho(t)S(\theta(t)))S(\theta(t))}{r\rho(t)^r} - \frac{q(t)|S(\theta(t))|^{r+1}}{r} \\ \rho'(t) &= \rho(t)\left[1 + \frac{q(t)}{r}\right]\phi\left(S(\theta(t))\right)C(\theta(t)) - \frac{w(t)f(\rho(t)S(\theta(t)))C(\theta(t))}{r\rho(t)^{r-1}} \end{aligned} \tag{3.6}$$

and the initial condition

$$\theta(t_0) = T^{-1}\left(\frac{y_0}{z_0}\right), \quad \rho(t_0) = (|y_0|^{r+1} + |z_0|^{r+1})^{\frac{1}{r+1}}. \tag{3.7}$$

The following result is on the global existence of the solution of IVP (3.1), (3.2).

Proposition 3.1. For any $t_0 \in [a,b]$ and $y_0, z_0 \in \mathbb{R}$, IVP (3.1), (3.2) has a solution which exists on the whole interval $[a,b]$.

Proof. The existence of a local solution of IVP (3.1), (3.2) is guaranteed by the Cauchy-Peano existence theorem. Now we show that the solution exists on the whole interval $[a,b]$. Assume the contrary, without loss of generality, we may assume $y(t)$ exits on a maximal right interval $[t_0, c)$ for some $c \in (t_0, b)$. Then $y(t)$ is unbounded on $[t_0, c)$.

(I) Consider the case when $f_\infty < \infty$. For any $\epsilon > 0$, there exists $M > 0$ such that

$$|f(y)| \le (f_\infty + \epsilon)\phi(|y|) \quad \text{for } |y| \ge M. \tag{3.8}$$

Let

$$K = \max_{\tau\in[t_0,c)} \{(f_\infty + \epsilon)w(\tau) + |q(\tau)|\}. \tag{3.9}$$

Case (i). Assume $y(t)$ is not oscillatory about M or $-M$ at c. Then there exists $t_* \in [t_0, c)$ such that $|y(t)| \ge M$ for all $t \in [t_*, c)$ and $|y'(t_*)| > 0$. From Eq. (3.1), for $t \in [t_*, c)$ we have

$$y(t) = y(t_*) + \int_{t_*}^{t} \phi^{-1}\left(\phi(y'(t_*)) - \int_{t_*}^{s} [w(\tau)f(y(\tau)) - q(\tau)\phi(y(\tau))]d\tau\right) ds. \tag{3.10}$$

Then from (3.8) and (3.9),

$$|y(t)| \le |y(t_*)| + \int_{t_*}^{t} \phi^{-1}\left(\phi(|y'((t_*)|) + K\int_{t_*}^{s} \phi(|y(\tau)|)d\tau\right) ds.$$

By Lemma 2.1 (i), $|y(t)|$ is bounded on $[t_0, c)$, contradicting the assumption.

Case (ii). Assume $y(t)$ is oscillatory about M at c. Then there exist sequences $\{t_n\}_{n=1}^\infty$ and $\{\tau_n\}_{n=1}^\infty$ in $[t_0, c)$ such that for each $n \in \mathbb{N}, t_n < \tau_n, t_n, \tau_n \to c-$, t_n is a local maximal point of y, $y(t_n) \to \infty$, $y(\tau_n) = M$, and $y(t)$ is decreasing on $[t_n, \tau_n]$. Note that by (3.8) and (3.9) that for $t \in [t_n, \tau_n]$

$$\begin{aligned}\int_{t_n}^{t} [w(s)f(y(s)) - q(s)\phi(y(s))]ds &\le \int_{t_n}^{t} [(f_\infty + \epsilon)w(s) + |q(s)|]\phi(y(s))ds \\ &\le \left(\int_{t_n}^{t} [(f_\infty + \epsilon)w(s) + |q(s)|]ds\right)\phi(y(t_n)) \le K(t - t_n)\phi(y(t_n)).\end{aligned}$$

Then (3.10) with $t_* = t_n$ and $t = \tau_n$ implies that

$$\begin{aligned}M = y(\tau_n) &= y(t_n) - \int_{t_n}^{\tau_n} \phi^{-1}\left(\int_{t_n}^{s} [w(\tau)f(y(\tau)) - q(\tau)\phi(y(\tau))]d\tau\right) ds \\ &\ge y(t_n) - \int_{t_n}^{\tau_n} \phi^{-1}\left(K(s - t_n)\phi(y(t_n))\right) ds \\ &\ge y(t_n) - \int_{t_n}^{\tau_n} \phi^{-1}(K(\tau_n - t_n))\phi(y(t_n))ds\end{aligned}$$

$$= \left(1 - [\phi^{-1}(K(\tau_n - t_n))](\tau_n - t_n)\right) y(t_n).$$

Letting $n \to \infty$ in the above inequality we get $\lim_{n\to\infty} y(t_n) \le M$, contradicting the assumption that $y(t_n) \to \infty$.

Case (iii). Assume $y(t)$ is oscillatory about $-M$ at c. The proof is similar to Case (ii) and hence is omitted.

(II) Consider the case when $f_\infty = \infty$. Define a generalized energy function for $y(t)$ by

$$[E(y)](t) = \frac{r}{r+1}|y'(t)|^{r+1} - \frac{1}{r+1}q(t)|y(t)|^{r+1} + w(t)F(y(t)), \tag{3.11}$$

where $F(y) = \int_0^y f(\sigma)d\sigma$. In view of Eq. (3.1) we find that

$$[E(y)]'(t) = -\frac{1}{r+1}q'(t)|y(t)|^{r+1} + w'(t)F(y(t)). \tag{3.12}$$

Let $k = \max\{|w'(t)|/w(t) : t \in [a,b]\}$. From (3.12) we have that for $t \in [a,b]$

$$\begin{aligned}[E(y)]'(t) &= -\frac{1}{r+1}q'(t)|y(t)|^{r+1} + \frac{w'(t)}{w(t)}[w(t)F(y(t))] \\ &\le -\frac{k+1}{r+1}q(t)|y(t)|^{r+1} + \frac{1}{r+1}[(k+1)q(t) - q'(t)]|y(t)|^{r+1} \\ &\quad + kw(t)F(y(t)).\end{aligned} \tag{3.13}$$

Because $w(t) > 0$ is continuous and $q(t)$, $q'(t)$ are bounded on $[a,b]$, we can find a constant $h > 0$ such that for $t \in [t_0, c)$

$$\frac{h}{r+1}[(k+1)q(t) - q'(t)] \le w(t). \tag{3.14}$$

Since $f_\infty = \infty$, we have $|y|^{r+1} = o(F(y))$ as $|y| \to \infty$. This means that there exists $M > 0$ such that $|y|^{r+1} \le hF(y)$ for $|y| \ge M$. Define

$$I_1 = \{t \in [t_0, c) : |y(t)| \le M\} \quad \text{and} \quad I_2 = \{t \in [t_0, c) : |y(t)| > M\}.$$

Then from (3.13) and (3.14), there exists $N > 0$ such that $[E(y)]'(t) \le N$ for $t \in I_1$, and for $t \in I_2$

$$[E(y)]'(t) \le (k+1)\left[-\frac{1}{r+1}q(t)|y(t)|^{r+1} + w(t)F(y(t))\right] \le (k+1)[E(y)](t).$$

Hence for $t \in [t_0, c)$ we have

$$\begin{aligned}[E(y)](t) &= [E(y)](t_0) + \int_{t_0}^t [E(y)]'(s) \\ &\le [E(y)](t_0) + \int_{[t_0,t]\cap I_1} N ds + \int_{[t_0,t]\cap I_2} (k+1)[E(y)](s)ds \\ &\le [E(y)](t_0) + N(c - t_0) + \int_{[t_0,t]\cap I_2} (k+1)[E(y)](s)ds.\end{aligned}$$

Let $N_* = |[E(y)](t_0)| + |N(c - t_0)|$. Then for $t \in [t_0, c)$

$$|[E(y)](t)| \le N_* + \int_{[t_0,t]\cap I_2} (k+1)|[E(y)](s)|ds \le N_* + \int_{t_0}^{t} (k+1)|[E(y)](s)|ds.$$

By the Gronwall inequality, for $t \in [t_0, c)$

$$|[E(y)](t)| \le N_* e^{(k+1)(t-t_0)} \le N_* e^{(k+1)(c-t_0)}.$$

Therefore, $\limsup_{t\to c-} |([E(y)](t)| < \infty$. On the other hand, since $y(t)$ is unbounded on $[t_0, c)$, there exists a sequence $t_n \to c-$ such that $|y(t_n)| \to \infty$. Since $\lim_{y\to\infty} F(y)/|y|^{r+1} = f_\infty = \infty$, by (3.11)

$$[E(y)](t_n) \ge \left(-\frac{1}{r+1}q(t_n) + w(t_n)\frac{F(y(t_n))}{|y(t_n)|^{r+1}}\right)|y(t_n)|^{r+1} \to \infty \quad \text{as } n \to \infty,$$

i.e. $|[E(y)](t_n)| \to \infty$ as $n \to \infty$. We have reached a contradiction. □

Proposition 3.2. For any $t_0 \in [a, b]$ and $y_0, z_0 \in \mathbb{R}$, the solution of IVP (3.1), (3.2) is unique.

Proof. Since $\phi^{-1}(z)$, $\phi(y)$ and $f(y)$ are locally Lipschitz continuous in $y, z \in \mathbb{R}\backslash\{0\}$, from the equivalent form (3.3), (3.4) to IVP (3.1), (3.2), we see that the local solution of IVP (3.1), (3.2) is unique for the case when $y_0 \neq 0$ and $z_0 \neq 0$. Thus it suffices to show the uniqueness of a local solution of the problem in the case when $y_0 z_0 = 0$. We divide the proof into the following three cases: (i) $y_0 = 0, z_0 \neq 0$; (ii) $y_0 \neq 0, z_0 = 0$; (iii) $y_0 = 0, z_0 = 0$. We will show the uniqueness in a right-neighborhood of t_0 only. The proof for the uniqueness in a left-neighborhood of t_0 is similar and hence is omitted.

(i) Assume $y_0 = 0, z_0 \neq 0$. We may assume that $z_0 > 0$ without loss of generality. Let y_1 and y_2 be solutions of IVP (3.1), (3.2). Then there exists a $t_1 \in (t_0, b]$ such that $y_j'(t) \ge z_0/2$ for $t \in [t_0, t_1]$, $j = 1, 2$. Let $[E(y_i)](t)$ be the generalized energy function for $y_i(t)$, $i = 1, 2$, as defined by (3.11). In view of (3.12), we obtain that for $j = 1, 2$,

$$[E(y_j)](t) - [E(y_j)](t_0) = \int_{t_0}^{t} \left(-\frac{1}{r+1}q'(s)|y_j(s)|^{r+1} + w'(s)F(y_j(s))\right) ds.$$

Note that $[E(y_1)](t_0) = [E(y_2)](t_0)$, we have

$$\begin{aligned} &[E(y_1)](t) - [E(y_2)](t) \\ &= \int_{t_0}^{t} \left[-\frac{1}{r+1}q'(s)(|y_1(s)|^{r+1} - |y_2(s)|^{r+1})\right] ds \\ &\quad + \int_{t_0}^{t} \left[w'(s)\left(F(y_1(s)) - F(y_2(s))\right)\right] ds. \end{aligned}$$

Therefore, by (3.11)

$$(3.15) \qquad |y_1'(t)|^{r+1} - |y_2'(t)|^{r+1}$$

$$
\begin{aligned}
&= \frac{1}{r}q(t)(|y_1(t)|^{r+1} - |y_2(t)|^{r+1}) - \frac{r+1}{r}w(t)(F(y_1(t)) - F(y_2(t))) \\
&+ \int_{t_0}^{t} \left[-\frac{1}{r}q'(s)(|y_1(s)|^{r+1} - |y_2(s)|^{r+1})\right] ds \\
&+ \int_{t_0}^{t} \left[\frac{r+1}{r}w'(s)(F(y_1(s)) - F(y_2(s)))\right] ds.
\end{aligned}
$$

By the Mean value theorem, there exist continuous functions $\xi_1(t)$ and $\xi_2(t)$ between $y_1(t)$ and $y_2(t)$ such that for $t \in [t_0, t_1]$

$$\left||y_1(t)|^{r+1} - |y_2(t)|^{r+1}\right| = (r+1)|\phi(\xi_1(t))||y_1(t) - y_2(t)|$$

and

$$|F(y_1)(t) - F(y_2)(t)| = |f(\xi_2(t))||y_1(t) - y_2(t)|.$$

Note that $y_1(t)$ and $y_2(t)$ are bounded on $[t_0, t_1]$, there exists $A > 0$ such that for $t \in [t_0, t_1]$, $|\phi(\xi_1(t)| \le A$ and $|f(\xi_2(t)| \le A$. Then

$$\left||y_1(t)|^{r+1} - |y_2(t)|^{r+1}\right| \le A|y_1(t) - y_2(t)| \tag{3.16}$$

and

$$|F(y_1)(t) - F(y_2)(t)| \le A|y_1(t) - y_2(t)|. \tag{3.17}$$

From (3.15), (3.16), and (3.17), we have that for $t \in [t_0, t_1]$

$$\left||y_1'(t)|^{r+1} - |y_2'(t)|^{r+1}\right| \le C_1|y_1(t) - y_2(t)| + \int_{t_0}^{t} C_2|y_1(s) - y_2(s)|ds, \tag{3.18}$$

where

$$
\begin{aligned}
C_1 &= \frac{A}{r}\left(\max_{t\in[t_0,t_1]}\{|q(t)|\} + (r+1)\max_{t\in[t_0,t_1]}\{|w(t)|\}\right), \\
C_2 &= \frac{A}{r}\left(\max_{t\in[t_0,t_1]}\{|q'(t)|\} + (r+1)\max_{t\in[t_0,t_1]}\{|w'(t)|\}\right).
\end{aligned}
$$

Since $y_j'(t) \ge z_0/2$ for $t \in [t_0, t_1]$, then

$$\left||y_1'(t)|^{r+1} - |y_2'(t)|^{r+1}\right| \ge (r+1)\left(\frac{z_0}{2}\right)^r |y_1'(t) - y_2'(t)|, \quad t \in [t_0, t_1]. \tag{3.19}$$

Let $Y(t) = y_1(t) - y_2(t)$ and $C_0 = (r+1)(z_0/2)^r$. Using (3.18) and (3.19) we see that for $t \in [t_0, t_1]$

$$C_0|Y'(t)| \le C_1|Y(t)| + \int_{t_0}^{t} C_2|Y(t)|dt. \tag{3.20}$$

Let $C = \max\{C_1/C_0, C_2/C_0\}$ and $|Y(t)|_R'$ the right derivative of $|Y(t)|$. Then

$$|Y(t)|_R' \le |Y'(t)| \le C|Y(t)| + C\int_{t_0}^{t} |Y(s)|ds, \quad t \in [t_0, t_1]. \tag{3.21}$$

Integrating (3.21) over $[t_0, t]$ we get that for $t \in [t_0, t_1]$

$$|Y(t)| \le C\int_{t_0}^{t} |Y(s)|ds + C\int_{t_0}^{t} (t-s)|Y(s)|ds$$

$$\leq C(1+t_1-t_0)\int_{t_0}^{t}|Y(s)|ds.$$

By the Gronwall inequality, we have $Y(t)\equiv 0$, i.e., $y_1(t)\equiv y_2(t)$ for $t\in[t_0,t_1]$.

(ii) Assume $y_0\neq 0, z_0=0$. Let $y(t)$ be a solution of IVP (3.1), (3.2), and $\theta(t)$, $\rho(t)$ the generalized Prüfer angle and distance of $y(t)$. Since $y_0\neq 0$, there exists $t_1\in(t_0,b]$ such that $S(\theta(t))\neq 0$ and $\rho(t)\neq 0$ for $t\in[t_0,t_1]$. Hence the right hand sides of (3.6) are locally Lipchitz in θ and ρ which garantees that the solution of IVP (3.6), (3.7) is unique, and so is of IVP (3.1), (3.2).

(iii) Assume $y_0=0, z_0=0$. We will show $y\equiv 0$ and hence the solution of the IVP is unique.

First, we consider the case when $f_0<\infty$. For any $\epsilon>0$, there exists $\delta>0$ such that

$$|f(y)|\leq (f_0+\epsilon)\phi(|y|)\quad \text{ for } |y|<\delta.$$

Since $y_0=0$, there exists $t_1\in(t_0,b]$ such that $|y(t)|<\delta$ for $t\in[t_0,t_1)$. From (3.1), (3.2) with $y_0=z_0=0$, we have that for $t\in[t_0,t_1)$,

$$y(t)=\int_{t_0}^{t}\phi^{-1}\left(-\int_{t_0}^{s}[w(\tau)f(y(\tau))-q(\tau)\phi(y(\tau))]d\tau\right)ds.$$

Hence

$$|y(t)|\leq\int_{t_0}^{t}\phi^{-1}\left(\max_{\tau\in[a,b]}\{(f_0+\epsilon)w(\tau)+|q(\tau)|\}\int_{t_0}^{s}\phi(|y(\tau)|)d\tau\right)ds.$$

By Lemma 2.1 (ii), we have $y(t)\equiv 0$ for $t\in[t_0,t_1)$.

Then we consider the case when $f_0=\infty$. Let $E(y)$ be defined by (3.11). Then (3.13) holds on $[t_0,b]$. Choose $h>0$ such that (3.14) holds on $[t_0,b]$. Since $f_0=\infty$, we have $|y|^{r+1}=o(F(y))$ as $y\to 0$. This means there exists $\delta>0$ such that $|y|^{r+1}<hF(y)$ for $|y|<\delta$. Since $y_0=0$, there exists $c\in(t_0,b]$ such that $|y(t)|<\delta$ for $t\in(t_0,c)$. It follows from (3.13) that for $t\in(t_0,c)$

$$[E(y)]'(t)\leq (k+1)[E(y)](t).$$

Note that $[E(y)](t_0)=0$, so we have $[E(y)](t)\leq 0$ for $t\in(t_0,c)$. This implies that $y(t)\equiv 0$ on $t\in(t_0,c)$. For otherwise, there exists $t_1\in(t_0,c)$ such that $y(t_1)\neq 0$ and $y(t_1)$ is sufficiently close to 0. Then from (3.11)

$$[E(y)](t_1)\geq\left(-\frac{1}{r+1}q(t_1)+w(t_1)\frac{F(y(t_1))}{|y(t_1)|^{r+1}}\right)|y(t_1)|^{r+1}>0.$$

We have reached a contradiction. $\square$

Note that IVP (1.1), (1.2) can be transformed to IVP (3.1), (3.2). Then Theorem 1.1 is the combination of Propositions 3.1 and 3.2.

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