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ENERGY FUNCTION METHOD FOR SOLVING NONLINEAR DIFFERENTIAL EQUATIONS

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ABSTRACT. A very general conceptual algorithm for finding a solution process of a first order nonlinear differential equation is presented. The scope of this method is exhibited by showing the existing methods of solving nonlinear differential equations as special cases. Moreover, this approach extends for solving a larger class of nonlinear differential equations.

1. INTRODUCTION

Historically, it is well-known [2–8] that the energy/Lyapunov function method has played a very significant role in the qualitative and quantitative analysis of nonlinear and nonstationary systems of dynamic systems biological, engineering, physical and social sciences. In this work, we extend its usage for finding close form solutions (explicit or implicit form).

In this work, we present a very general conceptual algorithm for finding a solution process of a first order nonlinear differential equation. The method seeks an energy function associated with a given dynamic process. By knowing the existence of a solution process, we assume that there is an energy function associated with a given dynamic system. The basic ideas are: (1) to seek an unknown energy function, (2) to associate a simpler differential equation with an unknown energy function and the original nonlinear differential equation, (3) to determine an energy function in the context of a simpler differential equation and the original nonlinear differential equation, and (4) to find a representation of a solution of the original differential equation in the context of the energy function and the solution process of a simpler differential equation. We note that during the reduction process (to a simpler differential equation), the energy and rate functions of a simpler differential equation are determined. A solution of an original nonlinear differential equation is recasted

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in the context of the energy function and the solution of easily solvable differential equations like: (a) the directly integrable differential equation, (b) a first order linear differential equation, and (c) nonlinear solvable differential equations.

Furthermore, the scope of this method is demonstrated by solving the variable separable differential equations, homogeneous differential equations, the Bernoulli type differential equations and essentially time invariant nonlinear differential equations [1, 2] in a systematic and unified way. Thus the energy function method unifies the existing methods of solving nonlinear differential equations.

The procedure uses an Energy/Lyapunov type function, in order to create a new and simpler (reduced) differential equation, whose solution will in turn produce an *implicit primitive* for the original differential equation.

2. GENERAL PROBLEM

Let us consider the following first order nonlinear differential equation:

$$dx = f(t, x)\, dt, \tag{2.1}$$

where f is a continuous function defined on $J \times R$ into R, $J = [\alpha, \beta]$. We further assume that the initial value problem corresponding to (2.1) has a unique solution. The goal is to discuss a general procedure to find a representation of a general solution of (2.1).

3. GENERAL ALGORITHM

We impose conditions on an (unknown) energy function $V(t, x)$; we then conduct a search for a suitable $V(t, x)$ with the goal of eventually producing a reduced (solvable) differential equation, which in turn shall provide a closed form implicit/explicit solution or primitive for the nonlinear equation (2.1).

Step 1. We assume the existence of $V(t, x)$ satisfying:

a) $V(t, x)$ is continuous on $J \times \mathbb{R}$.
b) $V(t, x)$ is monotonic in x, for each $t \in J$.
c) V is continuously differentiable with respect to t and x.
d) For each $t \in J$, V has an "inverse" $E(t, x)$ such that

$$V(t, E(t, x)) = x = E(t, V(t, x)).$$

Step 2. Define differential operator L associated with (2.1):

$$L = \frac{\partial}{\partial t} + f(t, x)\, \frac{\partial}{\partial x}$$

and apply L to V thus:

$$dV(t, x(t)) = LV(t, x(t))\, dt \tag{3.1}$$

or simply $dV = V_t dt + fV_x dt$.

Step 3. Define composite $m(t) = V(t, x(t))$. Study the structure of (3.1) and select a useful form or class of rate function $F(t, m)$ for which the following reduced differential equation can be readily solved:

$$dm = F(t, m)\, dt. \tag{3.2}$$

Step 4. Combine (3.1) and (3.2) to produce

$$F(t, V(t, x)) = \frac{\partial}{\partial t} V(t, x) + f(t, x) \frac{\partial}{\partial x} V(t, x). \tag{3.3}$$

Next analyze and search for such a $V(t, x)$ whose associated composite $m(t)$ solves the reduced (3.2).

Step 5. Recover the solution $x(t)$ of (2.1) from the (usually implicit) equation:

$$V(t, x) = m(t) + C. \tag{3.4}$$

Let us approach the analysis of this method by considering various classes of the resulting *reduced* form $F(t, m)$. A starting place is the simplest class of explicit integrable functions. We begin by considering the class:

4. **INTEGRABLE DIFFERENTIAL EQUATIONS**

In this section we demonstrate the general procedure described in Section 2 for the class of differential equations (2.1) which can be reduced to an explicitly integrable rate function $F(t, m) = p(t)$ in (3.2). The simpler $p(t)$ which results from the method will be continuous and therefore integrable.

(a) This resulting $p(t)$, or F, shall have been required to satisfy (3.3).
(b) From (3.1), the original ODE (2.1) shall be reduced to the form

$$dm = p(t)\, dt \tag{4.1}$$

which is reduced integrable differential equation.

Procedure. Perform the general steps 1 and 2 in Section 3.

Now, using the chosen class $F(t,m) = p(t)$, (3.3) becomes

$$p(t) = \frac{\partial}{\partial t} V(t,x) + f(t,x) \frac{\partial}{\partial x} V(t,x). \tag{4.2}$$

Step 4. If $x(t)$ is to be a solution of (2.1), then (4.1) imposes condition (3.1) on energy $V(t,x)$ which in turn produces

$$dV = V_t dt + f V_x dt = p(t)\, dt. \tag{4.3}$$

Step 5.

$$V(t,x(t)) = \int dV(t,x(t)) = \int p(t)\, dt + C \tag{4.4}$$

where C is a constant of integration.

Observation 4.1. Differential equation (2.1) is the most general type of explicit nonlinear ODE. Let us consider the application of the method to the subclass of the form

$$f(t,x) = -\frac{M(t,x)}{N(t,x)}; \quad dx = f(t,x)\, dt \tag{4.5}$$

and choice of reduced ODE rate $F(t,m) = p(t)$ in (3.2).

Note (4.5) also has form $N\, dx + M\, dt = 0$. M, N, and p are continuous.

Here (4.2) becomes

$$p(t) = \frac{\partial}{\partial t} V(t,x) - \frac{M(t,x)}{N(t,x)} \frac{\partial}{\partial x} V(t,x). \tag{4.6}$$

We proceed to search for a useful combination pair $p(t)$ and $V(t,x)$.

For form (4.5) we approach the search by considering a choice of energy function of the form

$$V(t,x) = \int u(t,x)\, N(t,x)\, dx \tag{4.7}$$

where now the nonzero factor $u(t,x)$ becomes our search goal.

For the sake of clarity, and using suppressed notation where feasible, we note $V_x = uN$; condition (4.6) now becomes

$$p(t) = V_t + fV_x = V_t - \frac{M}{N} V_x = V_t + fV_x = V_t - uM. \tag{4.8}$$

This implies

$$\frac{\partial}{\partial x}\left(\frac{\partial}{\partial t}\left(\int u(t,x)N(t,x)\, dx\right) - u(t,x)M(t,x)\right) = 0. \tag{4.9}$$

Thus the given functions $M(t,x)$, $N(t,x)$, and the unknown target $u(t,x)$ must together satisfy (4.9) in order to reduce the ODE (4.5) to the integrable class (4.1).

Observation 4.2. Manipulating (4.8) with (4.7) and (4.5) as follows, we find

$$\frac{\partial}{\partial t}\int u\,N\,dx - u\,M = \int \frac{\partial}{\partial t}(u\,N)\,dx - \int \frac{\partial}{\partial x}(u\,M)\,dx + q(t),$$

where q results from partial antiderivative. Thus the condition (3.2) can be guaranteed by the vanishing of

$$\int \frac{\partial}{\partial t}(u\,N)\,dx - \int \frac{\partial}{\partial x}(u\,M)\,dx = \int \left[\frac{\partial}{\partial t}(u\,N) - \frac{\partial}{\partial x}(u\,M)\right] dx$$

or the vanishing of the integrand

$$\frac{\partial}{\partial t}(u\,N) - \frac{\partial}{\partial x}(u\,M)$$

Thus, in this case, the factor $u(t,x)$ plays the role of an integrating factor [1, 2], and our method reduces to the *Generalized Method of Integrating Factor*. Further if $u(t,x) \equiv u(t)$ or if $u(t,x) \equiv u(x)$, then the Energy Function Method is equivalent to the usual *Method of Integrating Factor*.

Observation 4.3. In the further case where $u(t,x) \equiv 1$, we see the Energy Function Method reduces to the usual *Method of Exact Differential Equation* [1, 2].

In the following, we present examples to illustrate this approach.

Example 4.1. $dx = -\left(2\sec(tx) + \frac{x}{t}\right) dt$.

Note

$$f(t,x) = -\frac{2t + x\cos(tx)}{t\cos(tx)}$$

in the

$$-\frac{M(t,x)}{N(t,x)}$$

form.

Assuming there exists a $V(t,x)$, we formally write

$$dV = V_t dt + V_x dx$$

and make a choice for $dm = F(t,m)\,dt$. Suppose we choose reduced form $dm = p(t)\,dt$. We are now seeking a $V = V(t,x)$ such that

$$V_t + fV_x = V_t - \frac{M}{N}V_x = p(t). \tag{4.10}$$

One approach is to transfer the search for $V(t,x)$ to a search for some $u(t,x)$ such that $V = \int uN\,dx$ or formally

$$V(t,x) = \int_a^x u(t,y)\,N(t,y)\,dy.$$

Also note

$$V_x = uN = u(t,x)\,t\cos(tx). \tag{4.11}$$

Denoting $\int N\,dx$ by $\tilde{N}$; here $\tilde{N} = \sin(tx)$ and $\dfrac{\partial \tilde{N}}{\partial x} = N$. We compute by parts

$$V = \int uN\,dx = u\tilde{N} - \int u\tilde{N}\,dx,$$

and

$$V_t = u_t\tilde{N} + u\tilde{N}_t - \frac{\partial}{\partial t}\int u\tilde{N}\,dx. \tag{4.12}$$

Thus (3.1) becomes

$$V_t - \frac{M}{N}\,V_x = p(t),$$

or

$$u_t \sin(tx) + ux\cos(tx) - \frac{\partial}{\partial t}\int u\tilde{N}\,dx - uM = p(t).$$

By $M = 2t + x\cos(tx)$ we seek $u(t,x)$ to satisfy:

$$u_t \sin(tx) + ux\cos(tx) - \frac{\partial}{\partial t}\int u\tilde{N}\,dx - ux\cos tx - 2ut = p(t),$$

or simply

$$u_t \sin(tx) - 2ut - \frac{\partial}{\partial t}\int u\tilde{N}\,dx = p(t).$$

Setting $u \equiv 1$ will reduce the solution to $p(t) = -2t$.

Step 6. Reduced ODE $m(t) = -2t\,dt$ implies $m(t) = -t^2 + C$.

The key step now is to recall $m(t)$ is defined as the composite $V(t, x(t))$. And Energy $V = \int uN\,dx = \int N\,dx = \tilde{N} = \sin(tx)$.

Thus we have the implicit primitive $-t^2 + C = \sin(tx)$ solving Example 4.1.

Remark 4.1. This example is actually an *exact* form, but we shall see further cases.

Example 4.2.

$$x' = -\frac{2t\tan x + 2xt^2 + x - 2t}{t + \sec^2 x}$$

Here

$$f(t,x) = -\frac{M(t,x)}{N(t,x)}$$

is *not exact*.

Again set

$$V = \int uN\,dx = u\tilde{N} - \int u_x\tilde{N}\,dx$$

where

$$\tilde{N} = \int N\,dx = tx + \tan x.$$

Thus,

$$V_t = u_t\tilde{N} + ux - \frac{\partial}{\partial t}\int u_x\tilde{N}\,dx.$$

Also note again $V_x = uN$.

We are attempting to reach the reduced form $dm = p(t)\,dt$ where $m(t) = V(t,x(t))$. Thus we seek

$$V = \int uN\,dx$$

to satisfy

$$m' = V_t + fV_x = p(t).$$

That is

$$u_t(tx + \tan x) + ux - \frac{\partial}{\partial t}\int u_x\tilde{N}\,dx - u\left[2t\tan x + 2xt^2 + x - 2t\right] = p(t). \tag{4.13}$$

Here $u \equiv$ constant does not work. Try $u(t,x) = u(t)$; $u_x = 0$. Condition (4.13) becomes

$$(u' - 2tu)\,(tx + \tan x) + 2tu = p(t).$$

Setting $u' - 2tu = 0$ produces $2tu = p(t)$. Note $u = e^{t^2}$ suffices. We have $m' = p(t) = 2te^{t^2}$ which gives $m(t) = e^{t^2} + C$. But $m(t) = V(t,x(t))$ and

$$V = \int uN\,dx = e^{t^2}\tilde{N} = e^{t^2}[tx + \tan x].$$

Thus implicitly we have $e^{t^2} + C = e^{t^2}[tx + \tan x]$ or primitive of Example 4.2 as

$$tx + \tan x - 1 = Ce^{-t^2}. \tag{4.14}$$

Remark 4.2. We remark that this example, while not exact, can be made exact by the introduction of integrating factor e^{t^2}. Now other methods would indeed have developed the same integrating factor. However it is interesting to note how our accommodating factor $u(t)$ produced the energy function

$$V(t,x) = \int u(t)\,N(t,x)\,dx,$$

which produced $u' - 2tu = 0$, which generated the integrating factor [1,2]. Thus this general Energy Function Method does incorporate exactness and integrating factors [1,2] and, as we shall see, other classes of equations in the subsequent sections.

5. LINEAR NONHOMOGENEOUS EQUATIONS

We now consider the Energy Function Method approach to the problems of reducing nonlinear equations (2.1) into the class of linear nonhomogeneous differential equations of the form

$$dm = F(t,m)\,dt = [\mu(t)\,m + p(t)]\,dt \tag{5.1}$$

here $\mu(t)$ and $p(t)$ are continuous real-valued rate coefficients.

Procedure. Preliminary Steps 1 and 2 are parallel.

Having chosen the class of forms $F(t, V(t,x)) = \mu(t)\, V(t,x) + p(t)$, Step 3 is to compute the differential of $m(t) = V(t, x(t))$ along $x(t)$. Step 4 becomes

$$\mu(t)\, V(t,x) + p(t) = \frac{\partial}{\partial t} V(t,x) + f(t,x)\, \frac{\partial}{\partial x} V(t,x),$$

which Energy Function $V(t,x)$ must satisfy in order to reduce (3.1) to (5.1). Since the solution of (5.1) is

$$m(t) = C \exp\left[\int^t \mu(r)\, dr + \int^t \exp\left[\int_s^t \mu(r)\, dr\right]\right] p(s)\, ds$$

we have Step 5:

$$V(t, x(t)) = C \exp\left[\int^t \mu(r)\, dr + \int^t \exp\left[\int_s^t \mu(r)\, dr\right]\right] p(s)\, ds$$

forms the implicit solution of (2.1).

Observation 5.1. Let us consider the application of the Method to the subclass of the form

$$f(t,x) = -\frac{M(t,x) + R(t,x)}{N(t,x)};$$

$$dx = f(t,x)\, dt \tag{5.2}$$

and a choice of reduced ODE rate

$$F(t,m) = \mu(t)\, m + p(t)$$

in (5.1).

Note (5.2) also has the form $N\, dx + (M + R)\, dt = 0$. M, N, R and p are continuous. Equation (5.2) becomes

$$\mu(t)\, V + p(t) = \frac{\partial}{\partial t} V(t,x) - \frac{M(t,x) + R(t,x)}{N(t,x)}\, \frac{\partial}{\partial x} V(t,x). \tag{5.3}$$

We search for a useful combination triple $\mu(t)$, $p(t)$ and $V(t,x)$ to satisfy (5.3). As before, we approach the search by considering a choice of energy function of the form

$$V(t,x) = \int u(t,x)\, N(t,x)\, dx \tag{5.4}$$

where again the nonzero function u becomes our goal.

Since $V_x = u\, N$, condition (5.3) becomes

$$p(t) = V_t + f V_x = V_t - \frac{M}{N} V_x = V_t - u\, M,$$

which implies

$$\frac{\partial}{\partial x}\left(\frac{\partial}{\partial t}\left(\int u(t,x)\, N(t,x)\, dx\right) - u(t,x)\, M(t,x)\right) = 0 \tag{5.5}$$

and also we seek to have

$$-R(t,x)u(t,x) = \mu(t)\,V(t,x) \tag{5.6}$$

Thus, given functions $M(t,x)$, $N(t,x)$, $R(t,x)$ and the unknown $u(t,x)$ together must satisfy (5.5) and (5.6) in order to reduce the ODE (5.2) to its linear form (5.1).

Example 5.1. Consider

$$x' = -\frac{2t + x\cos tx + 2\sin tx}{t\cos tx}.$$

We shall attempt to use the Energy Method to reduce Example 5.1 to the simpler linear form (5.1)

$$m'(t) = \mu(t)\,m(t) + p(t)$$

where again $m(t)$ is to be the composite of our Energy function $V(t,x(t))$. Form (5.1) is suggested by writing (5.2) as $N\,x' = -M - R$ where $N(t,x) = t\cos tx$, $M(t,x) = 2t + x\cos tx$, and $R(t,x) = 2\sin tx$ and noting $R_x = 2N$. Thus (5.1) arranged $m' = p + \mu m$ has a pattern such that a derivative of the last expression on the RHS resembles the first term on the LHS. Proceeding, we seek energy $V = \int \mu N\,dx$ for some useful u.

Also note the association $uN = V_x$; thus N is a “derivative” of V; while in original (5.2), N is a derivative of R; while in (5.1) m' is a derivative of the last term $\mu\,m = \mu V(t,x(t))$; which is a sort of transitive identification.

Continuing, as before, we formulate V by parts:

$$V = \int uN\,dx = u\tilde{N} - \int u_x\tilde{N}\,dx$$

and differentiate

$$V_t = u_t\tilde{N} + u\tilde{N}_t - \frac{\partial}{\partial t}\int u_x\tilde{N}\,dx.$$

Here,

$$\tilde{N} = \int N\,dx = \int t\cos tx\,dx = \sin tx = \frac{1}{2}\,R.$$

And again $V_x = uN$ gives us

$$x'V_x = uNx' = u(-M - R)$$

which in turn produces

$$\begin{aligned} V_t + x'V_x &= u_t\frac{1}{2}\,R + ux\cos tx - \frac{\partial}{\partial t}\int u_x\tilde{N}dx - uM - uR \\ &= u_t\sin tx + ux\cos tx - \frac{\partial}{\partial t}\int u_x\tilde{N}dx - u(2t + x\cos tx) - u2\sin tx \qquad (5.7) \\ &= (u_t - 2u)\sin tx - 2ut - \frac{\partial}{\partial t}\int u_x\tilde{N}dx. \end{aligned}$$

At this point, interestingly, we can proceed in two ways:

a) Suppose we let $u(t,x) \equiv 1$. Then (5.7) simplifies to

$$m' = V_t + x'V_x = 2\sin tx - 2t,$$

making $m = V = \tilde{N} = \sin tx$; in other terms

$$m' = -2m - 2t \tag{5.8}$$

has the chosen linear form. But also we can approach (5.7) by trying:

b) Let $u(t,x) = u(t)$ *while* setting $u' - 2u = 0$. Thus $u = e^{2t}$, and $m = V = e^{2t}\tilde{N} = e^{2t}\sin tx$, and (5.7) becomes $m' = -2t\, e^{2t}$, simple integrable class.

Now reduction a) has solution

$$e^{2t}m = C - t\, e^{2t} + \frac{1}{2}\, e^{2t}.$$

Together with $m(t) = V(t, x(t)) = \sin tx$, we have the implicit primitive

$$e^{2t}\left[2\sin tx + 2t - 1\right] = C.S \tag{5.9}$$

Similarly, reduction b) has solution

$$m = e^{2t}\left(\frac{1}{2} - t\right) + C,$$

which leads to (5.9) also (of course).

6. NONLINEAR SOLVABLE DIFFERENTIAL EQUATIONS

The following examples illustrate the scope of this approach beyond the *linear reducible* differential equations.

Example 6.1. Suppose

$$x' = \frac{\left(t^2x + \sin x\right)^2 - 2t\, x}{t^2 + \cos x}.$$

Let $N = t^2 + \cos x$, and let $J = t^2x + \sin x$. Then x' has the form

$$-\frac{J^2 + 2t\, x}{J_x}$$

so that we might try for separable reduced form $m' = \mu(t)m^2$. Next we compute the defining condition on the Energy function

$$V = \int uN\, dx = uJ - \int u_x J\, dx.$$

V must satisfy:

$$\begin{aligned} V_t &= u_tJ + uJ_t - \frac{\partial}{\partial t}\int u_xJ\, dx \\ &= u_tJ + u2tx - \frac{\partial}{\partial t}\int u_xJ\, dx \end{aligned}$$

while $V_x = uN = uJ_x$.

These imply

$$\begin{aligned} m' &= V_t + x'V_x \\ &= u_t J + 2u\,tx - uJ^2 - 2u\,tx - \frac{\partial}{\partial t}\int u_x J\,dx \\ &= u_t\left[t^2 x + \sin x\right] - uJ^2 - \frac{\partial}{\partial t}\int u_x J\,dx. \end{aligned}$$

We see $u \equiv 1$ works here, producing $m' = -J^2$ and making energy function $V = J$. Our reduced ODE is

$$m' = -m^2, \tag{6.1}$$

whose solution is $m(t)[t + C] = 1$. Thus the general solution of Example 6.1 is

$$\left(t^2 x + \sin x\right)(t + C) = 1. \tag{6.2}$$

Example 6.2.

$$dx = -\frac{1}{2}\,\frac{(tx + \tan x)^3 + 2x}{\sec^2 x + t}\,dt.$$

We set

$$M(t,x) = -\frac{1}{2}\,(tx + \tan x)^3 - x$$

and

$$N(t,x) = \sec^2 x + t.$$

We note that

$$\frac{\partial}{\partial x}\,M(t,x) = -\frac{3}{2}\,(tx + \tan x)^2\left(\sec^2 x + t\right) - 1$$

and

$$\frac{\partial}{\partial t}\,N(t,x) = 1.$$

Thus Example 6.2 is neither exact nor reducible to exact by an integrating factor. However, we initiate the Energy Function Method procedure. Following the argument used in Example 4.2, we arrive at:

$$\begin{aligned} V(t,x) &= \int u\left(\sec^2 x + t\right)dt \\ &= u(tx + \tan x) - \int (tx + \tan x)\,\frac{\partial}{\partial x}\,u\,dx \text{ (integration by parts).} \end{aligned}$$

We note that

$$\frac{\partial}{\partial x}\,V(t,x) = u\,\frac{\partial}{\partial x}\,(tx + \tan x) = u\left(\sec^2 x + t\right),$$

$$\frac{\partial}{\partial t}\,V(t,x) = \frac{\partial}{\partial t}\,[u(tx + \tan x)] - \frac{\partial}{\partial t}\left[\int (tx + \tan x)\,\frac{\partial}{\partial x}\,u\,dx\right].$$

Thus

$$\begin{aligned}
&\frac{\partial}{\partial t} V(t,x(t)) + f(t,x(t)) \frac{\partial}{\partial x} V(t,x(t)) \\
&\qquad = \frac{\partial}{\partial t}\left(\int uN(t,x)\,dx\right) - uM(t,x) \\
&\qquad = \frac{\partial}{\partial t}[u(tx+\tan x)] - \frac{\partial}{\partial t}\left[\int (tx+\tan x)\frac{\partial}{\partial x} u\,dx\right] \\
&\qquad\quad - u\frac{1}{2}(tx+\tan x)^3 - ux.
\end{aligned}$$

Again we *choose* $u(t) = 1$. Thus $V(t,x) = tx + \tan x$.

$$\begin{aligned}
\frac{dm}{dt} &= \frac{\partial}{\partial t} V(t,x(t)) + f(t,x(t)) \frac{\partial}{\partial x} V(t,x(t)) \\
&= \frac{\partial}{\partial t}[(tx+\tan x)] - \frac{1}{2}(tx+\tan x)^3 - x \\
&= -\frac{1}{2}(tx+\tan x)^3 = -\frac{1}{2}[V(t,x(t))]^3.
\end{aligned} \tag{6.3}$$

The reduced autonomous separable form is

$$dm = -\frac{1}{2} m^3 dt. \tag{6.4}$$

Combining (6.3), (6.4) and, of course, definition $m(t) = V(t,x(t))$ the general implicit solution to Example 6.2 is

$$1 = (t+C)(tx+\tan x)^2.$$

Remark 6.1. The presented examples in this section illustrate the scope of the Energy/Lyapunov Function Method for solving nonlinear differential equations in a systematic and unified way. Moreover, by imitating the above procedure, one can solve the following nonlinear differential equation

$$dx = \frac{-\alpha J^m + J_t}{J_x} \tag{6.5}$$

where J is a smooth function of (t,x),

$$J_x = \frac{\partial}{\partial x} J, \quad J_t = \frac{\partial}{\partial t} J,$$

$m \in R$ and α is a real number or function of t only.

7. ENERGY FUNCTION METHODS AND EXISTING METHODS OF SOLVING NONLINEAR DIFFERENTIAL EQUATIONS

The slope of energy function method is further demonstrated by solving variable separable differential equations, homogeneous differential equations, the Bernoulli type differential equations, and essentially time invariant differential equations [1,2].

7.1. **Separable Differential Equations [1, 2].** This class of differential equations are easily reducible to integrable differential equations. Each *separable* differential equation is characterized by a rate function $f(t,x)$ which is in fact decomposable into a product of two functions, one of which is a function of the independent variable t, and the other is a function of the dependent variable x. *Thus*, we assume that $f(t,x)$ is a separable function in t and x variables:

$$f(t,x) = a(t)\, b(x)$$

and

$$G(x) = \int_c^x \frac{ds}{b(s)}$$

is invertible. In this case, the original problem (2.1) structure now becomes

$$dx = f(t,x)\, dt = a(t)\, b(x)\, dt. \tag{7.1}$$

The basic calculations of our so-called Energy Method become:

$$F(t,m) = p(t) = V_t + f(t,x)\, V_x$$

and hence

$$V_t + ab\, V_x = p(t). \tag{7.2}$$

Since the RHS is independent of x, we consider a choice of energy function $V(t,x)$ to make the LHS independent of x. Here we see that we can satisfy this condition by choosing $V(t,x) = V(x)$. In this case, $V_t = 0$ and (7.2) reduces to

$$a(t)\, b(x)\, V_x(x) = p(t). \tag{7.3}$$

But this would mean $b(x)\, V_x(x)$ is a constant. We choose 1 for simplicity. Altogether we now have reduced the situation to $p(t) = a(t)$. Also we have

$$V(t,x) = V(x) = \int_c^x \frac{1}{b(u)}\, du.$$

In other words, as we have seen several times, the energy function V is in the form of an *integral* over x. And we finish by solving both $m'(t) = p(t) = a(t)$ which is the *reduced* ODE. Also we solve the energy integral

$$V(x) = \int_c^x \frac{1}{b(u)}\, du;$$

and this produces the solution $m(t) = V(t, x(t))$, i.e.

$$\int_q^t a(\tau)\, d\tau = \int_c^x \frac{1}{b(u)}\, du + C.$$

Remark 7.1. Of course we already knew this was to be the solution to a separable ODE, but it is still interesting to see this approach and power of systematic unification under one methodology.

7.2. **Homogeneous Differential Equations [1,2].** The class of equations referred to as *homogeneous* are reducible to the separable class by known methods; here we shall analyze them with respect to the Energy Function Approach.

Definition 7.1. A differential equation (2.1) is said to be *homogeneous* if the rate function $f(t,x)$ in (2.1) is a homogeneous function of degree zero, that is, $f(kt,kx) = f(t,x)$ for any nonzero k.

Assumptions. We assume that rate function $f(t,x)$ in (2.1) is homogeneous of degree zero. Further we shall need to assume that $(f(1,u)-u)$ does not vanish; and that the indefinite integral

$$G(u) := \int^{u} \frac{ds}{f(1,s)-s}$$

is invertible.

Let $v = \dfrac{x}{t}$. Note $f(t,x) = f(1,v)$ by homogeneity. Also note $\dfrac{\partial v}{\partial t} = \dfrac{-x}{t^2}$ and $\dfrac{\partial v}{\partial x} = \dfrac{1}{t}$. Consider the type of energy function $V(t,x)$ *also homogeneous*:

$$V(t,x) = P(v) = P\left(\frac{x}{t}\right)$$

where P has yet to be determined.

The problem of seeking unknown energy function $V(t,x)$ is equivalent to the problem of seeking unknown function P.

Compute dV as follows:

$$\frac{\partial}{\partial t} V(t,x) = \frac{\partial}{\partial t} P\left(\frac{x}{t}\right) = P'(v)\left(-\frac{x}{t^2}\right)$$

and

$$\frac{\partial}{\partial x} V(t,x) = \frac{\partial}{\partial x} P\left(\frac{x}{t}\right) = P'(v)\left(\frac{1}{t}\right).$$

Now $dx = f(t,x)$ and $dt = f(1,v)\,dt$. Hence,

$$\begin{aligned} dV(t,x) &= P'(v)\left[-\frac{x}{t^2}\,dt + \frac{1}{t}\,dx\right] \\ &= P'(v)\left[-\frac{x}{t^2}\,dt + \frac{1}{t}\,dx\right] \\ &= P'(v)\left[-\frac{v}{t} + \frac{1}{t}\,f(1,v)\right] dt. \end{aligned}$$

We try the indefinite integral function

$$G(v) := \int^{v} \frac{ds}{f(1,s)-s}$$

for $P(v)$.

Here we have Energy

$$V(t,x) = G\left(\frac{x}{t}\right) = \int^{\frac{x}{t}} \frac{ds}{f(1,s)-s}.$$

We compute

$$\begin{aligned}\frac{dG}{dt} &= \frac{dG}{dv}\left[v_t + v_x x'\right] \\ &= \frac{1}{[f(1,v)-v]}\left[\frac{-x}{t^2} + f(1,v)\,\frac{1}{t}\right]\end{aligned}$$

Thus the reduced form is integrable differential equation

$$m' = \frac{1}{t}. \tag{7.4}$$

Altogether we have

$$\log t = \int^{\frac{x}{t}} \frac{ds}{f(1,s)-s} + C \tag{7.5}$$

for the general solution.

Example 7.1.

$$dx = \frac{x^2}{t^2}\,dt.$$

We note the homogeneity.

Further we see

$$G(u) := \int^{u} \frac{ds}{f(1,s)-s} = \int^{u} \frac{ds}{s^2 - s}$$

is invertible by partial fractions, i.e.

$$z = G(u) = \log\left(1 - \frac{1}{u}\right)$$

has inverse

$$u = \frac{1}{1-e^z}$$

Altogether we have

$$dm(t) = dV(t,x) = \frac{dG}{dv}\,dv = \frac{1}{v^2 - v}\,\frac{x(x-t)}{t^3}\,dt = \frac{1}{t}\,dt.$$

Solve for $m(t) = \log(t) + C$. Equate to

$$V = G\left(\frac{x}{t}\right) = \log\left(1 - \frac{1}{x}\right).$$

The general solution is

$$c^2 t = 1 - \frac{t}{x} \quad \text{or } k\,tx = x - t,\ k > 0.$$

Remark 7.2 ([1,2])**.** Of course being separable, this result is also obtainable through classical methods.

7.3. **The Bernoulli Type Differential Equations [1,2].** We present another subclass of differential equations reducible to (5.1). This class of equations are referred to as *Bernoulli* differential equations. First, we introduce a definition of a Bernoulli differential equation.

Definition 7.2. A differential equation is said to be a *Bernoulli* differential equation if the rate function $f(t,x)$ in $dx = f(t,x)\,dt$ is of the following form:

$$f(t,x) = K(t)\,x + Q(t)\,x^n$$

for *some* real number $n \neq 0, 1$.

We consider

$$dx = [K(t)\,x + Q(t)\,x^n]\,dt. \tag{7.6}$$

It is assumed that K and Q are continuous nonzero functions.

We initiate the procedure to reduce the Bernoulli type equation using the Energy Function Method. We associate a suitable natural Energy/Lyapunov function in a unified and coherent way. We propose a differential of the form:

$$\begin{aligned} dV(t,x) &= \mu(t)\,V(t,x) + p(t) \\ &= \frac{\partial}{\partial t}V(t,x) + [K(t)\,x + Q(t)\,x^n]\,\frac{\partial}{\partial x}V(t,x), \end{aligned} \tag{7.7}$$

In minimal notation

$$dV = \mu V + p = V_t + Kx\,V_x + Qx^n\,V_x. \tag{7.8}$$

We attempt $\mu V = Kx\,V_x$ with $\mu(t) = \delta K(t)$, which gives

$$\frac{V_x}{V} = \frac{\delta}{x}. \tag{7.9}$$

From this, it is clear that the quotient of $\dfrac{\partial}{\partial x}V(t,x)$ with $V(t,x)$ is *independent of* t. Therefore, we can assume that $V(t,x) \equiv V(x)$, that is, $V(t,x)$ is *independent of* t. This means that $V_t = 0$ and $p(t) = Qx^nV_x$ from (7.7). Solving (7.9)

$$V(x) = Cx^\delta, \quad C > 0. \tag{7.10}$$

We compute

$$\frac{d}{dx}V(x) = V_x = \delta C x^{\delta-1}$$

and

$$p(t) = \delta C Q(t)\,x^{n+\delta-1}. \tag{7.11}$$

Separating, we have

$$x^{n+\delta-1} = \frac{p(t)}{\delta Q(t)C}. \tag{7.12}$$

We note that the right-hand side of (7.12) is a function of t only. Therefore, we let $\delta = 1 - n$. Thus (7.11) becomes

$$\begin{aligned} p(t) &= (1-n)\, CQ(t) \\ dV(t,x) &= \mu(t)\, V(t,x) + p(t) \\ &= (1-n)\, K(t)\, Cx^{1-n} + (1-n)\, CQ(t). \end{aligned}$$

Letting $C = 1$ the reduced form is linear:

$$m'(t) = (1-n)\, K(t)\, m(t) + (1-n)\, Q(t) \tag{7.13}$$

which is a linear solvable form in $m(t)$. Thus the general solution becomes

$$x^{1-n} = m(t) = \Phi(t)\, C + (1-n) \int \Phi(t,s)\, Q(s)\, ds \tag{7.14}$$

where

$$\Phi(t) = \exp\left[(1-n) \int P(t)\, dt\right]$$

and of course $m(t)$ solves (7.13) through classical procedures.

7.4. **Essentially Time Invariant Differential Equations [1, 2].** This class of differential equations is of the following form

$$dx = F(ax + bt + c)\, dt \tag{7.15}$$

where F is smooth enough to assure the existence of solution of (7.15); a, b and c are given arbitrarily real numbers. Further assume that

$$G(x) = \int^x \frac{du}{aF(u) + b}$$

is in this case, the Energy Function is as follows:

$$V(t,x) = G(ax + bt + c) \tag{7.16}$$

where G is an unknown smooth Energy/Lyapunov like function to be determined.

By setting $v = ax + bt + c$, we have

$$\frac{\partial v}{\partial t}(t,n) = \frac{d}{dv}\, G(v)\, b, \quad \frac{\partial v}{\partial x}(t,x) = \frac{d}{dv}\, G(v)\, a \tag{7.17}$$

$$\begin{aligned} \frac{\partial v}{\partial t}(t,n) + F(ax+bt+c)\, \frac{\partial v}{\partial x}(t,x) &= b\, \frac{d}{dv}\, G(v) + F(v)\, \frac{d}{dv}\, G(v)\, a \\ &= LG(v) \\ &= [aF(v) + b]\, \frac{d}{dv}\, G(v). \end{aligned} \tag{7.18}$$

In this case, the reduced form is assumed to be

$$\mu(t)\, G(v) + P(t) = [aF(v) + b]\, \frac{d}{dx}\, G(x) \tag{7.19}$$

where μ and P are arbitrary functions. We know that the right-hand side is independent of t, therefore, one can choose $P(t) = P(\text{constant})$ and $\mu(t) = 0$. In this situation we have

$$P = [aF(v) + b] \frac{d}{dv} G(v) \tag{7.20}$$

which implies v

$$G(v) = \int^{v} \frac{P\, ds}{aF(s) + b} + C \tag{7.21}$$

and hence

$$V(t,x) = G(ax + bt + c) = \int^{ax+bt+c} \frac{P\, ds}{aF(s) + b} + C. \tag{7.22}$$

The reduced differential equation is

$$dm(t) = P\, dt. \tag{7.23}$$

Hence, the solution of (7.15) is given by

$$G(ax + bt + c) = Pt + C. \tag{7.24}$$

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