

STABILITY CRITERIA FOR CERTAIN SECOND ORDER NEUTRAL DELAY DIFFERENTIAL EQUATIONS

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ABSTRACT. In this paper we study the asymptotic stability of the zero solution of second order neutral delay differential equation of the form

$$y''(t) + \alpha y''(t - \tau) + ay'(t) + by'(t - \tau) + cy(t) + dy(t - \tau) = 0,$$

where $a, b, c, d, \alpha \in (-1, 0) \cup (0, 1)$, and $\tau > 0$ are constants. In this paper, we obtain a new necessary condition and obtain robust method of determining whether the zero solution is asymptotically stable. In proving our results we make use of Pontryagin's theory for quasi-polynomials.

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1. INTRODUCTION

The aim of this paper is to study the asymptotic stability of the zero solution of the neutral delay differential equation

$$(1.1) \quad y''(t) + \alpha y''(t - \tau) + ay'(t) + by'(t - \tau) + cy(t) + dy(t - \tau) = 0$$

where $\tau > 0$, $\alpha \in (-1, 0) \cup (0, 1)$, a, b, c , and d are real constants. There are many applications of neutral differential equations in scientific models such as of masses attached to an elastic bar [1] and population growth [2]. An interesting application of neutral equations appears in [3] and involves an interplay between physical observation and simulation (called “real-time dynamic substructuring”) for seismic testing. There are many studies of neutral equations mainly dealing with oscillations or sufficient conditions of stability of the zero solution. See [4–8]. In these studies necessary conditions or sufficient conditions are derived using Lyapunov's direct (or second) method. In all of these works, there is a sizable gap between sufficiency and necessity. In our previous papers [9,10], we considered non-neutral second order delay differential equation (i.e., $\alpha = 0$) and obtained an algorithmic method as a robust means testing for asymptotic stability. We also obtained new complete criteria for asymptotic stability in special cases of the non-neutral problem. In this paper, we

attack the more difficult neutral problem (1.1) and obtain robust algorithmic tests parallel to those in [9,10]. In addition, we obtain criteria for special cases leading to asymptotic stability regions for some cases. We also obtain a new and far reaching necessary condition (Theorem 3.3) for asymptotic stability which plays a role throughout this paper. Generally, including delays in a differential equation has a destabilizing effect. Our work on non-neutral delay equations certainly upholds this, but when the order is 2 or higher there are rare cases when the delay has a stabilizing effect. We raise the same question as whether inclusion of a “neutral term” can have a stabilizing effect, and we give an example that gives an affirmative answer.

This paper is organized as follows. In Section 2, we present the tools used in our asymptotic stability analysis. In Sections 3 we give our main results. In Section 4 we present some examples. Some of the results of this paper derived in a different way by Kuang (see [2]). Many of our results are new and derived using our approaches developed in our earlier papers.

2. BACKGROUND

In this section, we identify the characteristic function of (1.1), and cite results of Pontryagin [11] related to asymptotic stability and applications of Pontryagin’s results [12, §13.7–13.9].

The characteristic function of (1.1) (derived by searching for solutions of the form $e^{\lambda t}$ or by using Laplace transforms) is given by

$$(2.1) \quad \widehat{H}(s) = s^2 + \alpha s^2 e^{-s\tau} + as + bse^{-s\tau} + c + de^{-s\tau}.$$

We rewrite (2.1) by multiplying by $e^{s\tau}$ and letting $s = \frac{z}{\tau}$ to get

$$(2.2) \quad H(z) = \tau^2 e^z \widehat{H}\left(\frac{z}{\tau}\right) = z^2 e^z + \alpha z^2 + Aze^z + Bz + Ce^z + D$$

where

$$(2.3) \quad A = a\tau, \quad B = b\tau, \quad C = c\tau^2, \quad D = d\tau^2.$$

The following can be found in [13, Theorem 6.1].

Theorem 2.1 *In order that all solutions of (1.1) approach zero as $t \rightarrow \infty$ it is necessary and sufficient that all zeros of (2.1), or equivalently (2.2), have negative real parts and are bounded away from the imaginary axis, i.e., there is a positive real number ν such that $\operatorname{Re} z \leq -\nu$ for every zero z of $H(z)$.*

We first determine the conditions under which all zeros of (2.1), or equivalently (2.2), have negative real parts and then find conditions under which the zeros are bounded uniformly away from the imaginary axis. The function (2.2) is a special

function, usually called an exponential polynomial or a quasi-polynomial. The problem of analyzing the distribution of the zeros in the complex plane of such functions has received considerable attention.

Definition 2.1 Let $h(z, w)$ be a polynomial in the two variables z and w (with complex coefficients),

$$(2.4) \quad h(z, w) = \sum_{m,n} a_{mn} z^m w^n, \quad (m, n \text{ nonnegative integers}).$$

We call the term $a_{rs} z^r w^s$ the principal term of $h(z, w)$ if $a_{rs} \neq 0$, and for every term $a_{mn} z^m w^n$ with $a_{mn} \neq 0$, we have $m \leq r$ and $n \leq s$.

Note that $H(z) = h(z, e^z)$ where

$$(2.5) \quad h(z, w) = z^2 w + \alpha z^2 + Azw + Bz + Cw + D$$

It is clear from Definition 2.1 that $h(z, w)$ in (2.5) has principal term $z^2 w$. We now cite two theorems of Pontryagin, see [11,12].

Theorem 2.2 *Let $H(z) = h(z, e^z)$, where $h(z, w)$ is a polynomial with a principal term. We separate $H(iy)$ into real and imaginary parts; that is, we set $H(iy) = F(y) + iG(y)$. (Of course, $F(y)$ and $G(y)$ have entire extensions.) If all the zeros of the function $H(z)$ lie in the open left half plane, then the zeros of the functions $F(y)$ and $G(y)$ are real, are interlacing, and*

$$(2.6) \quad \Delta(y) = G'(y)F(y) - G(y)F'(y) > 0$$

for all real y . Moreover, in order that all the zeros of the function $H(z)$ lie in the open left half plane, it is sufficient that any one of the following conditions be satisfied:

- (a): All the zeros of the functions $F(y)$ and $G(y)$ are real and interlace, and the inequality (2.6) is satisfied for at least one value of y .
- (b): All the zeros of the function $F(y)$ are real and for each of these zeros $y = y_0$ (2.6) is satisfied, i.e., $F'(y_0)G(y_0) < 0$.
- (c): All the zeros of the function $G(y)$ are real and for each of these zeros the (2.6) is satisfied, i.e., $G'(y_0)F(y_0) > 0$.

In our case,

$$(2.7) \quad H(iy) = -y^2 e^{iy} - \alpha y^2 + iAye^{iy} + Biy + Ce^{iy} + D = F(y) + iG(y)$$

where

$$(2.8) \quad F(y) = -y^2 \cos y - \alpha y^2 - Ay \sin y + C \cos y + D$$

and

$$(2.9) \quad G(y) = -y^2 \sin y + Ay \cos y + By + C \sin y.$$

To study the location of the zeros of $H(z)$, we study the zeros of F and G . To do so, we need the following result which is useful in determining whether all roots of F and G are real. Let $f(z, u, v)$ be a polynomial in z, u , and v which we write in the form

$$(2.10) \quad f(z, u, v) = \sum_{m,n} z^m \phi_m^{(n)}(u, v)$$

where $\phi_m^{(n)}(u, v)$ is a polynomial of degree n , homogeneous in u and v , and let $z^r \phi_r^{(s)}(u, v)$ be the principal term of $f(z, u, v)$, and let $\phi^{*(s)}(u, v)$ denote the coefficient of z^r in $f(z, u, v)$, so that

$$\phi^{*(s)}(u, v) = \sum_{n \leq s} \phi_r^{(n)}(u, v).$$

(The Principal term for the polynomials of the form (2.10) are analogous to that defined in Definition 2.1, see [20, pages]). Also we let

$$\Phi^{*(s)}(z) = \phi^{*(s)}(\cos z, \sin z).$$

Theorem 2.3 *Let $f(z, u, v)$ be a polynomial with principal term $z^r \phi_r^{(s)}(u, v)$ and assume that $u^2 + v^2$ is not a factor of $\phi_r^{(s)}(u, v)$. If ϵ is such that $\Phi^{*(s)}(\epsilon + iy) \neq 0$ for all real y , then in the strip $-2\pi k + \epsilon \leq \operatorname{Re} z \leq 2\pi k + \epsilon$, the function $F(z) = f(z, \cos z, \sin z)$ has, for all sufficiently large values of k , exactly $4sk + r$ zeros. Thus, in order for the function $F(z)$ to have only real roots, it is necessary and sufficient that in the real interval $-2\pi k + \epsilon \leq x \leq 2\pi k + \epsilon$, it has exactly $4sk + r$ real roots for all sufficiently large k .*

Note that the functions $F(y)$ and $G(y)$ in (2.10) and (2.11) have principal terms $-z^2u$ and z^2v , respectively. The condition that $u^2 + v^2$ not be a factor of $z^r \phi_r^{(s)}(u, v)$ is frequently overlooked. When $s = 1$, it is not an issue. This condition is satisfied by polynomials $f(z, u, v)$ derived from function $h(z, w)$ in (2.12) with a principal term. As well, if $f(z, u, v)$ is derived from function involving $\sin z$ and $\cos z$, the Pythagorean identity could be used to make this condition satisfied. Nonetheless this condition is needed for Theorem 2.3 to be true as stated.

For the case in point, $r = 2$ and $s = 1$. Therefore $F(z)$ (given in (2.8)) has all real zeros if and only if $F(z)$ has $4k + 2$ zeros in $(-2k\pi, 2k\pi)$ for k sufficiently large, and the same holds for G given in (2.9) with $(-2k\pi, 2k\pi)$ replaced by $(-2k\pi + \epsilon, 2k\pi + \epsilon)$ where $0 < \epsilon < \pi$.

3. MAIN RESULTS

In this section we present the main results of this paper. We first describe the asymptotic behavior of the zeros of G . Throughout this paper for x real and $a > 0$, $[x]_a$ denotes the unique real number in the interval $[0, a)$ for which $x - [x]_a$ is an

integer multiple of a . We will use $a = \pi$ and $a = 2\pi$. See Kuang [2, p. 65] for the following result:

Lemma 3.1 *A necessary condition for the zero solution of (1.1) to be asymptotically stable is that $|\alpha| \leq 1$.*

In this paper we will mostly only consider $|\alpha| < 1$. We have the following necessary conditions.

Lemma 3.2 *If the zero solution of (1.1) is asymptotically stable, then $(C + D)(A + B + C) > 0$.*

Proof. Theorem 2.2 and the fact that $y = 0$ is a zero of G yield $\Delta(0) = G'(0)F(0) = (C + D)(A + B + C) > 0$.

Lemma 3.3 *For n sufficiently large, the interval $(n\pi - \pi/2, n\pi + \pi/2)$ contains exactly one zero r_n of G and $\lim_{n \rightarrow \infty} (r_n - n\pi) = 0$.*

Proof. From (2.9), $y = 0$ is a zero of G and

$$(3.1) \quad G(n\pi + \pi/2) = -(n\pi + \pi/2)^2(-1)^{n+1} + B(n\pi + \pi/2) + C(-1)^{n+1}.$$

Thus there can be at most four zeros of G of the form $n\pi + \pi/2$. All other zeros of G are roots of the equation

$$(3.2) \quad w(y) = \zeta(y)$$

where

$$(3.3) \quad w(y) = (y^2 - C) \tan y - By \sec y$$

and

$$(3.4) \quad \zeta(y) = Ay.$$

For n sufficiently large, w resembles the tangent function on $(n\pi - \pi/2, n\pi + \pi/2)$ in that w has limits $-\infty$ and ∞ at $n\pi - \pi/2$ and $n\pi + \pi/2$ when the limits are taken from inside the interval. For n sufficiently large these yield existence of a solution of (3.2) in $(n\pi - \pi/2, n\pi + \pi/2)$. Now (3.2) yields

$$(3.5) \quad \sin y = \frac{\frac{B}{y} + \frac{A \cos y}{y}}{1 - \frac{C}{y}}.$$

It follows from (3.5) that

$$\lim_{\substack{G(y)=0 \\ y \rightarrow \infty}} \sin y = 0,$$

and thus the roots of G in $(n\pi - \pi/2, n\pi + \pi/2)$ tends to the center of the interval as n goes to infinity. Also in this case it is easy to see that $w'(y) > |A|$ for $y \in (n\pi - \pi/4, n\pi + \pi/4)$ and n sufficiently large, and thus uniqueness holds.

The following is a very useful necessary condition for the asymptotic stability of the zero solution of (1.1). It refines the necessary condition in Lemma 3.2.

Theorem 3.1 *Assume $-1 < \alpha \leq 1$. If the zero solution of (1.1) is asymptotically stable, then $A + B + C > 0$ and $C + D > 0$.*

Proof. Assume the zero solution of (1.1) is asymptotically stable. From Lemma 3.2

$$(3.6) \quad \Delta(0) = (C + D)(A + B + C) > 0.$$

It follows from Theorems 2.1-2.3 that G has all real zeros and for k sufficiently large $(-2k\pi + \epsilon, 2k\pi + \epsilon)$ contains precisely $4k + 2$ zeros of G . We are taking $0 < \epsilon < \pi/4$. Since $y = 0$ is a zero of G and G is odd, $(0, 2k\pi + \epsilon)$ contains precisely $2k + 1$ zeros $r_1 < r_2 < \dots < r_{2k+1}$ of G where k is sufficiently large. By Lemma 3.3, $r_{2k+1} \in (2k\pi - \epsilon, 2k\pi + \epsilon)$ for k sufficiently large. From (2.8) and the hypothesis $-1 < \alpha \leq 1$, it follows that $F(r_{2k+1}) < 0$ for k sufficiently large. By Theorems 2.1 and 2.2, the zeros of F and G interlace and thus the $F(r_j)$ must strictly alternate in sign (where $r_0 = 0$). Thus $F(0)F(r_{2k+1}) < 0$, and since $F(r_{2k+1}) < 0$, $F(0) = C + D > 0$. By (3.6) $A + B + C > 0$, and the proof is complete.

We first consider special cases where some of the coefficients A, B, C , or D are zero. In this paper Z^+ denotes the set of all nonnegative integers.

Theorem 3.2 *Suppose that $|\alpha| < 1$ and that all zeros of $H(z)$ are in the open left half plane (i.e. $\text{Re}z < 0$ for every zero z of $H(z)$). Then all zeros of $H(z)$ are bounded away from imaginary axis (i.e. there $\eta > 0$ for which $\text{Re}z < -\eta$ for every zero z of $H(z)$).*

Proof. Assume otherwise. Then there is a sequence $z_n = \alpha_n + i\beta_n$ of zeros of $H(z)$ where $\alpha_n < 0$, and $\alpha_n \rightarrow 0$. If $\{\beta_n\}$ were bounded, then $H(z)$ would have a zero on the imaginary axis. Thus we may assume that $\beta_n \rightarrow \infty$ and $\beta_n > 0$. From $H(z) = 0$ and (2.2)

$$(3.7) \quad |1 + \alpha e^{-z_n}| = \left| \frac{Az_n + Bz_n e^{-z_n} + C + D e^{-z_n}}{z_n^2} \right|.$$

Since $\alpha_n \rightarrow 0$ and $\beta_n \rightarrow \infty$, $|z_n| \rightarrow \infty$ and the right hand side of (3.7) tends to 0 as $n \rightarrow \infty$. But

$$(3.8) \quad \begin{aligned} |1 + \alpha e^{-z_n}|^2 &= (1 + \alpha e^{-\alpha_n} \cos \beta_n)^2 + (\alpha e^{-\alpha_n} \sin \beta_n)^2 \\ &\geq (1 - |\alpha| e^{-\alpha_n})^2 \rightarrow (1 - |\alpha|)^2. \end{aligned}$$

Since $|\alpha| < 1$, (3.8) yields a contradiction.

We consider some special cases:

Theorem 3.3 *Assume that $|\alpha| < 1$, $A = B = 0$, $C > 0$, and $C + D > 0$.*

(i) *Suppose $D - \alpha C > 0$. The zero solution of (1.1) is asymptotically stable if and only if and if there exists a positive integer k such that*

$$\begin{aligned} (2k - 1)\pi &< \sqrt{C} < 2k\pi, \\ (1 + \alpha)((2k - 2)\pi)^2 &< C + D < (1 + \alpha)(2k\pi)^2, \end{aligned}$$

and

$$((2k - 1)\pi)^2(1 - \alpha) < C - D < (1 - \alpha)((2k + 1)\pi)^2.$$

(ii) Suppose $D - \alpha C < 0$. The zero solution of (1.1) is asymptotically stable if and only if there is a positive integer k such that

$$(2k - 2)\pi < \sqrt{C} < (2k - 1)\pi,$$

and

$$\begin{aligned} C + D &< (1 + \alpha)(2\pi)^2, \\ C - D &< (1 - \alpha)\pi^2 \end{aligned}$$

when $k = 1$ and

$$\begin{aligned} (1 + \alpha)((2k - 2)\pi)^2 &< C + D < (1 + \alpha)(2k\pi)^2, \\ ((1 - \alpha)(2k - 3)\pi)^2 &< C - D < (1 - \alpha)((2k - 1)\pi)^2 \end{aligned}$$

when $k > 1$.

(iii) If $D - \alpha C = 0$, then the zero solution of (1.1) is not asymptotically stable.

With $A = B = 0$, $C > 0$ and $C + D > 0$ are the necessary conditions of Theorem 3.2.

Proof. For $A = B = 0$, (2.8) and (2.9) yield

$$\begin{aligned} G(y) &= -\sin y(y^2 - C), \\ G'(y) &= -\cos y(y^2 - C) - 2y \sin y, \end{aligned}$$

and

$$F(y) = -y^2(\cos y + \alpha) + C \cos y + D.$$

The zeros of G are $y = \pm\sqrt{C}$ and $y = n\pi (n \in Z)$. If y is a zero of G , then

$$(3.9) \quad \Delta(y) = [-y^2(\cos y + \alpha) + C \cos y + D][- \cos y(y^2 - C) - 2y \sin y],$$

and, in particular,

$$(3.10) \quad \Delta(-\sqrt{C}) = \Delta(\sqrt{C}) = -2\sqrt{C} \sin \sqrt{C} [D - \alpha C].$$

If $D - \alpha C = 0$, then $\Delta(\pm\sqrt{C}) = 0$, and by Theorems 2.2-2.3, the zero solution of (1.1) is not asymptotically stable, and (iii) is proven. Suppose $D - \alpha C > 0$. By (3.10), $\Delta(\sqrt{C}) > 0$ if and only if $\sin \sqrt{C} < 0$, or there exists a positive integer k such that

$$(2k - 1)\pi < \sqrt{C} < 2k\pi.$$

At the points $y = n\pi (n \in Z^+)$ we have

$$(3.11) \quad \Delta(n\pi) = [-(n\pi)^2((-1)^n + \alpha) + C(-1)^n + D][-(-1)^n((n\pi)^2 - C)].$$

Suppose that $n \geq 2k$. Then $(n\pi)^2 - C > 0$. If, in addition, n is even, then

$$(3.12) \quad \Delta(n\pi) = -[-(n\pi)^2(1 + \alpha) + C + D]((n\pi)^2 - C) > 0$$

if and only if

$$C + D < (1 + \alpha)(n\pi)^2.$$

Thus $\Delta(n\pi) > 0$ for all even $n \geq 2k$ if and only if

$$C + D < (1 + \alpha)(2k\pi)^2.$$

For odd $n \geq 2k$,

$$(3.13) \quad \Delta(n\pi) = [(n\pi)^2(1 - \alpha) - C + D]((n\pi)^2 - C) > 0.$$

if and only if

$$C - D < (1 - \alpha)(n\pi)^2.$$

Thus $\Delta(n\pi) > 0$ for all odd $n > 2k$ if and only if

$$C - D < (1 - \alpha)((2k + 1)\pi)^2.$$

Suppose that $0 \leq n \leq 2k - 1$. Then $(n\pi)^2 - C < 0$. As above, $\Delta(n\pi) > 0$ for all even n with $0 \leq n < 2k - 1$ if and only if

$$C + D > (1 + \alpha)((2n - 2)\pi)^2,$$

and $\Delta(n\pi) > 0$ for all odd n with $0 < n \leq 2k - 1$ if and only if

$$C - D > (1 - \alpha)((2k - 1)\pi)^2.$$

Since Δ is an even function of y , (i) is now proven using Theorems 2.2, 2.3 and 3.2.

Suppose that $D - \alpha C < 0$. By (3.11), $\Delta(\sqrt{C}) > 0$ if and only if $\sin \sqrt{C} > 0$, or there exist a positive integer k such that

$$(2k - 2)\pi < \sqrt{C} < (2k - 1)\pi.$$

The remainder of the proof of (ii) is similar to that of (i) above. Note that when $k = 1$, the only nonnegative multiple of π less than or equal to $(2k - 2)\pi$ is 0 and $\Delta(0) = (C + D)C > 0$ by hypothesis.

We consider the pure delay case, i.e. $A = 0$ and $C = 0$.

Theorem 3.4 *Assume that $A = 0$, $C = 0$, $B > 0$, and $D > 0$. Then G has all real zeros if and only if $0 < B < B^*$ where $B^* = y^* \sin y^*$ and y^* is the unique solution of $\tan y = -y$ in $(0, \pi)$ ($B^* \approx 1.819705741$).*

With $A = C = 0$, $B > 0$ and $D > 0$ are the necessary conditions of Theorem 3.2, and thus Theorem 3.4 and Theorem 3.5 below represent the general pure delay case.

Proof. When $A = 0$ and $C = 0$, (2.9) yields

$$G(y) = -y^2 \sin y + By.$$

Thus $y = 0$ is a zero of G , and the nonzero zeros of G are the roots of the equation

$$(3.14) \quad \sin y = \frac{B}{y}.$$

For the function G to have all real and distinct zeros it is necessary and sufficient that $\sin y$ and $\frac{B}{y}$ agree at two distinct points in the interval $(0, \pi)$. The choice of B^* yields $\sin y = \frac{B^*}{y}$ having one root in $(0, \pi)$ with multiplicity 2, and $\sin y = \frac{B}{y}$ has two roots in $(0, \pi)$ precisely when $0 < B < B^*$. Note that in this case $(2k\pi, 2k\pi + \epsilon)$ contains a zero of G while $(-2k\pi, -2k\pi + \epsilon)$ does not for k sufficiently large so that G has $4k + 2$ zeros in $(-2k\pi + \epsilon, 2k\pi + \epsilon)$ for k sufficiently large.

With $0 < B < B^*$, the positive zeros of G , $0 < r_1 < r_2 < \dots$, satisfy $2(n - 1)\pi < r_{2n-1} < r_{2n} < (2n - 1)\pi$ for $n = 1, 2, \dots$, and $[r_{2n-1}]_{2\pi} \downarrow 0$ and $[r_{2n}]_{2\pi} \uparrow \pi$ as $n \rightarrow \infty$

Theorem 3.5 *Let $-1 < \alpha < 1$, $A = 0$, $C = 0$, $0 < B < B^*$, and $D > 0$. Let $r_1 < r_2 < r_3 < r_4 \dots$ be the positive zeros of G . If $-1 < \alpha \leq 0$, the zero solution of (1.1) is asymptotically stable if and only if $F(r_1) < 0$. If $0 < \alpha < 1$, then the zero solution of (1.1) is asymptotically stable if and only if $F(r_1) < 0$, and $F(r_{2j}) > 0$ for $j = 1, 2, \dots, m - 1$ where m is the smallest index for which $\cos r_{2m} + \alpha < 0$.*

Proof. By Theorem 3.5, G has all real zeros. Note that $F(0) = D > 0$. In either case, the sign conditions on F are necessary for the zeros of F and G to interlace and thus for the zero solution of (1.1) to be asymptotically stable. Thus we need to prove sufficiency in both cases. To this end, it suffices to show that $(-1)^n F(r_n) > 0$ for $n = 1, 2, \dots$. Since $F(0) = D > 0$, this would imply the interlacing of the zeros of F and G and the zero solution of (1.1) would be asymptotically stable. Note that from (2.8)

$$F(y) = -y^2(\cos y + \alpha) + D$$

Suppose $-1 < \alpha \leq 0$ and $F(r_1) < 0$. From the proof of Theorem 3.4, $r_{2n-1}, r_{2n} \in ((2n - 2)\pi, (2n - 1)\pi)$ and $r_{2n} \in ((2n - 3/2)\pi, (2n - 1)\pi)$ for $n = 1, 2, \dots$, and $r_{2n-1} \in ((2n - 2)\pi, (2n - 3/2)\pi)$ for $n = 2, 3, \dots$. Since $\alpha \leq 0$ and $\cos y < 0$ for $y \in ((2n - 3/2)\pi, (2n - 2)\pi)$, $F(y) > 0$ for such y . Thus $F(r_{2n}) > 0$ for all $n = 1, 2, \dots$. Since $F(r_1) < 0$, $r_1 \in (0, \pi/2)$ and $\cos r_1 + \alpha > 0$. Also, $[r_3]_{2\pi} < [r_1]_{2\pi}$ while $r_3 > r_1$. It follows that

$$(3.15) \quad F(r_3) = -r_3^2(\cos r_3 + \alpha) + D < -r_1^2(\cos r_1 + \alpha) + D = F(r_1) < 0.$$

The same argument yields $F(r_{2n-1}) < 0$ for all $n = 1, 2, \dots$. It follows then that the zero solution of (1.1) is asymptotically stable.

Now we assume that $0 < \alpha < 1$. Since $F(r_1) = -r_1^2(\cos r_1 + \alpha) + D < 0$, $\cos r_1 + \alpha > 0$. The argument in the first case yields $F(r_{2n-1}) < 0$ for all $n = 1, 2, \dots$. For the even roots, note that $[r_{2k}]_{2\pi} \uparrow \pi$ as $k \rightarrow \infty$. Thus there is a first index m where $\cos r_{2m} + \alpha < 0$. Since $[r_{2m}]_{2\pi}$ increases it follows that $\cos r_{2n} + \alpha < 0$ and $F(r_{2n}) > 0$ for all $m \geq n$. Asymptotic stability of the zero solution of (1.1) now follows.

In the next case, we take $B = C = 0$, $A > 0$, and $D > 0$. Of course, if $B = C = 0$, then $A > 0$, and $D > 0$ are the necessary conditions of Theorem 3.1. For $B = C = 0$,

(2.8) and (2.9) yield

$$(3.16) \quad G(y) = -y^2 \sin y + Ay \cos y$$

and

$$(3.17) \quad F(y) = -y^2 \cos y - \alpha y^2 - Ay \sin y + D.$$

The real zeros of G are $y = 0$ and the roots of $\cot y = \frac{y}{A}$. With $A > 0$, $\cot y = \frac{y}{A}$ has precisely one root in each open interval between successive multiples of π . When k is sufficiently large, $(2k\pi, 2k\pi + \epsilon)$ contains a root of $\cot y = \frac{y}{A}$ and $(-2k\pi, -2k\pi + \epsilon)$ does not, and so G has $4k+2$ zeros in $(-2k\pi + \epsilon, 2k\pi + \epsilon)$. Here $0 < \epsilon < \pi$. Thus G has all real zeros. For $n = 1, 2, \dots$, let r_n be the zero of G in the interval $((n-1)\pi, n\pi)$. In fact, $[r_n]_\pi \in (0, \pi/2)$ and $[r_n]_\pi$ decreases to zero as $n \rightarrow \infty$. As such $\cos r_{2j-1} > 0$ and $\cos r_{2j-1}$ increases to 1 as $j \rightarrow \infty$, and $\cos r_{2j} < 0$ and decreases to -1 as $j \rightarrow \infty$.

Theorem 3.6 *Let $-1 < \alpha < 1$, $B = C = 0$, $A > 0$, and $D > 0$. If $0 < \alpha < 1$, the zero solution of (1.1) is asymptotically stable if and only if $F(r_1) < 0$ and $F(r_{2j}) > 0$ for $j = 1, \dots, m-1$ where m is the smallest index for which $\cos r_{2m} + \alpha < 0$. If $-1 < \alpha \leq 0$, the zero solution of (1.1) is asymptotically stable if and only if $F(r_{2j-1}) < 0$ for $j = 1, \dots, m$ where m is the smallest index for which $\cos r_{2m-1} + \alpha > 0$.*

Proof. As in the proof of Theorem 3.5, necessity is evident, and for sufficiency we need to argue that under the stated conditions, we have

$$(3.18) \quad (-1)^n F(r_n) > 0 \quad \text{for } n = 1, 2, \dots$$

From (3.16),

$$(3.19) \quad r_n \sin r_n = A \cos r_n,$$

and simple calculations yield

$$(3.20) \quad F(r_n) = -[\cos r_n (r_n^2 + A^2) + \alpha r_n^2 - D].$$

Suppose $0 < \alpha < 1$. The first expression in (3.20) yields that $F(r_{2k-1})$ is decreasing in k . Thus if $F(r_1) < 0$, then $F(r_{2k-1}) < 0$ for all $k = 2, 3, \dots$. Also (3.20) yields that $F(r_{2k}) > 0$ for all k where $\cos r_{2k} + \alpha < 0$. Thus sufficiency holds when $0 < \alpha < 1$. Suppose $-1 < \alpha \leq 0$. The first expression in (3.20) yields that $F(r_{2k}) > 0$ for all $k = 1, 2, \dots$. The second expression in (3.20) yields that $F(r_{2k-1})$ is decreasing over those k where $\cos r_{2k-1} + \alpha > 0$. Thus sufficiency holds for $-1 < \alpha \leq 0$.

Now we consider the case where $A = 0$ and B and C are nonzero. In the first lemma we consider the case $C > 0$ and $D > 0$, and in the second lemma we consider the case $CD < 0$. Note that the necessary conditions of Theorem 3.1 become $B + C > 0$ and $C + D > 0$ so that $C < 0$ and $D < 0$ is ruled out for asymptotic stability and thus both lemmas cover all cases of interest.

Lemma 3.5 *Assume that $A = 0$, $B \neq 0$, $C > 0$, $D > 0$, and $B + C > 0$. Necessary conditions for the zero solution of (1.1) to be asymptotically stable are*

1. *If $B < 0$, then*
 - (i) *G has exactly one zero in $(0, \pi)$ (in fact, it is in $(\pi/2, \pi)$ when $-1 < \alpha < 0$),*
 - (ii) *G has exactly two zeros in $(2j\pi, (2j + 1)\pi)$ for $j = 1, \dots, m$ and G has two zeros in $((2m + 1)\pi, (2m + 2)\pi)$ if $\sqrt{C} \in (2m\pi, (2m + 2)\pi)$.*
2. *If $B > 0$, then*
 - (i) *G has exactly two zeros in $(0, \pi)$ if $\sqrt{C} \in (0, \pi)$*
 - (ii) *G has exactly two zeros in $((2j + 1)\pi, (2j + 2)\pi)$ for $j = 0, \dots, m$ and G has exactly two zeros in $((2m + 2)\pi, (2m + 3)\pi)$ when $\sqrt{C} \in ((2m + 1)\pi, (2m + 3)\pi)$.*

Note that in 1(ii) the first condition is empty when $m = 0$, and in fact it is replaced by 1(i).

Proof. With $A = 0$, (2.8) and (2.9) yield

$$(3.21) \quad G(y) = -y^2 \sin y + By + C \sin y$$

and

$$(3.22) \quad F(y) = -y^2 \cos y - \alpha y^2 + C \cos y + D.$$

The zeros of G are $y = 0$ and the roots of the equation $\csc y = \frac{1}{B}(y - \frac{C}{y})$. Note that $y = \sqrt{C}$ is the positive zero of $\zeta(y) = \frac{1}{B}(y - \frac{C}{y})$.

In the remainder of this proof, we will mark some arguments with Roman numerals for subsequent reference, as they will be repeated later.

Suppose $B < 0$. Since $B + C > 0$, $-C/B > 1$. Thus $\csc y = \zeta(y)$ has an odd number of roots in $(0, \pi)$ and even number of roots in each interval $(n\pi, (n + 1)\pi)$ for $n = 1, 2, \dots$. Notice that if r is a zero of G , then

$$(3.23) \quad r^2 - C = B \frac{r}{\sin r},$$

and by (3.21)

$$(3.24) \quad F(r) = -Br \cot(r) - \alpha r^2 + D.$$

Let W denote the function whose values $W(r)$ are given by the right side of (3.24).

For $-1 < \alpha < 0$, $W(r) > 0$ in $(0, \pi/2]$. Since $F(0) = C + D > 0$, G cannot have a zero in $(0, \pi/2]$. Otherwise interlacing of the zeros of F and G would fail and the zero solution of (1.1) would not be asymptotically stable. Since $\csc y$ is increasing in $(\pi/2, \pi)$ and $\zeta(y)$ is decreasing in $(\pi/2, \pi)$, G has exactly one zero in $(\pi/2, \pi)$.

(Argument I) Suppose $0 < \alpha < 1$. Now $W'(r) = B \csc^2 r(r - \sin r \cos r) - 2\alpha r < 0$ in every interval $(n\pi, (n + 1)\pi)$ for $n = 0, 1, \dots$. Now G has an odd number of zeros in $(0, \pi)$. If G has three zeros in $(0, \pi)$, then in order for the zeros of F and G to

interlace, F (and thus W) would have three points of sign change in $(0, \pi)$. As such W' would have a sign change in $(0, \pi)$ which is false. Thus 1(i) is proven. **(I.)**

On any interval $((2\ell - 1)\pi, 2\ell\pi)$, $\ell = 1, 2, \dots$, the concavity properties of $\csc y$ and $\frac{1}{B}(y - \frac{C}{y})$ yield that G can have at most two zeros. The arguments above for $-1 < \alpha < 0$ and $0 < \alpha < 1$ also yield that G can have at most two zeros in each interval $(2\ell\pi, (2\ell + 1)\pi)$, $\ell = 1, 2, \dots$. Part 1(ii) now follows because G must have $4k + 2$ zeros in $(-2k\pi + \epsilon, 2k\pi + \epsilon)$ for all sufficiently large k .

Now suppose $B > 0$. With $C > 0$, it follows that G has an even number of zeros in each interval $(n\pi, (n+1)\pi)$, $n = 0, 1, \dots$. Suppose $\sqrt{C} \in ((2m+1)\pi, (2m+3)\pi)$ for some $m = 0, 1, 2, \dots$. We have that $\zeta(y) < 0$ if $0 < y < \sqrt{C}$ and $\zeta(y) > 0$ if $y > \sqrt{C}$. As such on $(0, \sqrt{C})$, $\zeta(y)$ can only meet $\csc y$ on lower branches of $\csc y$. That is, on $(0, \sqrt{C})$, G can only have zeros in intervals of the form $((2\ell + 1)\pi, (2\ell + 2)\pi)$. Likewise, on (\sqrt{C}, ∞) , G can only have zeros in intervals of the form $(2\ell\pi, (2\ell + 1)\pi)$. Due to opposing concavities G can meet the upper branches in at most two points. We show that in lower branches of $\csc y$, G can have at most two zeros. In this case, the necessary conditions will follow from Theorem 2.3 so G has all real zeros.

Assume $0 < \alpha < 1$. There are 3 cases based on the sign of $D - \alpha C$.

(Argument II.) If $D - \alpha C = 0$, the zeros of F are $\pm\sqrt{C}$ and the roots of $\cos y = -\alpha$. If G has four zeros in any interval between consecutive multiples of π , F would need to have at least three zeros in this interval which is not the case. **(II.)**

The zeros of F are the roots of the equation $y^2 = \phi(y)$ where $\phi(y) = \frac{C \cos y + D}{\cos y + \alpha}$. Note that $\phi(y)$ has period 2π and vertical asymptotes corresponding to roots of $\cos y = -\alpha$. We have $\phi'(y) = \frac{(D - \alpha C) \sin y}{(\cos y + \alpha)^2}$.

(Argument III.) If $D - \alpha C > 0$, $\phi'(y)$ agrees in sign with $\sin y$. In the interval $((2\ell + 1)\pi, (2\ell + 2)\pi)$, $\phi(y)$ is decreasing on $((2\ell + 1)\pi, \rho)$ and on $(\rho, (2\ell + 2)\pi)$ where ρ is the root of $\cos y = -\alpha$ in this interval. Since y^2 is increasing on $(0, \infty)$, F can have at most two zeros in $((2\ell + 1)\pi, (2\ell + 2)\pi)$, and the proof in this case is complete. **(III.)**

(Argument IV.) Now assume $D - \alpha C < 0$. In this case $\phi'(y)$ and $-\sin y$ agree in sign. A typical sketch of y^2 and $\phi(y)$ is shown in Figure 1 for $D - \alpha C < 0$.

Notice that $\phi(0) = \frac{C+D}{1+\alpha} < C < \frac{-C+D}{-1+\alpha} = \phi(\pi)$, which is reflected in Figure 1. If $y < \sqrt{C}$, then $y^2 < \phi(\pi)$ and y^2 can only meet the lower branches of $\phi(y)$. Due to opposite concavities, in any interval $((2\ell + 1)\pi, (2\ell + 2)\pi) \subseteq (0, \sqrt{C})$, $y^2 = \phi(y)$ can only have two roots. If in any of these intervals G had four zeros, then F would need three zeros for interlacing to hold. The case with $B > 0$ and $0 < \alpha < 1$ is now complete. **(IV.)**

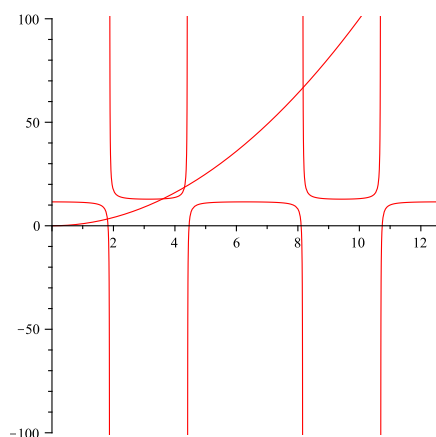


Figure 1 y^2 and $\phi(y)$ when $D - \alpha C < 0$

Suppose $B > 0$ and $-1 < \alpha < 0$. In this case $W'(r) = B \csc^2 r(r - \sin r \cos r) - 2\alpha r > 0$ and as in Argument I interlacing would fail if G has 4 zeros in $((2\ell + 1)\pi, (2\ell + 2)\pi)$ and 2(ii) is proven.

Remark 3.1 If \sqrt{C} were an odd multiple of π when $B > 0$ or if \sqrt{C} were an even multiple of π when $B < 0$, the zero count for G and Theorem 2.3 would yield that the zero solution of (1.1) is not asymptotically stable.

Lemma 3.6 Assume that $A = 0$, $B \neq 0$, $B + C > 0$, $C + D > 0$, and $CD < 0$. Necessary conditions for the zero solution of (1.1) to be asymptotically stable are

1. If $B < 0$ (and thus $C > 0$), then
 - (i) G has exactly one zero in $(0, \pi)$ (in fact, it is in $(\pi/2, \pi)$ when $-1 < \alpha < 0$)
 - (ii) G has exactly two zeros in $(2j\pi, (2j + 1)\pi)$ for $j = 1, \dots, m$ when $\sqrt{C} \in (2m\pi, (2m + 2)\pi)$ and G has two zeros in $((2m + 1)\pi, (2m + 2)\pi)$
2. If $B > 0$, then
 - (i) G has exactly two zeros in $(0, \pi)$ if $\sqrt{C} \in (0, \pi)$
 - (ii) G has exactly two zeros in $((2j + 1)\pi, (2j + 2)\pi)$ for $j = 0, \dots, m$ if $\sqrt{C} \in ((2m + 1)\pi, (2m + 2)\pi)$ and G has two zeros in $((2m + 2)\pi, (2m + 3)\pi)$.
 - (iii) G has exactly two zeros in $((2j)\pi, (2j + 1)\pi)$ for $j = 0, \dots, m$ if $\sqrt{-C} \in ((2m + 1)\pi, (2m + 2)\pi)$ and $C < 0$.

Proof. The functions $G(y)$ and $F(y)$ are given by (3.21) and (3.22), and as in Lemma 3.5, the zeros of G are $y = 0$ and the roots of the equation $\csc y = \frac{1}{B}(y - \frac{C}{y})$. The proof uses the arguments given in Lemma 3.5.

Suppose $B < 0$ and $C > 0$. Then $D < 0$ and $\frac{-C}{B} > 1$. When $0 < \alpha < 1$, $W'(r) = B \csc^2 r(r - \sin r \cos r) - 2\alpha r < 0$ on every interval $(n\pi, (n + 1)\pi)$, and Argument I yields the result. Suppose $-1 < \alpha < 0$. Notice that $\phi'(y) = \frac{(D - \alpha C) \sin y}{(\cos y + \alpha)^2}$, and if $D - \alpha C < 0$, then $-\sin y$ and $\phi'(y)$ agree in sign. Here we apply Argument IV. When $D - \alpha C = 0$, Argument II applies, and when $D - \alpha C > 0$, Argument III applies.

Suppose $B > 0$ and $C > 0$, then $D < 0$ with $-1 < \alpha < 0$, $W'(r) > 0$ and Argument I applies. For $0 < \alpha < 1$, $D - \alpha C < 0$ and we apply Argument IV.

We consider $B > 0$, $C < 0$. Then $D > 0$ and $-\frac{C}{B} < 1$. If $-1 < \alpha < 0$, then $W'(r) > 0$, and we apply Argument I. See Figures 2a-2b for the roots of $\csc y = \frac{1}{B}(y - \frac{C}{y})$.

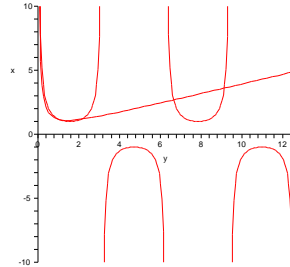


Figure 2a $\csc(y)$ and $\zeta(y)$ in $(0, 4\pi)$ when $C < 0$, $B > 0$

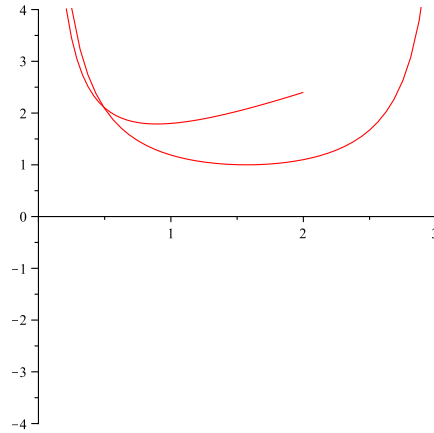


Figure 2b $\csc(y)$ and $\zeta(y)$ in $(0, \pi)$ when $C < 0$, $B > 0$

Suppose $0 < \alpha < 1$. Then $D - \alpha C > 0$. With $B > 0$, $C < 0$, and $-\frac{C}{B} < 1$, $\csc y = \frac{1}{B}(y - \frac{C}{y})$ and thus G has an even number of zeros in the interval $(2m\pi, (2m + 1)\pi)$ and no zeros in the intervals $((2m + 1)\pi, (2m + 2)\pi)$ for $m = 0, 1, \dots$ (see Figures 2a-2b). We will show that each interval $(2m\pi, (2m + 1)\pi)$ contains precisely two zeros of G . If we rule out four or more zeros, the standard counting argument yields the result. Assume that G has four zeros in $(2m\pi, (2m + 1)\pi)$. Since $F(0) > 0$ and $G(0) = 0$, at the first four zeros of $G(y)$ in $(2m\pi, (2m + 1)\pi)$, $F(y)$ has values that are negative, positive, negative, and positive, respectively, in order that the zeros of G and F interlace. Now F must have three zeros between successive pairs of these zeros of G , and F must change sign from negative to positive, positive to negative, and negative to positive at these zeros, respectively. Since $\phi(y) < 0 < y^2$ on $(2m\pi + \rho, (2m + 1)\pi)$ where $\rho \in (0, \pi)$ and $\cos \rho + \alpha = 0$, these zeros of F are in $(2m\pi, 2m\pi + \rho)$ (see Figure 3). Since $\cos y + \alpha > 0$ for $y \in (2m\pi, 2m\pi + \rho)$, $\bar{F}(y) = y^2 - \phi(y)$ must change sign from positive to negative, negative to positive, and positive to negative, respectively

at these zeros of F . Thus $\bar{F}'(y)$ has values that are negative, positive, and negative successively. Thus $\bar{F}''(y)$ has values that are positive and negative, successively and so \bar{F}''' has a negative value. But

$$(3.25) \quad \bar{F}'''(y) = -\frac{\sin y(D - \alpha C)(\alpha^2 - 4 \cos y\alpha - 6)}{(\cos y + \alpha)^4} > 0$$

for $y \in (2m\pi, 2m\pi + \pi)$, a contradiction.

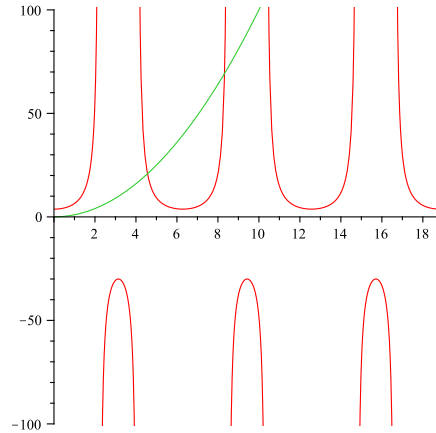


Figure 3 y^2 and $\phi(y)$ when $D - \alpha C > 0$

Remark 3.2 We assume $A = 0$ and the necessary conditions of Lemmas 3.5–3.6 hold. Under these conditions G has all real zeros, and we denote the positive zeros of G as $r_1 < r_2 < \dots$. Of course, these are the positive roots of $\csc y = \frac{1}{B}(y - \frac{C}{y})$. With $C > 0$ and $B > 0$ the odd numbered roots in (\sqrt{C}, ∞) are decreasing to 0 modulo 2π , i.e. $[r_{2j+1}]_{2\pi} \downarrow 0$, and the even numbered roots increase to π modulo 2π , i.e. $[r_{2j}]_{2\pi} \uparrow \pi$. With $C > 0$ and $B < 0$, in (\sqrt{C}, ∞) , $[r_{2j+1}]_{2\pi} \uparrow 2\pi$ while $[r_{2j}]_{2\pi} \downarrow \pi$. With $C < 0$ and $B > 0$, r_{2j+1} in $(\sqrt{-C}, \infty)$ $[r_{2j+1}]_{2\pi} \downarrow 0$ and $[r_{2j}]_{2\pi} \uparrow \pi$.

By Remark 3.2 and Lemmas 3.5 and 3.6, there are 3 cases to consider for the zero configurations of G .

Theorem 3.8 *Assume that $A = 0$ and that the necessary conditions of Lemmas 3.5 and 3.6 hold. Let $r_j, j = 0, 1, 2, \dots$ be the nonnegative zeros of G where $r_0 = 0$. The zero solution of (1.1) is asymptotically stable if and only if*

$$(-1)^j F(r_j) > 0, \quad j = 1, 2, \dots$$

We omit the proof of Theorem 3.8 as the conditions are precisely those needed to guarantee interlacing of the zeros of F and G and the conditions in Lemmas 3.5 and 3.6 guarantee F and G have all real zeros.

It is clear that one cannot use Theorem 3.8 with an infinite number of conditions, and thus we obtain a stability test with a finite number of conditions.

Theorem 3.9 (Algorithmic Stability Test I) *Assume that $A = 0$ and that the necessary conditions of Lemmas 3.5 and 3.6 hold. The zero solution of (1.1) is asymptotically stable if and only if*

1: $F(r_{2j+1}) < 0$ for $j = 1, 2, \dots, P_1$, and

2: $F(r_{2j}) > 0$ for $j = 1, 2, \dots, P_2$

where $P_1 = \max(M_1, N_1, L_1)$ $P_2 = \max(M_2, N_2, L_2)$ and $M_1, M_2, N_1, N_2, L_1, L_2$ are as follows.

Here M_1 and M_2 are the first positive integers such that $r_{2M_1+1} > \sqrt{|C|}$ and $r_{2M_2} > \sqrt{|C|}$. The values L_1, L_2, N_1 and N_2 are based on the parameters C and α as follows.

I. If $C > 0$, L_1 and L_2 are the first positive integers such that

1: $-r_{2L_1+1}^2(\cos r_{2L_1+1} + \alpha) + C + D < 0$ and

2: $-r_{2L_2}^2(\cos r_{2L_2} + \alpha) - C + D > 0$.

For $0 < \alpha < 1$ $N_1 = 1$ and N_2 is the first positive integer such that $\cos r_{2N_2} + \alpha < 0$. For $-1 < \alpha < 0$, N_1 is the first positive integer such that $\cos r_{2N_1+1} + \alpha > 0$, and $N_2 = 1$.

II. If $C < 0$, then $L_1 = M_1$ and $L_2 = M_2$. For $-1 < \alpha < 0$, $N_1 = 1$ and N_2 is the first positive integer such that $\cos r_{2N_2} + \alpha < 0$ and for $0 < \alpha < 1$, $N_1 = 1$ and N_2 is the first positive index such that $\cos r_{2N_2} + \alpha > 0$.

The monotone convergence of the residues of the even and odd numbered roots of G guarantee the existence of these numbers. These monotonicities guarantee that if 1 and 2 in Theorem 3.9 hold, then these inequalities hold for all $j > P_1$ and $j > P_2$, respectively. Asymptotic stability would then follow from Theorem 3.8.

In the case where $D = \alpha C$, we can obtain asymptotic stability criteria based on the parameters of the problem rather than an algorithm. We also obtain stability regions. We will use the following notations: ρ_1 is the root of $\cos y = -\alpha$ in $(0, \pi)$ and ρ_2 is the root of $\cos y = -\alpha$ in $(\pi, 2\pi)$. Notice that $\rho_2 = 2\pi - \rho_1$, $\rho_{2j+1} = \rho_1 + 2j\pi$, and $\rho_{2j+2} = \rho_2 + 2j\pi$, $j = 1, 2, \dots$, constitute all positive roots of $\cos y = -\alpha$. For convenience, we let $\rho_{-1} = \rho_0 = 0$, and $\beta = \frac{B}{\sqrt{1-\alpha^2}}$.

Theorem 3.10 *Assume $A = 0$, $B \neq 0$, $D = \alpha C$, $-1 < \alpha < 1$, and $\alpha \neq 0$. The zero solution of (1.1) is asymptotically stable if and only if*

$$(3.26) \quad 0 < \beta < \rho_1 \quad \text{and} \quad 0 < C < \rho_1^2 - \beta\rho_1,$$

or $0 < \beta < 2\rho_1$, and there exist $L \geq 1$ such that

$$(3.27) \quad \rho_{2L}^2 + \beta\rho_{2L} < C < \rho_{2L+1}^2 - \beta\rho_{2L+1},$$

or $-2(\pi - \rho_1) < \beta < 0$ and there exists $L \geq 1$ such that

$$(3.28) \quad \rho_{2L-1}^2 - \beta\rho_{2L-1} < C < \rho_{2L}^2 + \beta\rho_{2L}.$$

Proof. First note that with $D = \alpha C$, $C > 0$ is a necessary condition from Theorem 3.1. With $A = 0$ and $D = \alpha C$, the positive zeros of F are $y = \sqrt{C}$ and the positive roots $\cos y = -\alpha$. It is easy to see from Theorem 2.3 that F has all real zeros. From (3.22)

$$(3.29) \quad F'(y) = -2y(\cos y + \alpha) + (y^2 - C) \sin y.$$

By Theorem 2.2 the zero solution of (1.1) is asymptotically stable if and only if

$$(3.30) \quad \Delta(\sqrt{C}) = 2BC(\cos \sqrt{C} + \alpha) > 0,$$

$$(3.31) \quad \Delta(\rho_{2j-1}) = -\sin(\rho_{2j-1})(\rho_{2j-1}^2 - C) \left(\sin \rho_{2j-1}(C - \rho_{2j-1}^2) + B\rho_{2j-1} \right) > 0,$$

and

$$(3.32) \quad \Delta(\rho_{2j}) = -\sin(\rho_{2j})(\rho_{2j}^2 - C) \left(\sin \rho_{2j}(C - \rho_{2j}^2) + B\rho_{2j} \right) > 0$$

for all $j = 1, 2, \dots$

Notice that $\sin \rho_{2j-1} = \sqrt{1 - \alpha^2}$ and $\sin \rho_{2j} = -\sqrt{1 - \alpha^2}$ for $j = 1, 2, \dots$ and thus equations (3.32) and (3.33) are equivalent to

$$(3.33) \quad \Delta(\rho_{2j-1}) = (1 - \alpha^2)(\rho_{2j-1}^2 - C)(\rho_{2j-1}^2 - \beta\rho_{2j-1} - C) > 0$$

and

$$(3.34) \quad \Delta(\rho_{2j}) = (1 - \alpha^2)(\rho_{2j}^2 - C)(\rho_{2j}^2 + \beta\rho_{2j} - C) > 0$$

for all $j = 1, 2, \dots$

If $B > 0$, then (3.31) is equivalent to $\cos \sqrt{C} + \alpha > 0$, or equivalently, $\rho_{2L} < \sqrt{C} < \rho_{2L+1}$ for some integer $L \geq 0$. Now (3.34) and (3.35) hold for all $j = 1, 2, \dots$ if and only if the positive zeros of the quadratic functions $\rho^2 - \beta\rho - C$ and $\rho^2 + \beta\rho - C$ lie in the interval $(\rho_{2L-1}, \rho_{2L+1})$ and (ρ_{2L}, ρ_{2L+2}) , respectively. That is,

$$(3.35) \quad \rho_{2L-1} < \frac{\beta + \sqrt{\beta^2 + 4C}}{2} < \rho_{2L+1}$$

and

$$(3.36) \quad \rho_{2L} < \frac{-\beta + \sqrt{\beta^2 + 4C}}{2} < \rho_{2L+2}.$$

We have that (3.36) and (3.37) are equivalent to

$$(3.37) \quad 2\rho_{2L-1} - \beta < \sqrt{\beta^2 + 4C} < 2\rho_{2L+1} - \beta$$

and

$$(3.38) \quad 2\rho_{2L} + \beta < \sqrt{\beta^2 + 4C} < 2\rho_{2L+2} + \beta.$$

With $\beta = \frac{B}{\sqrt{1-\alpha^2}} > 0$, (3.38) and (3.39) are equivalent to

$$(3.39) \quad 2\rho_{2L} + \beta < \sqrt{\beta^2 + 4C} < 2\rho_{2L+1} - \beta.$$

In this case, (3.40) is equivalent to $0 < \beta < \rho_{2L+1} - \rho_{2L}$ and

$$(3.40) \quad \rho_{2L}^2 + \beta\rho_{2L} < C < \rho_{2L+1}^2 - \beta\rho_{2L}.$$

If $L = 0$, these are equivalent to $0 < \beta < \rho_1$ and $0 < C < \rho_1^2 - \beta\rho_1$. If $L \geq 1$, these are equivalent to $0 < \beta < 2\rho_1$ and (3.40). See Example 4.1 for region of stability. The proof for $B < 0$ is similar.

For the general case we will use the following lemma. The proof is essentially the same as the proof of Lemma 3.3 in [9], and we omit the proof.

Lemma 3.7 *Assume $-1 < \alpha < 1$. Let $\delta = (1 + |\alpha|)/2$. Let $n \in \mathbb{Z}^+$. If $n \geq M := \max(M_1, M_2, M_3, M_4)$ where M_1, M_2, M_3 and M_4 be the smallest positive integers such that*

$$(3.41) \quad \frac{|B| + \sqrt{B^2 + 4C}}{2M_1\pi - \pi} < 1$$

$$(3.42) \quad \frac{|C|}{(M_2\pi - \pi/2)^2} + \frac{\delta|A| + |B|}{(M_2\pi - \pi/2)\sqrt{1 - \delta^2}} < 1$$

$$(3.43) \quad \frac{|C| + \delta|B| + \delta^2|A|}{\delta^2(M_3\pi - \pi/2)^2} + \frac{(2\delta + |B|)\sqrt{1 - \delta^2}}{\delta^2(M_3\pi - \pi/2)} < 1$$

$$(3.44) \quad \frac{|C|}{(M_4\pi - \pi/2)^2} + \frac{|B| + \delta|A|}{(M_4\pi - \pi/2)\sqrt{1 - \delta^2}} < 1,$$

then the interval $[n\pi - \pi/2, n\pi + \pi/2]$ contains exactly one zero r of G and $(n\pi - \cos^{-1} \delta < r < n\pi + \cos^{-1} \delta)$.

Remark 3.3 Recall that G has all real zeros if and only if G has $4k + 2$ zeros in $(-2j\pi + \epsilon, 2k\pi + \epsilon)$ (or, equivalently, $2k + 1$ zeros in $(0, 2k\pi + \epsilon)$ for all sufficiently large k where $0 < \epsilon < \pi/2$). From Lemma 3.7, G has all real zeros if and only if G has $M + 1$ zeros in $(0, M\pi + \epsilon)$. In this case let $r_1 < r_2 < r_3 < \dots$ denote the positive zeros of G . The zero solution of (1.1) is asymptotically stable if in addition $(-1)^j F(r_j) > 0$ for all $j = 1, 2, \dots$. It follows that $r_{2k+1} - 2k\pi \rightarrow 0$ and $r_{2k} - (2k - 1)\pi \rightarrow 0$ as $k \rightarrow \infty$.

Suppose $0 < \alpha < 1$. We have

$$(3.45) \quad F(r_j) = -r_j^2 \left(\cos r_j + \alpha + A \frac{\sin r_j}{r_j} - C \frac{\cos r_j}{r_j^2} - \frac{D}{r_j^2} \right).$$

If $2k \geq M$, then Lemma 3.7 yields $\cos r_{2k} < -\delta$ so that

$$(3.46) \quad \cos r_{2k} + \alpha < -\frac{1 - \alpha}{2} = -\frac{1 - |\alpha|}{2}$$

and if $2k + 1 \geq M$, then $\cos r_{2k-1} > \delta$ so that

$$(3.47) \quad \cos r_{2k+1} + \alpha > \frac{1 + 3|\alpha|}{2} > \frac{1 - |\alpha|}{2}.$$

Let J be the smallest positive integer for which

$$(3.48) \quad \frac{|A|}{(J\pi - \pi/2)} + \frac{|C| + |D|}{(J\pi - \pi/2)^2} < \frac{1 - |\alpha|}{2}$$

It follows that if

$$(3.49) \quad j > N := \max(M, J), \quad \text{then} \quad (-1)^j F(r_j) > 0.$$

If $-1 < \alpha < 0$, the analysis is similar with the sensitive inequality being for odd indices rather than the even ones.

In this analysis, we established the following general asymptotic stability test.

Theorem 3.11 (Algorithmic Stability Test II, General Test) *Assume that $A + B + C > 0$ and $C + D > 0$, and that $-1 < \alpha < 0$ or $0 < \alpha < 1$. Let M be as in Lemma 3.7, and assume that G has $M + 1$ zeros in $(0, M\pi + \pi/2)$. Then the zero solution of (1.1) is asymptotically stable if and only if*

$$(3.50) \quad (-1)^j F(r_j) > 0, \quad j = 1, 2, \dots, N$$

where N defined in (3.50).

4. EXAMPLES

Example 4.1 Consider (1.1) with $A = 0$, $\alpha = 0.5$, $D = \alpha C$, $C > 0$ i.e.

$$(4.1) \quad y''(t) + \alpha y''(t - \tau) + by'(t - \tau) + cy(t) + \alpha cy(t - \tau) = 0.$$

Recall $B = b\tau$ and $C = c\tau^2$. We apply Theorem 3.10 and Theorem 3.3 (iii). In this example $\rho_1 = \arccos(-0.5) = 2.094395102$. In the figure below the portion of the asymptotic stability region for $0 < C < \rho_4^2$ is shown. Specifically, the asymptotic stability region is the union of the interiors of the derived triangles in the right half plane in (C, β) -space. See the bounds in (3.27) and (3.28) for the slopes of the boundary lines. We also note that in Theorem 3.10 the inequalities $\beta < \rho_1$, $\beta < 2\rho_2$, and $\beta > -2(\pi - \rho_1)$ are redundant upon other inequalities in their respective cases. We include them because they reveal the upper and lower vertices of the triangles in the asymptotic stability region. Figure 4 is typical for all α with $-1 < \alpha < 0$ or $0 < \alpha < 1$. The region is shown in (C, β) -space, the upper vertices lie on the line $\beta = 2\rho_1 = 2\arccos(-\alpha)$ (except the first one which lie on $\beta = \rho_1$) and the lower vertices lie on the line $y = -2(\pi - \rho_1) = -2(\pi - \arccos(-\alpha))$. As $\alpha \rightarrow 1$, the lower triangles narrow away (in (C, β) -space), and as $\alpha \rightarrow -1$, the upper triangles narrow away. Since $B = \beta\sqrt{1 - \alpha^2}$, all of the triangles narrow away in (C, B) -space as $\alpha \rightarrow \pm 1$.

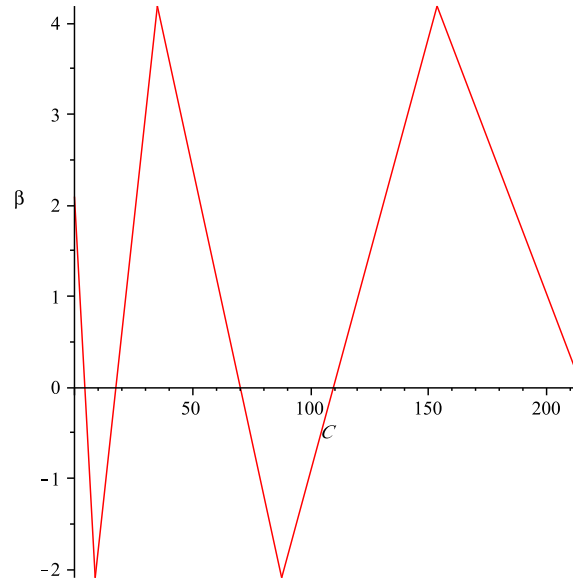


Figure 4- Region of asymptotic stability for $\alpha = 0.5$ in the (C, β) -plane.

Example 4.2 Consider (1.1) with $A = 0, B = 1, C = 0.5, D$ unspecified and $\alpha = 0.6$, i.e.,

$$(4.2) \quad y''(t) + \alpha y''(t - \tau) + by'(t - \tau) + cy(t) + dy(t - \tau) = 0$$

where

$$(4.3) \quad B = b\tau, \quad C = c\tau^2, \quad D = d\tau^2.$$

The zeros of $G(y)$ are independent of D . Table I gives values of the first four positive zeros of G and the values of the function F at these zeros.

Table I	
$r_1 = 1.38054$	$F(r_1) = -1.409410313 + D$
$r_2 = 2.73984$	$F(r_2) = 1.944804 + D$
$r_3 = 6.44098$	$F(r_3) = -65.36888 + D$
$r_4 = 9.3166$	$F(r_4) = 33.715318 + D$

In this example, we use Algorithmic Stability Test I. Here $-r_3^2(\cos r_3 + \alpha) + C + D = -65.3627 + D$ and $-r_2^2(\cos r_2 + \alpha) - C + D = 1.9050 + D$. We have $M_1 = 1, M_2 = 1, N_1 = 1, N_2 = 1$. The zero solution of (1.1) is asymptotically stable for all D where $-1.90499 < D < 1.409$. Note that $L_1 = 1$ and $L_2 = 1$ for these values of D .

Example 4.3 Consider (1.1) with $A = 3, B = 1, C = 0.5, D = 5.3$ and $\alpha = 0.7$, i.e.

$$(4.4) \quad y''(t) + \alpha y''(t - \tau) + ay'(t) + by'(t - \tau) + cy(t) + dy(t - \tau) = 0$$

where

$$(4.5) \quad A = a\tau, \quad B = b\tau, \quad C = c\tau^2, \quad D = d\tau^2.$$

In this example we will use the Algorithmic Stability Test II and Lemma 3.7. Using direct calculations we found that $M_1 = 1$, $M_2 = 3$, $M_3 = 2$, and $M_4 = 3$, and by Lemma 3.7, $M = 3$ and the value of J defined in (3.45) is 8. By (3.46), $N = 8$. In the Table II below we present r_j and the value of $F(r_j)$ for $j = 1, \dots, 8$.

Table II	
$r_1 = 1.511303125$	$F(r_1) = -0.930789465$
$r_2 = 3.632129007$	$F(r_2) = 12.39430676$
$r_3 = 6.836264110$	$F(r_3) = -77.52907973$
$r_4 = 9.628546078$	$F(r_4) = 36.55028535$
$r_5 = 12.87196732$	$F(r_5) = -279.8333553$
$r_6 = 15.8333687$	$F(r_6) = 83.98506486$
$r_7 = 19.05782871$	$F(r_7) = -615.6250571$
$r_8 = 22.08138467$	$F(r_8) = 155.0640468$

The function G has four zeros in $(0, 3\pi + \pi/2)$ and interlacing holds for $j = 1, \dots, 8$ and by Algorithmic Stability Test II the zero solution of (4.4) is asymptotically stable.

For the nonneutral case, i.e. $\alpha = 0$ we found that $F(r_1) = 0.668036530 > 0$. and interlacing fails. By Theorem 2.2 the zero solution is not asymptotically stable. Typically introducing delays and neutral terms has an unstabilizing effect. In nonneutral case when the order is 2 or higher, we have found rare cases when the delay has a stabilizing effect. This example provides a case where the addition of a neutral term has a stabilizing effect.

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