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A TURNPIKE PROPERTY FOR A CLASS OF DISCRETE-TIME OPTIMAL CONTROL SYSTEMS AND POROSITY

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ABSTRACT. In the paper we investigate the structure of solutions of discrete-time control systems with a compact metric space of states. We are interested in turnpike properties of the approximate solutions which are independent of the length of the interval, for all sufficiently large intervals. Using the porosity notion we show that most control systems possess the turnpike property.

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1. INTRODUCTION

The study of the existence and the structure of solutions of optimal control problems defined on infinite intervals and on sufficiently large intervals has recently been a rapidly growing area of research. See, for example, [5–7, 9, 10, 15, 16, 18, 25, 26, 28, 34–40] and the references mentioned therein. These problems arise in engineering [1, 19], in models of economic growth [2, 8, 9, 14, 18, 21, 23, 24, 31, 32, 38–40], in infinite discrete models of solid-state physics related to dislocations in one-dimensional crystals [4, 33] and in the theory of thermodynamical equilibrium for materials [11, 20, 22]. In this paper we study the structure of solutions of a discrete-time optimal control system considered in [34–40].

Let (K, ρ) be a compact metric space, R^n be the n-dimensional Euclidean space equipped with the Euclidean norm and let $C(K \times K)$ be the space of all continuous functions defined on the space $K \times K$.

For each $v \in C(K \times K)$ set

$$\|v\| = \sup\{|v(z_1, z_2)| : z_1, z_2 \in K\}. \tag{1.1}$$

Denote by $\mathcal{M}$ the set of all sequences $\{v_i\}_{i=0}^{\infty}$, where $v_i \in C(K \times K)$, $i = 0, 1, \dots$ and such that

$$\sup\{\|v_i\| : i = 0, 1, \dots\} < \infty. \tag{1.2}$$

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Let $\{v_i\}_{i=0}^{\infty} \in \mathcal{M}$. For each pair of integers T_1, T_2 satisfying $0 \le T_1 < T_2$ and each $y, z \in K$ we consider the optimization problems

$$\sum_{i=T_1}^{T_2-1} v_i(x_i, x_{i+1}) \to \min,$$

$$\{x_i\}_{i=T_1}^{T_2} \subset K, x_{T_1} = y, \quad x_{T_2} = z$$

and

$$\sum_{i=T_1}^{T_2-1} v_i(x_i, x_{i+1}) \to \min,$$

$$\{x_i\}_{i=T_1}^{T_2} \subset K, x_{T_1} = y.$$

The interest in discrete-time optimal problems of these types stems from the study of various optimization problems which can be reduced to it, e.g., continuous-time control systems which are represented by ordinary differential equations whose cost integrand contains a discounting factor [9], tracking problems in engineering [20], the study of Frenkel-Kontorova model related to dislocations in one-dimensional crystals [4, 33] and the analysis of a long slender bar of a polymeric material under tension in [11, 20, 22]. See also [18, 34–40] and the references mentioned therein.

We are interested in a turnpike property of the approximate solutions of these problems which is independent of the length of the interval $[T_1, T_2]$, for all sufficiently large intervals. To have this property means, roughly speaking, that the approximate solutions of the optimal control problems are determined mainly by the cost function v, and are essentially independent of T_2, T_1, y and z. Turnpike properties are well known in mathematical economics. The term was first coined by Samuelson in 1948 (see [31]) where he showed that an efficient expanding economy would spend most of the time in the vicinity of a balanced equilibrium path (also called a von Neumann path). This property was further investigated for optimal trajectories of models of economic dynamics (see, for example, [2, 14, 21, 23, 24, 32, 38] and the references mentioned there).

Recently it was shown that the turnpike property is a general phenomenon which holds for large classes of variational and optimal control problems [34–38]. For each of these classes we established that most problems (integrands, objective functions respectively) in the sense of Baire category possess the turnpike property.

In the sequel we say that a property of elements of a complete metric space Z is generic (typical) in Z if the set of all elements of Z which possess this property contains an everywhere dense G_δ subset of Z. In this case we also say that the property holds for a generic (typical) element of Z or that a generic (typical) element of Z possesses the property [3, 11, 27, 38].

In [34–38] we identified a class of problems with a complete metric space of objective functions (integrands) and showed that the turnpike property holds for a generic element of the space of objective functions (integrands, respectively). In this paper we use the concept of porosity which will enable us to obtain more refined results.

First we recall the concept of porosity. Let Z be a metric space. We denote by $B_Z(y,r)$ the closed ball of center $y \in Z$ and radius $r > 0$. A subset $E \subset Z$ is called porous in Z if there exist $\alpha \in (0,1)$ and $r_0 > 0$ such that for each $r \in (0, r_0]$ and each $y \in Z$, there exists $z \in Z$ for which

$$B_Z(z, \alpha r) \subset B_Z(y,r) \setminus E.$$

A subset of the space Z is called σ-porous in Z if it is a countable union of porous subsets in Z.

Other notions of porosity can be found in the literature. We use the rather strong concept of porosity which has already found application in, for example, [12], [13], [29], [30].

Since porous sets are nowhere dense, all σ-porous sets are of the first category. If Y is a finite-dimensional Euclidean space, then σ-porous sets are of Lebesgue measure zero. In fact, the class of σ-porous sets in such a space is much smaller than the class of sets which have measure zero and are of the first category.

In this paper we equip the space $\mathcal{M}$ with an appropriate complete metric and construct a set $\mathcal{F} \subset \mathcal{M}$ such that the complement $\mathcal{M} \setminus \mathcal{F}$ is porous set in the metric space $\mathcal{M}$ and that for any element of $\mathcal{F}$ the corresponding optimal control system possesses the turnpike property.

In the sequel we use the following notation and definitions.

Put

$$\operatorname{diam}(K) = \sup\{\rho(z_1, z_2) : z_1, z_2 \in K\}. \tag{1.3}$$

We assume that the sum over empty set is zero.

For each $\{v_i\}_{i=0}^{\infty} \in \mathcal{M}$, each $y, z \in K$ and each pair of integers T_1, T_2 satisfying $0 \le T_1 < T_2$ set

$$\begin{aligned} &U(\{v_i\}_{i=0}^{\infty}, y, z, T_1, T_2) \\ &= \inf\left\{\sum_{i=T_1}^{T_2-1} v_i(x_i, x_{i+1}) : \{x_i\}_{i=T_1}^{T_2} \subset K, x_{T_1} = y, x_{T_2} = z\right\}, \end{aligned} \tag{1.4}$$

$$\tilde{U}(\{v_i\}_{i=0}^{\infty}, y, T_1, T_2) = \inf\left\{\sum_{i=T_1}^{T_2-1} v_i(x_i, x_{i+1}) : \{x_i\}_{i=T_1}^{T_2} \subset K, x_{T_1} = y\right\}. \tag{1.5}$$

The following two results will be proved in Section 2.

Theorem 1.1. *Let $\{v_i\}_{i=0}^{\infty} \in \mathcal{M}$ and $y \in K$. Then there exists a sequence $\{x_i\}_{i=0}^{\infty} \subset K$ such that $x_0 = y$ and*

$$\sum_{i=0}^{T-1} v_i(x_i, x_{i+1}) = U(\{v_i\}_{i=0}^{\infty}, y, x(T), 0, T)$$

for all natural numbers T.

Theorem 1.2. *Let $\{v_i\}_{i=0}^{\infty} \in \mathcal{M}$, $0 \le T_1 < T_2$ be integers and let $\{x_i\}_{i=T_1}^{T_2} \subset K$ satisfy*

$$\sum_{i=T_1}^{T_2-1} v_i(x_i, x_{i+1}) = U(\{v_i\}_{i=0}^{\infty}, x_{T_1}, x_{T_2}, T_1, T_2).$$

Then for each $\{y_i\}_{i=T_1}^{T_2} \subset K$,

$$\sum_{i=T_1}^{T_2-1} v_i(y_i, y_{i+1}) \ge \sum_{i=T_1}^{T_2-1} v_i(x_i, x_{i+1}) - 4 \sup\{\|v_i\| : i = 0, 1, \dots\}. \tag{1.6}$$

Corollary 1.3. *Let $\{v_i\}_{i=0}^{\infty} \subset \mathcal{M}$ and let $\{x_i\}_{i=0}^{\infty} \subset K$ satisfy*

$$\sum_{i=0}^{T-1} v_i(x_i, x_{i+1}) = U(\{v_i\}_{i=0}^{\infty}, x_0, x_T, 0, T)$$

for any integer $T \ge 1$. Then for each $\{y_i\}_{i=0}^{\infty} \subset K$

$$\sum_{i=0}^{T-1} v_i(y_i, y_{i+1}) - \sum_{i=0}^{T-1} v_i(x_i, x_{i+1}) \ge -4 \sup\{\|v_i\| : i = 0, 1, \dots\}$$

for all integers $T \ge 1$.

Corollary 1.4. *Let $\{v_i\}_{i=0}^{\infty} \subset \mathcal{M}$, $\{x_i\}_{i=0}^{\infty} \subset K$, satisfy*

$$\sum_{i=0}^{T-1} v_i(x_i, x_{i+1}) = U(\{v_i\}_{i=0}^{\infty}, x_0, x_T, 0, T)$$

for all integers $T \ge 1$ and let $\{y_i\}_{i=0}^{\infty} \subset K$. Then either the sequence

$$\left\{ \sum_{i=0}^{T} v_i(y_i, y_{i+1}) - \sum_{i=0}^{T-1} v_i(x_i, x_{i+1}) : T = 0, 1, \dots \right\}$$

is bounded or $\sum_{i=0}^{T} v_i(y_i, y_{i+1}) - \sum_{i=0}^{T-1} v_i(x_i, x_{i+1}) \to \infty$ as $T \to \infty$.

Let $\{v_i\}_{i=0}^{\infty} \in \mathcal{M}$. A sequence $\{x_i\}_{i=0}^{\infty} \subset K$ is called $(\{v_i\}_{i=0}^{\infty})$-good [9, 14, 38] if for each $\{y_i\}_{i=0}^{\infty} \subset K$ there is $M > 0$ such that

$$\sum_{i=0}^{T-1} v_i(y_i, y_{i+1}) - \sum_{i=0}^{T-1} v_i(x_i, x_{i+1}) \ge -M \text{ for all integers } T \ge 1.$$

Let $\{v_i\}_{i=0}^\infty \in \mathcal{M}$. A sequence $\{x_i\}_{i=0}^\infty \subset K$ is called ($\{v_i\}_{i=0}^\infty$)-overtaking optimal if for any sequence $\{y_i\}_{i=0}^\infty \subset K$ satisfying $y_0 = x_0$,

$$\limsup_{T\to\infty} \left[\sum_{i=0}^{T} v_i(x_i, x_{i+1}) - \sum_{i=0}^{T} v_i(y_i, y_{i+1})\right] \le 0.$$

We use the convention that $\infty/\infty = 1$. For each $\{v_i\}_{i=0}^\infty$, $\{u_i\}_{i=0}^\infty \in \mathcal{M}$ put

$$\tilde{d}(\{v_i\}_{i=0}^\infty, \{u_i\}_{i=0}^\infty) = \sup\{|v_i(z_1, z_2) - u_i(z_1, z_2)| : z_1, z_2 \in K, i = 0, 1, \dots\}$$

$$+ \sup\{|(v_i - u_i)(y_1, y_2) - (v_i - u_i)(z_1, z_2)|(\rho(z_1, y_1) + \rho(z_2, y_2))^{-1} :$$

$$y_1, y_2, z_1, z_2 \in K \text{ such that } (y_1, y_2) \ne (z_1, z_2) \text{ and } i = 0, 1, \dots\}, \tag{1.7}$$

$$d(\{v_i\}_{i=0}^\infty, \{u_i\}_{i=0}^\infty) = \tilde{d}(\{v_i\}_{i=0}^\infty, \{u_i\}_{i=0}^\infty)(1 + \tilde{d}(\{v_i\}_{i=0}^\infty, \{u_i\}_{i=0}^\infty))^{-1}. \tag{1.8}$$

It is not difficult to see that $(\mathcal{M}, d)$ is a complete metric space. In view of (1.8) for each $\{v_i\}_{i=0}^\infty, \{u_i\}_{i=0}^\infty \in \mathcal{M}$ such that $d(\{v_i\}_{i=0}^\infty, \{u_i\}_{i=0}^\infty) < 1$ we have

$$\tilde{d}(\{v_i\}_{i=0}^\infty, \{u_i\}_{i=0}^\infty) = d(\{v_i\}_{i=0}^\infty, \{u_i\}_{i=0}^\infty)(1 - d(\{v_i\}_{i=0}^\infty, \{u_i\}_{i=0}^\infty))^{-1}. \tag{1.9}$$

Denote by $\mathcal{M}_u$ the set of all sequences $\{v_i\}_{i=0}^\infty \in \mathcal{M}$ for which the following property holds:

for each $\epsilon > 0$ there exists $\delta > 0$ such that if an integer $i \ge 0$ and if $y_1, y_2, z_1, z_2 \in X$ satisfy $\rho(y_1, z_1), \rho(y_2, z_2) \le \delta$, then $|v_i(y_1, y_2) - v_i(z_1, z_2)| \le \epsilon$.

Denote by $\mathcal{M}_L$ the set of all $\{v_i\}_{i=0}^\infty \in \mathcal{M}$ such that for each integer $i \ge 0$,

$$\sup\{|v_i(y_1, y_2) - v_i(z_1, z_2)|(\rho(y_1, z_1) + \rho(y_2, z_2))^{-1} :$$

$$y_1, y_2, z_1, z_2 \in X \text{ such that } (y_1, y_2) \ne (z_1, z_2)\} < \infty$$

and denote by $\mathcal{M}_{uL}$ the set of all $\{v_i\}_{i=0}^\infty \in \mathcal{M}$ such that

$$\sup\{|v_i(y_1, y_2) - v_i(z_1, z_2)|(\rho(y_1, z_1) + \rho(y_2, z_2))^{-1} :$$

$$y_1, y_2, z_1, z_2 \in K \text{ such that } (y_1, y_2) \ne (z_1, z_2), i \ge 0 \text{ is an integer}\} < \infty.$$

Clearly, $\mathcal{M}_u$, $\mathcal{M}_L$ and $\mathcal{M}_{uL}$ are closed subsets of the complete metric space $(\mathcal{M}, d)$.

Denote by $\mathcal{M}_a$ the set of all $\{v_i\}_{i=0}^\infty \in \mathcal{M}$ such that $v_i = v_0$ for all integers $i \ge 0$. Clearly, $\mathcal{M}_a$ is a closed subset of $(\mathcal{M}, d)$ and it is identified with the the space $C(K \times K)$.

Let $v \in C(K \times K)$ and $v_i = v$ for all integers $i \ge 0$. Set

$$\mu(v) = \inf\left\{\liminf_{N\to\infty} \sum_{i=0}^{N-1} v(z_i, z_{i+1}) : \{z_i\}_{i=0}^\infty \subset K\right\}. \tag{1.10}$$

Clearly,

$$\mu(v) \le \inf\{v(x, x) : x \in K\}. \tag{1.11}$$

Denote by $\mathcal{M}_{ar}$ the set of all $\{v_i\}_{i=0}^{\infty} \in \mathcal{M}_a$ such that

$$\mu(v_0) = \inf\{v_0(z,z) : z \in K\}. \tag{1.12}$$

Clearly $\mathcal{M}_{ar}$ is a closed subset of the metric space $(\mathcal{M}, d)$.

It is not difficult to see that the following assertion holds.

Theorem 1.5. *Let $\{v_i\}_{i=0}^{\infty} \in \mathcal{M}_{ar}$. Then the following assertions hold.*

1. *There is $x \in K$ such that $\mu(v_0) = v_0(x,x)$.*
2. *Assume that $x \in K$ satisfies $\mu(v_0) = v_0(x,x)$. Then for each integer $T \geq 0$,*

$$T\mu(v_0) = U(\{v_i\}_{i=0}^{\infty}, x, x, 0, T).$$

Set $\mathcal{M}_{Lar} = \mathcal{M}_L \cap \mathcal{M}_{ar}$. Clearly, $\mathcal{M}_{Lar}$ is a closed subset of the metric space$(\mathcal{M}, d)$.

If K is a compact subset of a normed space $(X, \|\cdot\|)$ and $\rho(x,y) = \|x-y\|$, $x, y \in K$, then we denote by $\mathcal{M}_{ac}$ the set of all $\{v_i\}_{i=0}^{\infty} \in \mathcal{M}_a$ such that v_0 is convex.

Note that in this case $\mathcal{M}_{ac}$ is a closed subset of $(\mathcal{M}, d)$ and in view of Proposition 2.1 of [36], $\mathcal{M}_{ac} \subset \mathcal{M}_{ar}$.

The following theorem is our main result.

Theorem 1.6. *Let $\mathcal{A}$ be $\mathcal{M}_u$ or $\mathcal{M}_{uL}$ or $\mathcal{M}_{ar}$ or $\mathcal{M}_{Lar}$ or $\mathcal{M}_{ac}$ equipped with the metric d. Let $\mathcal{F}$ be the set of all $\{v_i\}_{i=0}^{\infty} \in \mathcal{A}$ for which there exist $\{\bar{x}_i\}_{i=0}^{\infty} \subset K$ and a neighborhood $\mathcal{U}$ of $\{v_i\}_{i=0}^{\infty}$ in $\mathcal{A}$ such that the following properties hold:*

(i) For each $\{w_i\}_{i=0}^{\infty} \in \mathcal{U}$ and each pair of integers $T_2 > T_1 \geq 0$,

$$U(\{w_i\}_{i=0}^{\infty}, \bar{x}_{T_1}, \bar{x}_{T_2}, T_1, T_2) = \sum_{i=T_1}^{T_2-1} w_i(\bar{x}_i, \bar{x}_{i+1}).$$

(ii) For each $\{w_i\}_{i=0}^{\infty} \in \mathcal{U}$ and each $(\{w_i\}_{i=0}^{\infty})$-good sequence $\{y_i\}_{i=0}^{\infty} \subset K$,

$$\sum_{i=0}^{\infty} \rho(y_i, \bar{x}_i) < \infty.$$

(iii) For each $\{w_i\}_{i=0}^{\infty} \in \mathcal{U}$ and each $x_0 \in K$ there is an $(\{w_i\}_{i=0}^{\infty})$-overtaking optimal $\{x_i\}_{i=0}^{\infty} \subset K$ and moreover, any sequence $\{x_i\}_{i=0}^{\infty} \subset K$ satisfying

$$\sum_{i=0}^{T} w_i(x_i, x_{i+1}) = U(\{w_i\}_{i=0}^{\infty}, x_0, x_T, 0, T)$$

for all integers T, is $(\{w_i\}_{i=0}^{\infty})$-overtaking optimal.

(iv) For each $M_0 > 0$ there is $M_1 > 0$ such that for each $\{w_i\}_{i=0}^{\infty} \in \mathcal{M}$, each pair of integers $T_1 \geq 0$, $T_1 > T_1 + 1$ and each sequence $\{x_i\}_{i=T_1}^{T_2} \subset K$ satisfying

$$\sum_{i=T_1}^{T_2-1} w_i(x_i, x_{i+1}) \leq U(\{w_i\}_{i=0}^{\infty}, x_{T_1}, x_{T_2}, T_1, T_2) + M_0$$

the inequality $\sum_{i=T_1}^{T_2} \rho(x_i, \bar{x}_i) \leq M_1$ *holds.*

(v) For each $\epsilon \in (0,1)$ *there exist* $\delta \in (0,\epsilon)$ *and a natural number* L *such that for each* $\{w_i\}_{i=0}^{\infty} \in \mathcal{U}$*, each integer* $T_1 \geq 0$*, each integer* $T_2 > T_1 + 2L + 1$ *and each sequence* $\{x_i\}_{i=T_1}^{T_2} \subset K$ *which satisfies*

$$\sum_{i=T_1}^{T_2-1} w_i(x_i, x_{i+1}) \leq U(\{w_i\}_{i=0}^{\infty}, x_{T_1}, x_{T_2}, T_1, T_2) + \delta$$

there exist integers $\tau_1 \in [T_1, T_1 + L]$*,* $\tau_2 \in [T_2 - L, T_2]$ *such that*

$$\sum_{i=\tau_1}^{\tau_2} \rho(x_i, \bar{x}) \leq \epsilon.$$

Moreover, if $\rho(x_{T_1}, \bar{x}_{T_1}) \leq \delta$*, then* $\tau_1 = T_1$ *and if* $\rho(x_{T_2}, \bar{x}_{T_2}) \leq \delta$*, then* $\tau_2 = T_2$*.*

Then $\mathcal{A} \setminus \mathcal{F}$ *is a porous set in* $(\mathcal{A}, d)$ *with*

$$\alpha = 128^{-1}(2\mathrm{diam}(K) + 1)^{-1}.$$

2. PROOFS OF THEOREMS 1.1 AND 1.2

Proof of Theorem 1.1. For each natural number n there exists $\{x_i^{(n)}\}_{i=0}^{n} \subset K$ such that

$$x_0^{(n)} = y, \quad \sum_{i=0}^{n} v_i(x_i^{(n)}, \quad x_{i+1}^{(n)}) = \tilde{U}(\{v_i\}_{i=0}^{\infty}, y, 0, n). \tag{2.1}$$

By using a diagonalization process we obtain that there exists a strictly increasing sequence of natural numbers $\{n_k\}_{k=1}^{\infty}$ such that for each integer $i \geq 0$ there is

$$x_i = \lim_{k\to\infty} x_i^{(n_k)}. \tag{2.2}$$

Clearly,

$$x_0 = y. \tag{2.3}$$

Let T be a natural number. We show that

$$\sum_{i=0}^{T-1} v_i(x_i, x_{i+1}) = U(\{v_i\}_{i=0}^{\infty}, y, x_T, 0, T) > 0. \tag{2.4}$$

Let us assume the contrary. Then

$$\epsilon := \sum_{i=0}^{T-1} v_i(x_i, x_{i+1}) - U(\{v_i\}_{i=0}^{\infty}, y, x_T, 0, T) > 0 \tag{2.5}$$

and there exists $\{z_i\}_{i=0}^{T} \subset K$ such that

$$z_0 = y, \quad z_T = x_T,$$

$$\sum_{i=0}^{T-1} v_i(x_i, x_{i+1}) - \sum_{i=0}^{T-1} v_i(z_i, z_{i+1}) > \epsilon/2. \tag{2.6}$$

There exists $\delta > 0$ such that

$$|v_i(y_1, y_2) - v_i(\bar{y}_1, \bar{y}_2)| \le \epsilon(8 + 8T)^{-1} \tag{2.7}$$

for any integer $i \in [0, T+1]$ and each $y_1, y_2, \bar{y}_1, \bar{y}_2 \in K$ satisfying $\rho(y_i, \bar{y}_i) \le 2\delta, i = 0, \dots, T+1$. In view of (2.2) there exists a natural number k such that

$$n_k > T + 8, \quad \rho(x_i, x_i^{(n_k)}) \le \delta, \quad i = 0, \dots, T+8. \tag{2.8}$$

Define a sequence $\{\tilde{z}_i\}_{i=0}^{n_k} \subset K$ as follows:

$$\tilde{z}_i = z_i, \quad i = 0, \dots, T, \tilde{z}_i = x_i^{(n_k)}, \quad i = T+1, \dots, n_k. \tag{2.9}$$

By (2.1), (2.6) and (2.9),

$$\tilde{z}_0 = y = x_0^{(n_k)}, \quad \tilde{z}_{n_k} = x_{n_k}^{(n_k)}. \tag{2.10}$$

Relations (2.1), (2.9) and (2.10) imply that

$$0 \le \sum_{i=0}^{n_k - 1} v_i(\tilde{z}_i, \tilde{z}_{i+1}) - \sum_{i=0}^{n_k - 1} v_i(x_i^{(n_k)}, x_{i+1}^{(n_k)}) = \sum_{i=0}^{T} v_i(\tilde{z}_i, \tilde{z}_{i+1}) - \sum_{i=0}^{T} v_i(x_i^{n_k}, x_{i+1}^{(n_k)})$$

$$= \sum_{i=0}^{T-1} v_i(z_i, z_{i+1}) + v_T(z_T, x_{T+1}^{(n_k)}) - \sum_{i=0}^{T} v_i(x_i^{(n_k)}, x_{i+1}^{(n_k)}). \tag{2.11}$$

By (2.6) and (2.11),

$$0 \le \sum_{i=0}^{T-1} v_i(z_i, z_{i+1}) - \sum_{i=0}^{T-1} v_i(x_i, x_{i+1}) + \left[\sum_{i=0}^{T-1} v_i(x_i, x_{i+1}) - \sum_{i=0}^{T-1} v_i(x_i^{(n_k)}, x_{i+1}^{(n_k)})\right]$$

$$+ v_T(x_T, x_{T+1}^{(n_k)}) - v_T(x_T^{(n_k)}, x_{T+1}^{(n_k)}) \le -\epsilon/2 + \left[\sum_{i=0}^{T-1} v_i(x_i, x_{i+1}) - \sum_{i=0}^{T-1} v_i(x_i^{(n_k)}, x_{i+1}^{(n_k)})\right]$$

$$+ v_T(x_T, x_{T+1}^{(n_k)}) - v_T(x_T^{(n_k)}, x_{T+1}^{(n_k)}). \tag{2.12}$$

By (2.8) and the choice of δ (see (2.7)),

$$|v_T(x_T, x_{T+1}^{(n_k)}) - v_T(x_T^{(n_k)}, x_{T+1}^{(n_k)})| \le \epsilon(8(T+1))^{-1}$$

and for $i = 0, \dots, T$,

$$|v_i(x_i, x_{i+1}) - v_i(x_i^{(n_k)}, x_{i+1}^{(n_k)})| \le \epsilon(8(T+1))^{-1}.$$

Combined with (2.12) these two inequalities above imply that

$$0 \le -\epsilon/2 + (T+1)(\epsilon/8)(T+1)^{-1} < 0,$$

a contradiction. The contradiction we have reached proves that (2.4) holds. Theorem 1.1 is proved.

Proof of Theorem 1.2. Let $\{y_i\}_{i=T_1}^{T_2} \subset K$. We show that (1.6) holds. It is clear that we may only consider the case when $T_2 > T_1 + 1$. Set

$$z_{T_1} = x_{T_1}, \quad z_{T_2} = x_{T_2}, \quad z_i = y_i, \quad i \in \{T_1, \dots, T_2\} \setminus \{T_1, T_2\}. \tag{2.13}$$

By (2.13),

$$0 \le \sum_{i=T_1}^{T_2-1} v_i(z_i, z_{i+1}) - \sum_{i=T_1}^{T_2-1} v_i(x_i, x_{i+1})$$

$$\le \sum_{i=T_1}^{T_2-1} v_i(z_i, z_{i+1}) - \sum_{i=T_1}^{T_2-1} v_i(y_i, y_{i+1}) + \sum_{i=T_1}^{T_2-1} v_i(y_i, y_{i+1}) - \sum_{i=T_1}^{T_2-1} v_i(x_i, x_{i+1})$$

$$\le 4 \sup\{\|v_i\| : i = 0, 1, \dots\} + \sum_{i=T_1}^{T_2-1} v_i(y_i, y_{i+1}) - \sum_{i=T_1}^{T_2-1} v_i(x_i, x_{i+1}).$$

Theorem 1.2 is proved.

3. AUXILIARY RESULTS

Let $\{v_i\}_{i=0}^{\infty} \in \mathcal{M}$. By Theorem 1.1 there is $\{\bar{x}_i\}_{i=0}^{\infty} \in K$ such that

$$\sum_{i=0}^{T-1} v_i(\bar{x}_i, \bar{x}_{i+1}) = U(\{v_i\}_{i=0}^{\infty}, \bar{x}_0, \bar{x}_T, 0, T) \tag{3.1}$$

for all natural numbers T.

If $\{v_i\}_{i=0}^{\infty} \in \mathcal{M}_{ar}$, then in view of Theorem 1.5 we may assume that

$$\bar{x}_i = \bar{x}_0 \text{ for all integers } i \ge 0, \tag{3.2}$$

where

$$v_0(\bar{x}_0, \bar{x}_0) = \mu(v_0). \tag{3.3}$$

Let n be a natural number. Put

$$(v_n)_i(x, y) = v_i(x, y) + n^{-1}(\rho(x, \bar{x}_i) + \rho(y, \bar{x}_{i+1})), \tag{3.4}$$

$$x, y \in K, \quad i = 0, 1, \dots$$

Clearly, $\{(v_n)_i\}_{i=0}^{\infty} \in \mathcal{M}$ and if $\{v_i\}_{i=0}^{\infty} \in \mathcal{M}_a$ (respectively, $\mathcal{M}_L$, $\mathcal{M}_{uL}$,$\mathcal{M}_{ar}$, $\mathcal{M}_{Lar}$, $\mathcal{M}_{ac}$, $\mathcal{M}_{ac} \cap \mathcal{M}_L$), then $\{(v_n)_i\}_{i=0}^{\infty} \in \mathcal{M}_u$ (respectively, $\mathcal{M}_L$, $\mathcal{M}_{uL}$, $\mathcal{M}_{ar}$, $\mathcal{M}_{Lar}, \mathcal{M}_{ac}$, $\mathcal{M}_{ac} \cap \mathcal{M}_L$) and

$$\sup\{|(v_n)_i(x, y) - v_i(x, y)| : x, y \in K, i = 0, 1, \dots\}$$

$$\le 2n^{-1} \sup\{\rho(z_1, z_2) : z_1, z_2 \in K\} \le 2n^{-1} \mathrm{diam}(K) \tag{3.5}$$

and that for $x_1, x_2, y_1, y_2 \in K$ and $i \in \{0, 1, \dots\}$

$$|((v_n)_i - v_i)(x_1, y_1) - ((v_n)_i - v_i)(x_2, y_2)|$$

$$(3.6)$$
$$= n^{-1}|(\rho(x_1, \bar{x}_i) + \rho(y_1, \bar{x}_{i+1})) - (\rho(x_2, \bar{x}_i) + \rho(y_2, \bar{x}_{i+1}))| \le n^{-1}(\rho(x_1, x_2) + \rho(y_1, y_2)).$$

Lemma 3.1. *Let* $\{w_i\}_{i=0}^{\infty} \in \mathcal{M}$ *satisfy*

$$|w_i(z_1, z_2) - (v_n)_i(z_1, z_2)| \le (4n)^{-1} \tag{3.7}$$

for all $z_1, z_2 \in X$ *and all integers* $i \ge 0$ *and*

$$|(w_i - (v_n)_i)(z_1, z_2) - (w_i - (v_n)_i)(y_1, y_2)| \le (4n)^{-1}(\rho(z_1, y_1) + \rho(z_2, y_2)) \tag{3.8}$$

for all integers $i \ge 0$ *and all* $y_1, y_2, z_1, z_2 \in K$.

Then the following assertions hold.

1. Let $z_1, z_2 \in K$ *and an integer* $i \ge 0$. *Then*

$$w_i(z_1, z_2) - w_i(\bar{x}_i, \bar{x}_{i+1}) \ge v_i(z_1, z_2) - v_i(\bar{x}_i, \bar{x}_{i+1}) + 3(4n)^{-1}(\rho(z_1, \bar{x}_i) + \rho(z_2, \bar{x}_{i+1})).$$

2. Let integers T_1, T_2 *satisfy* $0 \le T_1 < T_2$ *and let* $\{z_i\}_{i=T_1}^{T_2} \subset K$. *Then*

$$\sum_{i=T_1}^{T_2-1} w_i(z_i, z_{i+1}) - \sum_{i=T_1}^{T_2-1} w_i(\bar{x}_i, \bar{x}_{i+1}) \ge \sum_{i=T_1}^{T_2-1} v_i(z_i, z_{i+1}) - \sum_{i=T_1}^{T_2-1} v_i(\bar{x}_i, \bar{x}_{i+1})$$
$$+3(4n)^{-1} \sum_{i=T_1}^{T_2-1} (\rho(z_i, \bar{x}_i) + \rho(z_{i+1}, \bar{x}_{i+1})).$$

3. For each pair of integers $T_2 > T_1 \ge 0$,

$$U(\{w_i\}_{i=0}^{\infty}, \bar{x}_{T_1}, \bar{x}_{T_2}, T_1, T_2) = \sum_{i=T_1}^{T_2-1} w_i(\bar{x}_i, \bar{x}_{i+1}).$$

4. For each $(\{w_i\}_{i=0}^{\infty})$*-good sequence* $\{y_i\}_{i=0}^{\infty}$, $\sum_{i=0}^{\infty} \rho(y_i, \bar{x}_i) < \infty$.

5. Let $\{w_i\}_{i=0}^{\infty} \in \mathcal{M}_u$. *Then for each* $x_0 \in K$ *there is a* $(\{w_i\}_{i=0}^{\infty})$*-overtaking optimal sequence* $\{x_i\}_{i=0}^{\infty} \in K$. *Moreover, any sequence* $\{x_i\}_{i=0}^{\infty} \subset K$ *satisfying*

$$\sum_{i=0}^{T} w_i(x_i, x_{i+1}) = U(\{w_i\}_{i=0}^{\infty}, x_0, x_T, 0, T) \text{ for all integers } T \ge 0$$

is $(\{w_i\}_{i=0}^{\infty})$*-overtaking optimal.*

6. Let $M_0 > 0$. *Then for each pair of integers* $T_1 \ge 0$, $T_2 > T_1 + 1$ *and each sequence* $\{x_i\}_{i=T_1}^{T_2} \subset K$ *satisfying*

$$\sum_{i=T_1}^{T_2-1} w_i(x_i, x_{i+1}) \le U(\{w_i\}_{i=0}^{\infty}, x_{T_1}, x_{T_2}, T_1, T_2) + M_0$$

the following inequality holds:

$$\sum_{i=T_1}^{T_2} \rho(x_i, \bar{x}_i) \le 2n(M_0 + 8\sup\{\|v_i\| : i = 0, 1, \dots\}) + 2 + 16\mathrm{diam}(K).$$

7. Let $M_0 > 0$, $\epsilon > 0$ and a natural number $L \geq 2$ satisfies

$$L\epsilon > 2n(M_0 + 8\sup\{\|v_i\| : i = 0, 1, \dots\}) + 16\mathrm{diam}(K).$$

Then for each integer $T \geq 0$ and each sequence $\{x_i\}_{i=L}^{T+L} \subset K$ satisfying

$$\sum_{i=T}^{T+L-1} w_i(x_i, x_{i+1}) \leq U(\{w_i\}_{i=0}^{\infty}, x_T, x_{T+L}, T, T+L) + M_0$$

the following inequality holds:

$$\min\{\rho(x_i, \bar{x}_i) : i = T, \dots, T+L\} \leq \epsilon.$$

Proof of Assertion 1. By (3.4) and (3.8),

$$\begin{aligned} w_i(z_1, z_2) - w_i(\bar{x}_i, \bar{x}_{i+1}) &= (v_n)_i(z_1, z_2) - (v_n)_i(\bar{x}_i, \bar{x}_{i+1}) \\ &\quad + [(w_i - (v_n)_i)(z_1, z_2) - (w_i - (v_n)_i)(\bar{x}_i, \bar{x}_{i+1})] \\ &\geq (v_n)_i(z_1, z_2) - (v_n)_i(\bar{x}_i, \bar{x}_{i+1}) \\ &\quad - (4n)^{-1}(\rho(z_1, \bar{x}_i) + \rho(z_2, \bar{x}_{i+1})) \\ &= v_i(z_1, z_2) + n^{-1}(\rho(z_1, \bar{x}_i) + \rho(z_2, \bar{x}_{i+1})) \\ &\quad - v_i(\bar{x}_i, \bar{x}_{i+1}) - (4n)^{-1}(\rho(z_1, \bar{x}_i) + \rho(z_2, \bar{x}_{i+1})) \\ &= v_i(z_1, z_2) - v_i(\bar{x}_i, \bar{x}_{i+1}) + 3(4n)^{-1}(\rho(z_1, \bar{x}_i) + \rho(z_2, \bar{x}_{i+1})). \end{aligned}$$

Assertion 1 is proved.

Assertion 2 follows from Assertion 1 while Assertion 3 follows from Assertion 2 and (3.1). It is easy to see that Assertion 4 follows from Assertion 2, (3.1) and Theorem 1.2.

Let us prove Assertion 5. Let $x_0 \in K$. By Theorem 1.1 there is a sequence $\{x_i\}_{i=0}^{\infty} \subset K$ such that for all natural numbers T

$$\sum_{i=0}^{T-1} w_i(x_i, x_{i+1}) = U(\{w_i\}_{i=0}^{\infty}, x_0, x_T, 0, T). \tag{3.9}$$

We show that $\{x_i\}_{i=0}^{\infty}$ is $(\{w_i\}_{i=0}^{\infty})$-overtaking optimal.

Assume the contrary. Then there exists $\{y_i\}_{i=0}^{\infty} \subset K$ such that

$$y_0 = x_0,$$

$$\limsup_{T\to\infty} \left[\sum_{i=0}^{T} w_i(x_i, x_{i+1}) - \sum_{i=0}^{T} w_i(y_i, y_{i+1})\right] > 0. \tag{3.10}$$

By (3.9) and Theorem 1.2, $\{x_i\}_{i=0}^{\infty}$ is $(\{w_i\}_{i=0}^{\infty})$-good. By (3.10), (3.9) and Corollaries 1.3 and 1.4, the sequence $\{y_i\}_{i=0}^{\infty}$ is $(\{w_i\}_{i=0}^{\infty})$-good. In view of Assertion 4,

$$\sum_{i=0}^{\infty} \rho(y_i, \bar{x}_i) < \infty, \sum_{i=0}^{\infty} \rho(x_i, \bar{x}) < \infty. \tag{3.11}$$

By (3.10) there exist $\epsilon > 0$ and a strictly increasing sequence of natural numbers $\{T_k\}_{k=0}^{\infty}$ such that $T_0 \geq 4$, $T_{k+1} \geq T_k + 4$ for all integers $k \geq 0$ and

$$\sum_{i=0}^{T_k-1} w_i(x_i, x_{i+1}) \geq \sum_{i=0}^{T_k-1} w_i(y_i, y_{i+1}) + 4\epsilon \text{ for all integers } k \geq 0. \tag{3.12}$$

Since $\{w_i\}_{i=0}^{\infty} \in \mathcal{M}_u$ there is $\delta > 0$ such that the following property holds:

(P1) for each integer $i \geq 0$ and each $z_1, z_2, \tilde{z}_1, \tilde{z}_2 \in K$ satisfying $\rho(z_j, \tilde{z}_j) \leq 2\delta$, $j = 1, 2$,

$$|w_i(z_1, z_2) - w_i(\tilde{z}_1, \tilde{z}_2)| \leq \epsilon.$$

By (3.11) there is a natural number k such that

$$\rho(y_{T_k}, \bar{x}_{T_k}) \leq \delta/2, \rho(x_{T_k}, \bar{x}_{T_k}) \leq \delta/2. \tag{3.13}$$

Define a sequence $\{z_i\}_{i=0}^{T_k} \subset K$ by

$$z_i = y_i, i = 0, \ldots, T_k - 1, z_{T_k} = x_{T_k}. \tag{3.14}$$

Clearly,

$$z_0 = y_0 = x_0. \tag{3.15}$$

By (3.15), (3.14), (3.9) and (3.12),

$$\begin{aligned}
0 &\leq \sum_{i=0}^{T_k-1} w_i(z_i, z_{i+1}) - \sum_{i=0}^{T_k-1} w_i(x_i, x_{i+1}) \\
&= \sum_{i=0}^{T_k-1} w_i(y_i, y_{i+1}) - \sum_{i=0}^{T_k-1} w_i(x_i, x_{i+1}) + [\sum_{i=0}^{T_k-1} w_i(z_i, z_{i+1}) - \sum_{i=0}^{T_k-1} w_i(y_i, y_{i+1})] \\
&\leq -4\epsilon + w_{T_k-1}(y_{T_k-1}, x_{T_k}) - w_{T_k-1}(y_{T_k-1}, y_{T_k}).
\end{aligned}$$

Together with property (P1) and (3.13) this implies that $0 \leq -4\epsilon + \epsilon$, a contradiction. The contradiction we have reached proves Assertion 5.

Let us prove Assertion 6. Let $T_1 \geq 0$, $T_2 > T_1 + 1$ be integers and $\{x_i\}_{T_1}^{T_2} \subset K$ satisfies

$$\sum_{i=T_1}^{T_2-1} w_i(x_i, x_{i+1}) \leq U(\{w_i\}_{i=0}^{\infty}, x_{T_1}, x_{T_2}, T_1, T_2) + M_0. \tag{3.16}$$

Define

$$y_{T_1} = x_{T_1}, y_{T_2} = x_{T_2}, y_i = \bar{x}_i \text{ for all integers } i \text{ satisfying } T_1 < i < T_2. \tag{3.17}$$

By (3.17), (3.8) and (3.4),

$$\sum_{i=T_1}^{T_2-1} w_i(y_i, y_{i+1}) = \sum_{i=T_1}^{T_2-1} w_i(\bar{x}_i, \bar{x}_{i+1})$$

$$+[w_{T_1}(y_{T_1}, y_{T_1+1}) - w_{T_1}(\bar{x}_{T_1}, \bar{x}_{T_1+1}) + w_{T_2-1}(y_{T_2-1}, y_{T_2}) - w_{T_2-1}(\bar{x}_{T_2-1}, \bar{x}_{T_2})]$$

$$\leq \sum_{i=T_1}^{T_2-1} w_i(\bar{x}_i, \bar{x}_{i+1}) + (4n)^{-1}4 + 4n^{-1}2\mathrm{diam}(K)$$

$$+[v_{T_1}(y_{T_1}, y_{T_1+1}) - v_{T_1}(\bar{x}_{T_1}, \bar{x}_{T_1+1}) + v_{T_2-1}(y_{T_2-1}, y_{T_2}) - v_{T_2-1}(\bar{x}_{T_2-1}, \bar{x}_{T_2})]$$

$$(3.18) \qquad \leq \sum_{i=T_1}^{T_2-1} w_i(\bar{x}_i, \bar{x}_{i+1}) + n^{-1} + 8n^{-1}\mathrm{diam}(K) + 4\sup\{\|v_i\| : i = 0, 1 \dots\}.$$

It follows from (3.18), (3.16) and (3.17) that

$$\sum_{i=T_1}^{T_2-1} w_i(\bar{x}_i, \bar{x}_{i+1}) + n^{-1}(1 + 8\mathrm{diam}(K)) + 4\sup\{\|v_i\| : i = 0, 1, \dots\}$$

$$\geq \sum_{i=T_1}^{T_2-1} w_i(y_i, y_{i+1}) \geq \sum_{i=T_1}^{T_2-1} w_i(x_i, x_{i+1}) - M_0.$$

Together with Assertion 2, (3.1) and Theorem 1.2 this implies that

$$\begin{aligned}
&M_0 + n^{-1}(1 + 8\mathrm{diam}(K)) + 4\sup\{\|v_i\| : i = 0, 1, \dots\} \\
&\geq \sum_{i=T_1}^{T_2-1} w_i(x_i, x_{i+1}) - \sum_{i=T_1}^{T_2-1} w_i(\bar{x}_i, \bar{x}_{i+1}) \\
&\geq \sum_{i=T_1}^{T_2-1} v_i(x_i, x_{i+1}) - \sum_{i=T_1}^{T_2-1} v_i(\bar{x}_i, \bar{x}_{i+1}) + 3(4n)^{-1}\left(\sum_{i=T_1}^{T_2-1} \rho(x_i, \bar{x}_i) + \rho(x_{i+1}, \bar{x}_{i+1})\right) \\
&\geq -4\sup\{\|v_i\| : i = 0, 1, \dots\} + 3(4n)^{-1}\left(\sum_{i=T_1}^{T_2-1} \rho(x_i, \bar{x}_i) + \rho(x_{i+1}, \bar{x}_{i+1})\right)
\end{aligned}$$

and

$$\sum_{i=T_1}^{T_2-1} \rho(x_i, \bar{x}_i) \leq 2n(M_0 + 8\sup\{\|v_i\| : i = 0, 1, \dots\}) + 16\mathrm{diam}(K).$$

Thus Assertion 6 is proved.

Assertion 7 follows from Assertion 6. Lemma 3.1 is proved.

Lemma 3.2. *Assume that $\{v_i\}_{i=0}^{\infty} \in \mathcal{M}_u$ and $\epsilon > 0$. Then there exists $\delta \in (0, \epsilon)$ such that for each $\{w_i\}_{i=}^{\infty} \in \mathcal{M}$ which satisfies*

$$(3.19) \quad |w_i(z_1, z_2) - (v_n)_i(z_1, z_2)| \leq (4n)^{-1} \text{ for all } z_1, z_2 \in K \text{ and all integers } i \geq 0,$$

$$|(w_i - (v_n)_i)(z_1, z_2) - (w_i - (v_n)_i)(y_1, y_2)| \leq (4n)^{-1}(\rho(z_1, y_1) + \rho(z_2, y_2))$$

$$(3.20) \qquad \text{for all integers } i \geq 0 \text{ and all } y_1, y_2, z_1, z_2 \in K$$

and each sequence $\{x_i\}_{i=T_1}^{T_2} \subset K$ with integers $T_1 \geq 0$, $T_2 > T_1 + 1$ satisfying

$$(3.21) \qquad \rho(x_{T_j}, \bar{x}_{T_j}) \leq \delta, j = 1, 2,$$

$$\sum_{i=T_1}^{T_2-1} w_i(x_i, x_{i+1}) \le U(\{w_i\}_{i=0}^{\infty}, x_{T_1}, x_{T_2}, T_1, T_2) + \delta \tag{3.22}$$

the inequality $\sum_{i=T_1}^{T_2} \rho(x_i, \bar{x}_i) \le \epsilon$ *holds.*

Proof. Since $\{v_i\}_{i=0}^{\infty} \in \mathcal{M}$ there is

$$\delta \in (0, (\epsilon/8)(6n)^{-1})$$

such that the following property holds:

(P2) for each integer $i \ge 0$ and each $z_1, z_2, \tilde{z}_1, \tilde{z}_2 \in K$ satisfying $\rho(z_j, \tilde{z}_j) \le 2\delta$, $j = 1, 2$ the inequality

$$|v_i(z_1, z_2) - v_i(\tilde{z}_1, \tilde{z}_2)| \le (64n)^{-1}\epsilon$$

holds.

Let $\{w_i\}_{i=0}^{\infty} \in \mathcal{M}$ satisfy (3.19) and (3.20), $T_1 \ge 0$, $T_2 > T_1 + 1$ be integers and let a sequence $\{x_i\}_{i=T_1}^{T_2} \subset K$ satisfy (3.21) and (3.22).

Define

$$z_{T_1} = \bar{x}_{T_1}, \quad z_{T_2} = \bar{x}_{T_2}, \quad z_i = x_i \text{ for all integers } i \text{ satisfying } T_1 < i < T_2,$$

$$y_{T_1} = x_{T_1}, \quad y_{T_2} = x_{T_2}, \quad y_i = \bar{x}_i \text{ for all integers } i \text{ satisfying } T_1 < i < T_2. \tag{3.23}$$

By (3.23) and (3.22),

$$\sum_{i=T_1}^{T_2-1} w_i(x_i, x_{i+1}) \le \sum_{i=T_1}^{T_2-1} w_i(y_i, y_{i+1}) + \delta. \tag{3.24}$$

In view of (3.23), (320) and (3,4),

$$\begin{aligned}
\sum_{i=T_1}^{T_2-1} w_i(y_i, y_{i+1}) &= \sum_{i=T_1}^{T_2-1} w_i(\bar{x}_i, \bar{x}_{i+1}) + [w_{T_1}(y_{T_1}, y_{T_1+1}) - w_{T_1}(\bar{x}_{T_1}, \bar{x}_{T_1+1})] \\
&\quad +[w_{T_2-1}(y_{T_2-1}, y_{T_2}) - w_{T_2-1}(\bar{x}_{T_2-1}, \bar{x}_{T_2})] \\
&= \sum_{i=T_1}^{T_2-1} w_i(\bar{x}_i, \bar{x}_{i+1}) + [(v_n)_{T_1}(y_{T_1}, y_{T_1+1}) - (v_n)_{T_1}(\bar{x}_{T_1}, \bar{x}_{T_1+1})] \\
&\quad +[(v_n)_{T_2-1}(y_{T_2-1}, y_{T_2}) - (v_n)_{T_2}(\bar{x}_{T_2-1}\bar{x}_{T_2})] \\
&\quad +[(w_{T_1} - (v_n)_{T_1})(y_{T_1}, y_{T_1+1}) - (w_{T_1} - (v_n)_{T_1})(\bar{x}_{T_1}, \bar{x}_{T_2})] \\
&\quad +[(w_{T_2-1} - (v_n)_{T_2-1})(y_{T_2-1}, y_{T_2}) - (w_{T_2-1} - (v_n)_{T_2-1})(\bar{x}_{T_2-1}, \bar{x}_{T_2})] \\
&\le \sum_{i=T_1}^{T_2-1} w_i(\bar{x}_i, \bar{x}_{i+1}) + [(v_n)_{T_1}(x_{T_1}, \bar{x}_{T_1+1}) - (v_n)_{T_1}(\bar{x}_{T_1}, \bar{x}_{T_1+1})] \\
&\quad +[(v_n)_{T_2-1}(\bar{x}_{T_2-1}, x_{T_2}) - (v_n)_{T_2}(\bar{x}_{T_2-1}, \bar{x}_{T_2})] + (4n)^{-1}[\rho(x_{T_1}, \bar{x}_{T_1}) + \rho(x_{T_2}, \bar{x}_{T_2})] \\
&= \sum_{i=T_1}^{T_2-1} w_i(\bar{x}_i, \bar{x}_{i+1}) + v_{T_1}(x_{T_1}, \bar{x}_{T_1+1}) - v_{T_1}(\bar{x}_{T_1}, \bar{x}_{T_1+1})
\end{aligned}$$

$$+n^{-1}\rho(x_{T_1},\bar{x}_{T_1})+v_{T_2-1}(\bar{x}_{T_2-1},x_{T_2})-v_{T_2}(\bar{x}_{T_2-1},\bar{x}_{T_2})+n^{-1}\rho(\bar{x}_{T_2},x_{T_2}). \tag{3.25}$$

By (3.24), (3.25), (3.21) and property (P2),

$$\sum_{i=T_1}^{T_2-1} w_i(x_i,x_{i+1})-\sum_{i=T_1}^{T_2-1} w_i(\bar{x}_i\bar{x}_{i+1})$$

$$\le \sum_{i=T_1}^{T_2-1} w_i(x_i,x_{i+1})-\sum_{i=T_1}^{T_2-1} w_i(y_i,y_{i+1})+\sum_{i=T_1}^{T_2-1} w_i(y_i,y_{i+1})-\sum_{i=T_1}^{T_2-1} w_i(\bar{x}_i,\bar{x}_{i+1})$$

$$\le \delta+(2/n)\delta+v_{T_1}(x_{T_1},\bar{x}_{T_1})-v_{T_1}(\bar{x}_{T_1},\bar{x}_{T_1+1})$$

$$+v_{T_2-1}(\bar{x}_{T_2-1},x_{T_2})-v_{T_2}(\bar{x}_{T_2-1},\bar{x}_{T_2})\le 3\delta+(32n)^{-1}\epsilon. \tag{3.26}$$

By Assertion 2 of Lemma 3.1,

$$\sum_{i=T_1}^{T_2-1} w_i(x_i,x_{i+1})-\sum_{i=T_1}^{T_2-1} w_i(\bar{x}_i,\bar{x}_{i+1})$$

$$\ge \sum_{i=T_1}^{T_2-1} v_i(x_i,x_{i+1})-\sum_{i=T_1}^{T_2-1} v_i(\bar{x}_i,\bar{x}_{i+1})+3(4n)^{-1}\sum_{i=T_1}^{T_2-1}(\rho(x_i,\bar{x}_i)+\rho(x_{i+1},\bar{x}_{i+1})). \tag{3.27}$$

In view of (3.22) and (3.1),

$$\sum_{i=T_1}^{T_2-1} v_i(x_i,x_{i+1})-\sum_{i=T_1}^{T_2-1} v_i(\bar{x}_i,\bar{x}_{i+1})=\sum_{i=T_1}^{T_2-1} v_i(x_i,x_{i+1})-\sum_{i=T_1}^{T_2-1} v_i(z_i,z_{i+1})$$

$$+\sum_{i=T_1}^{T_2-1} v_i(z_i,z_{i+1})-\sum_{i=T_1}^{T_2-1} v_i(\bar{x}_i,\bar{x}_{i+1})\ge \sum_{i=T_1}^{T_2-1} v_(x_i,x_{i+1})-\sum_{i=T_1}^{T_2-1} v_i(z_i,z_{i+1})$$

$$=v_{T_1}(x_{T_1},x_{T_1+1})-v_{T_1}(\bar{x}_{T_1},x_{T_1+1})+v_{T_2-1}(x_{T_2-1},x_{T_2})-v_{T_2-1}(x_{T_2-1},\bar{x}_{T_2}).$$

Together with (3.26), (3.27), (3.21) and property (P2) this implies that

$$3\delta+(32n)^{-1}\epsilon\ge 3(4n)^{-1}\sum_{i=T_1}^{T_2-1}(\rho(x_i,\bar{x}_i)+\rho(x_{i+1},\bar{x}_{i+1}))$$

$$+\sum_{i=T_1}^{T_2-1} v_i(x_i,x_{i+1})-\sum_{i=T_1}^{T_2-1} v_i(\bar{x}_i,\bar{x}_{i+1})$$

$$\ge 3(4n)^{-1}\sum_{i=T_1}^{T_2-1}(\rho(x_i,\bar{x}_i)+\rho(x_{i+1},\bar{x}_{i+1}))-(32n)^{-1}\epsilon,$$

$$\sum_{i=T_1}^{T_2-1}\rho(x_i,\bar{x}_i)\le 2n[(16n)^{-1}\epsilon+3\delta]\le \epsilon/4.$$

Lemma 3.2 is proved.

Lemma 3.3. *Assume that* $\{v_i\}_{i=0}^{\infty} \in \mathcal{M}_u$ *and* $\epsilon \in (0,1)$. *Then there exist* $\delta \in (0,\epsilon)$ *and a natural number* L *such that for each* $\{w_i\}_{i=0}^{\infty} \in \mathcal{M}$ *which satisfies* (3.19) *and* (3.20), *each integer* $T_1 \geq 0$, *each integer* $T_2 > T_1+2L+1$ *and each sequence* $\{x_i\}_{i=T_1}^{T_2} \subset K$ *which satisfies*

$$\sum_{i=T_1}^{T_2-1} w_i(x_i, x_{i+1}) \leq U(\{w_i\}_{i=0}^{\infty}, x_{T_1}, x_{T_2}, T_1, T_2) + \delta \tag{3.28}$$

there exist integers $\tau_1 \in [T_1, T_1+L]$, $\tau_2 \in [T_2-L, T_2]$ *such that*

$$\sum_{i=\tau_1}^{\tau_2} \rho(x_i, \bar{x}_{i+1}) \leq \epsilon.$$

Moreover, if $\rho(x_{T_1}, \bar{x}_{T_1}) \leq \delta$, *then* $\tau_1 = T_1$ *and if* $\rho(x_{T_2}, \bar{x}_{T_2}) \leq \delta$, *the* $\tau_2 = T_2$.

Proof. Let $\delta \in (0,\epsilon)$ be as guaranteed by Lemma 3.2. In view of Assertion 7 of Lemma 3.1 there exists a natural number $L \geq 2$ such that the following property holds:

(P3) for each $\{w_i\}_{i=0}^{\infty} \in \mathcal{M}$ which satisfies (3.19) and (3.20), each integer $T \geq 0$ and each sequence $\{x_i\}_{i=T}^{T+L} \subset K$ satisfying

$$\sum_{i=T}^{T+L-1} w_i(x_i, x_{i+1}) \leq U(\{w_i\}_{i=0}^{\infty}, x_T, x_{T+L}, T, T+L) + 1$$

the inequality $\min\{\rho(x_i, \bar{x}_i) : i = T, \dots, T+L\} \leq \delta$ holds.

Assume that $\{w_i\}_{i=0}^{\infty} \in \mathcal{M}$ satisfies (3.19) and (3.20), an integer $T_1 \geq 0$, an integer $T_2 > T_1+L+1$ and a sequence $\{x_i\}_{i=T_1}^{T_2} \subset K$ satisfies (3.18).

It follows from (P3) applied to the sequences $\{x_i\}_{i=T_1}^{T_1+L}$, $\{x_i\}_{i=T_2-L}^{T_2}$ that there exist integers

$$\tau_1 \in [T_1, T_1+L], \quad \tau_2 \in [T_2-L, T_2] \tag{3.29}$$

such that

$$\rho(x_{\tau_i}, \bar{x}_{\tau_i}) \leq \delta, \quad i = 1, 2. \tag{3.30}$$

If $\rho(x_{T_1}, \bar{x}_{T_1}) \leq \delta$, then put $\tau_1 = T_1$ and if $\rho(x_{T_2}, \bar{x}_{T_2}) \leq \delta$, then put $\tau_2 = T_2$.

By (3.30), (3.28), (3.29), the choice of δ and Lemma 3.2,

$$\sum_{i=\tau_1}^{\tau_2} \rho(x_i, \bar{x}_i) \leq \epsilon.$$

Lemma 3.3 is proved.

4. PROOF OF THEOREM 1.6

Put

$$\alpha_0 = 64^{-1}(2\mathrm{diam}(K)+1)^{-1}, \quad \alpha = \alpha_0/2. \tag{4.1}$$

Let $\{v_i\}_{i=0}^{\infty} \in \mathcal{A}, r \in (0,1]$. By Theorem 1.1 there is $\{\bar{x}_i\}_{i=0}^{\infty} \in K$ such that

$$\sum_{i=0}^{T-1} v_i(\bar{x}_i, \bar{x}_{i+1}) = U(\{v_i\}_{i=0}^{\infty}, \bar{x}_0, \bar{x}_1, 0, T) \tag{4.2}$$

for all natural numbers T, and if $\{v_i\}_{i=0}^{\infty} \in \mathcal{M}_{ar}$, then in view of Theorem 1.5

$$\bar{x}_i = \bar{x}_0 \tag{4.3}$$

for all integers $i \geq 0$, where

$$v_0(\bar{x}_0, \bar{x}_0) = \mu(v_0). \tag{4.4}$$

Choose a natural number n such that

$$n^{-1}(2\mathrm{diam}(K)+1) \in [r/8, r/2] \tag{4.5}$$

and put

$$(v_n)_i(x,y) = v_i(x,y) + n^{-1}(\rho(x,\bar{x}_i) + \rho(y, \bar{x}_{i+1})), \quad x, y \in K, i = 0, 1, \dots. \tag{4.6}$$

We noted in section 3 that $\{(v_n)_i\}_{i=0}^{\infty} \in \mathcal{A}$ and in view of (3.5) and (3.6),

$$\tilde{d}_{\mathcal{M}}(\{v_i\}_{i=0}^{\infty}, \{(v_n)_i\}_{i=0}^{\infty}) \leq 2n^{-1}\mathrm{diam}(K) + n^{-1} = n^{-1}(2\mathrm{diam}(K)+1),$$

$$d_{\mathcal{M}}(\{v_i\}_{i=0}^{\infty}, \{(v_n)_i\}_{i=0}^{\infty})$$
$$\leq n^{-1}(2\mathrm{diam}(K)+1)(1 + n^{-1}(2\mathrm{diam}(K)+1)^{-1}) \leq n^{-1}2\mathrm{diam}(K)+1).$$

By (4.5), (4.7) and (4.1),

$$\{\{w_i\}_{i=0}^{\infty} \in \mathcal{A} : d_{\mathcal{M}}(\{(v_n)_i\}_{i=0}^{\infty}, \{w_i\}_{i=0}^{\infty}) \leq \alpha_0 r\}$$

$$\subset \{\{w_i\}_{i=0}^{\infty} \in \mathcal{A} : d_{\mathcal{M}}(\{v_i\}_{i=0}^{\infty}, \{w_i\}_{i=0}^{\infty}) \leq r\}. \tag{4.8}$$

Assume that

$$\{w_i\}_{i=0}^{\infty} \in \mathcal{A}, \quad d_{\mathcal{M}}(\{(v_n)_i\}_{i=0}^{\infty}, \{w_i\}_{i=0}^{\infty}) \leq \alpha_0 r. \tag{4.9}$$

By (4.5), (1.9), (4.1) and (4.9),

$$\tilde{d}_{\mathcal{M}}(\{(v_n)_i\}_{i=0}^{\infty}, \{w_i\}_{i=0}^{\infty}) \leq \alpha_0 r(1-\alpha r)^{-1}$$
$$\leq 2\alpha_0 r \leq 32^{-1}(2\mathrm{diam}(K)+1)^{-1} r \leq 4^{-1}n^{-1}.$$

Together with (1.7) this implies that

$$\sup\{|(v_n)_i(z_1,z_2) - w_i(z_1,z_2)| : z_1, z_2 \in K, i = 0, 1, \dots\} \leq 2\alpha r \leq (4n)^{-1},$$

$$\sup\{|(v_n)_i - w_i)(y_1,y_2) - ((v_n)_i - w_i)(z_1,z_2)|(\rho(z_1,y_1) + \rho(z_2,y_2))^{-1} :$$
$$y_1, y_2, z_1, z_2 \in K \text{ such that } (y_1,y_2) \neq (z_1,z_2), i = 0, 1, \dots\} \leq 2\alpha r \leq (4n)^{-1}.$$

Thus

$$\{\{w_i\}_{i=0}^{\infty} \in \mathcal{A}, d_{\mathcal{M}}(\{(v_n)_i\}_{i=0}^{\infty}, \{w_i\}_{i=0}^{\infty}) \le \alpha_0 r\}$$
$$\subset \{\{w_i\}_{i=0}^{\infty} \in \mathcal{A} : \text{such that 3.19), (3.20) hold}\}.$$

By the relation above and Lemmas 3.1 and 3.3

$$\{\{w_i\}_{i=0}^{\infty} \in \mathcal{A} : d_{\mathcal{M}}(\{(v_n)_i\}_{i=0}^{\infty} : \{w_i\}_{i=0}^{\infty}\} < \alpha_0 r\} \subset \mathcal{F}.$$

Theorem 1.5 is proved.

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