HOMOTOPY EXTENSION TYPE MAPS AND FIXED POINTS

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ABSTRACT. Some new continuation theorems are presented for admissible (and more general) maps using the idea of extendability.

1. INTRODUCTION

In this paper we use the notion of extendability to establish new continuation theorems for admissible (and more general) maps. The ideas used are elementary, in particular the notion of homotopy and Urysohn functions. An added bonus is that the results hold for maps between Hausdorff topological spaces (i.e. the spaces need not be vector spaces). Our theory was motivated by the books [2, 3] and the references therein.

Now we introduce the maps we will discuss in Section 2. Let X and Z be subsets of Hausdorff topological spaces. We will consider maps $F : X \to K(Z)$; here $K(Z)$ denotes the family of nonempty compact subsets of Z. A nonempty topological space is said to be acyclic if all its reduced Čech homology groups over the rationals are trivial. Now $F : X \to K(Z)$ is acyclic (and we write $F \in AC(X, Z)$) if F is upper semicontinuous with acyclic values.

Let X, Y and Γ be Hausdorff topological spaces. A continuous single valued map $p : \Gamma \to X$ is called a Vietoris map (written $p : \Gamma \Rightarrow X$) if the following two conditions are satisfied:

(i). for each $x \in X$, the set $p^{-1}(x)$ is acyclic

(ii). p is a proper map i.e. for every compact $A \subseteq X$ we have that $p^{-1}(A)$ is compact.

Let $\phi : X \to Y$ be a multivalued map (note for each $x \in X$ we assume $\phi(x)$ is a nonempty subset of Y). A pair (p, q) of single valued continuous maps of the form $X \xleftarrow{p} \Gamma \xrightarrow{q} Y$ is called a selected pair of ϕ (written $(p, q) \subset \phi$) if the following two conditions hold:

(i). p is a Vietoris map

and

(ii). $q(p^{-1}(x)) \subset \phi(x)$ for any $x \in X$.

Now we define the admissible maps [2].

Definition 1.1. A upper semicontinuous map $\phi : X \to Y$ with closed values is said to be admissible (and we write $\phi \in Ad(X, Y)$) provided there exists a selected pair (p, q) of ϕ.

Now we define the permissible maps [2]. Let X and Y be Hausdorff topological spaces.

Definition 1.2. A multivalued map $F : X \to K(Y)$ is in the class $\mathcal{A}_m(X, Y)$ if (i). F is continuous, and (ii). for each $x \in X$ the set $F(x)$ consists of one or m acyclic components; here m is a positive integer. We say F is of class $\mathcal{A}_0(X, Y)$ if F is upper semicontinuous and for each $x \in X$ the set $F(x)$ is acyclic.

Definition 1.3. A decomposition $(F_1, \ldots, F_n)$ of a multivalued map $F : X \to 2^Y$ is a sequence of maps

$$X = X_0 \xrightarrow{F_1} X_1 \xrightarrow{F_2} X_2 \xrightarrow{F_3} \cdots \xrightarrow{F_{n-1}} X_{n-1} \xrightarrow{F_n} X_n = Y,$$

where $F_i \in \mathcal{A}_{m_i}(X_{i-1}, X_i)$, $F = F_n \circ \cdots \circ F_1$. One can say that the map F is determined by the decomposition $(F_1, \ldots, F_n)$. The number n is said to be the length of the decomposition $(F_1, \ldots, F_n)$. We will denote the class of decompositions by $\mathcal{D}(X, Y)$.

Definition 1.4. An upper semicontinuous map $F : X \to K(Y)$ is permissible provided it admits a selector $G : X \to K(Y)$ which is determined by a decomposition $(G_1, \ldots, G_n) \in \mathcal{D}(X, Y)$. We denote the class of permissible maps from X into Y by $\mathcal{P}(X, Y)$.

Now we define the maps of Park which include the above maps. Let X and Y be Hausdorff topological spaces. Given a class $\mathcal{X}$ of maps, $\mathcal{X}(X, Y)$ denotes the set of maps $F : X \to 2^Y$ (nonempty subsets of Y) belonging to $\mathcal{X}$, and $\mathcal{X}_c$ the set of finite compositions of maps in $\mathcal{X}$. We let

$$\mathcal{F}(\mathcal{X}) = \{Z : Fix F \neq \emptyset \text{ for all } F \in \mathcal{X}(Z, Z)\}$$

where $Fix F$ denotes the set of fixed points of F.

The class $\mathcal{U}$ of maps is defined by the following properties:

(i). $\mathcal{U}$ contains the class $\mathbf{C}$ of single valued continuous functions;

(ii). each $F \in \mathcal{U}_c$ is upper semicontinuous and compact valued; and

(iii). $B^n \in \mathcal{F}(\mathcal{U}_c)$ for all $n \in \{1, 2, \ldots\}$; here $B^n = \{x \in \mathbf{R}^n : \|x\| \leq 1\}$.

We say $F \in \mathcal{U}_c^k(X, Y)$ if for any compact subset K of X there is a $G \in \mathcal{U}_c(K, Y)$ with $G(x) \subseteq F(x)$ for each $x \in K$. Recall $\mathcal{U}_c^k$ is closed under compositions.

2. **CONTINUATION THEORY**

Throughout this section Y will be a completely regular topological space and U will be an open subset of Y. In applications we are usually interested in maps $F : \overline{U} \to Y$ (here $\overline{U}$ denotes the closure of U in Y) and in conditions which guarantee a fixed point in U. In our first (quite abstract) result assuming that we have a homotopy extension type property (i.e. a $H : Y \times [0,1] \to K(Y)$ with $H_1|_{\overline{U}} = F$) we will show that H_1 (so consequently F) has a fixed point in U.

Definition 2.1. We say $F \in cAd(Y,Y)$ if $F \in Ad(Y,Y)$ is a compact map.

Definition 2.2. If $F \in cAd(Y,Y)$ and $p \in Y$ then we say $F \cong \{p\}$ in $cAd(Y,Y)$ if there exists an upper semicontinuous compact map $\Omega : Y \times [0,1] \to K(Y)$ with $\Omega \in cAd(Y \times [0,1], Y)$, $\Omega_1 = F$ and $\Omega_0 = \{p\}$ (here $\Omega_t(x) = \Omega(x,t)$).

Theorem 2.1. *Let Y be a completely regular topological space, U an open subset of Y and $u_0 \in U$. Suppose there exists an upper semicontinuous compact map $H : Y \times [0,1] \to K(Y)$ with $H \in cAd(Y \times [0,1], Y)$, $H(x,0) = \{u_0\}$ for each $x \in Y$ and $x \notin H_t(x)$ for $x \in \partial U$ and $t \in (0,1]$. In addition assume the following property holds:*

(2.1) $\qquad \begin{cases} \text{for any } \Phi \in cAd(Y,Y) \text{ and any } p \in Y \text{ with } \Phi \cong \{p\} \\ \text{in } cAd(Y,Y) \text{ we have that } \Phi \text{ has a fixed point in } Y. \end{cases}$

Then H_1 has a fixed point in U.

PROOF: Let

$$B = \{x \in Y \backslash U : x \in H_t(x) \text{ for some } t \in [0,1]\}.$$

We consider two cases, namely $B \neq \emptyset$ and $B = \emptyset$.

Case (i). $B = \emptyset$.

Then for every $t \in [0,1]$ we have $x \notin H_t(x)$ for $x \in Y \backslash U$. Also from $H_1 \cong \{u_0\}$ in $cAd(Y,Y)$ and (2.1) we know there exists $y \in Y$ with $y \in H_1(y)$. Since $x \notin H_1(x)$ for $x \in Y \backslash U$ we deduce that $y \in U$.

Case (ii). $B \neq \emptyset$.

Now B is closed and compact and also note $B \cap \overline{U} \neq \emptyset$ (recall $x \notin H_t(x)$ for $x \in \partial U$ and $t \in [0,1]$). Then there exists a continuous map $\mu : Y \to [0,1]$ with $\mu(B) = 0$ and $\mu(\overline{U}) = 1$. Define a map $R : Y \to K(Y)$ by

$$R(x) = H(x, \mu(x)) = H \circ \eta(x)$$

where $\eta : Y \to Y \times [0,1]$ is given by $\eta(x) = (x, \mu(x))$. Since R is the composition of two admissible maps we have that $R \in Ad(Y,Y)$ is a compact map i.e. $R \in cAd(Y,Y)$. In fact $R \cong \{u_0\}$ in $cAd(Y,Y)$. To see this let $\Omega : Y \times [0,1] \to K(Y)$ be given by

$$\Omega(x,t) = H(x, t\mu(x)) = H \circ \tau(x,t)$$

where $\tau : Y \times [0,1] \to Y \times [0,1]$ is given by $\tau(x,t) = (x, t\mu(x))$. Note $\Omega \in cAd(Y \times [0,1], Y)$, $\Omega_1 = R$ and $\Omega_0 = \{u_0\}$ (note $H(x,0) = \{u_0\}$ for each $x \in Y$).

Now (2.1) guarantees that there exists $x \in Y$ with $x \in R(x) = H_{\mu(x)}(x)$. If $x \in Y \backslash U$ then since $x \in B$ we have $x \in H(x, \mu(x)) = H(x,0) = \{u_0\}$ which is a contradiction since $u_0 \in U$. Thus $x \in U$ and so $x \in R(x) = H(x, \mu(x)) = H(x, 1)$. $\square$

Remark 2.1. In (2.1) we can replace $p \in Y$ with $p \in U$ or indeed just p equals u_0.

Remark 2.2. We can replace the compactness of the maps H, Ω and in cAd with any assumption on the appropriate maps that guarantee that B is compact (see [4]).

Corollary 2.1. *Let Y be a completely regular topological space, U an open subset of Y and $u_0 \in U$. Suppose there exists a retraction (continuous) $r : Y \to \overline{U}$ and assume there exists an upper semicontinuous compact map $\Lambda : \overline{U} \times [0,1] \to K(Y)$ with $\Lambda \in cAd(\overline{U} \times [0,1], Y)$, $\Lambda(x,0) = \{u_0\}$ for each $x \in \overline{U}$ and $x \notin \Lambda_t(x)$ for $x \in \partial U$ and $t \in (0,1]$. In addition assume (2.1) holds. Then Λ_1 has a fixed point in U.*

PROOF: Let

$$H(x,t) = \Lambda(r(x), t) = \Lambda \circ \xi(x,t)$$

where $\xi : Y \times [0,1] \to \overline{U} \times [0,1]$ is given by $\xi(x,t) = (r(x), t)$. Notice $H : Y \times [0,1] \to K(Y)$ is a upper semicontinuous compact map, $H \in cAd(Y \times [0,1], Y)$, $H(x,0) = \Lambda(r(x), 0) = \{u_0\}$ for $x \in Y$ (note $r(x) \in \overline{U}$ for $x \in Y$). Also note $x \notin H_t(x)$ for $x \in \partial U$ and $t \in (0,1]$. To see this note if there exists $x \in \partial U$ and $t \in (0,1]$ with $x \in H_t(x)$ then $x \in \Lambda_t(r(x)) = \Lambda_t(x)$ (note $r(x) = x$ for $x \in \overline{U}$), a contradiction. Thus H satisfies the conditions in the statement of Theorem 2.1. As a result H_1 (so consequently Λ_1, note $r(x) = x$ for $x \in \overline{U}$) has a fixed point in U. $\square$

Remark 2.3. Let Y, U, u_0 and Λ be as in Corollary 2.1 and suppose (2.1) holds. To deduce that Λ_1 has a fixed point in U the idea is to apply Theorem 2.1 so we must construct an upper semicontinuous compact map $H : Y \times [0,1] \to K(Y)$ with $H \in cAd(Y \times [0,1], Y)$, $H(x,0) = \{u_0\}$ for each $x \in Y$ and $x \notin H_t(x)$ for $x \in \partial U$ and $t \in (0,1]$ and such that $H_1(x) = \Lambda_1(x)$ if $x \in U$ and $x \in H_1(x)$. If a retraction $r : Y \to \overline{U}$ exists then it is easy to construct the H as Corollary 2.1 shows. It is also possible to construct a H in certain situations when a retraction described above does not exist. As an example consider a subclass of the Ad maps, namely the acyclic (AC) maps. We say a map $F \in cAC(Y, Y)$ if $F \in AC(Y, Y)$ is a compact map. Let Y, U, u_0, Λ and (2.1) be as above with Ad replaced by AC and cAd replaced by cAC. Now let

$$D = \left\{ x \in \overline{U} : x \in \Lambda_t(x) \text{ for some } t \in [0,1] \right\}.$$

Note $D \neq \emptyset$ (since $u_0 \in U$) is closed and compact and $D \cap (Y \backslash U) = \emptyset$. Thus there exists a continuous map $\sigma : Y \to [0,1]$ with $\sigma(D) = 1$ and $\sigma(Y \backslash U) = 0$. Define

$H : Y \times [0, 1] \to K(Y)$ by

$$H(x, t) = \begin{cases} \Lambda(x, t\sigma(x)), x \in \overline{U} \\ \{u_0\}, x \in Y \backslash U. \end{cases}$$

Clearly $H : Y \times [0, 1] \to K(Y)$ is an upper semicontinuous compact map with $H \in cAC(Y \times [0, 1], Y)$, $H(x, 0) = \{u_0\}$ for each $x \in Y$ and $x \notin H_t(x)$ for $x \in \partial U$ and $t \in (0, 1]$ (to see this note if there exists an $x \in \partial U$ and $t \in (0, 1]$ with $x \in H_t(x)$ then $x \in \Lambda(x, t\sigma(x)) = \Lambda_{t\sigma(x)}(x)$ so $x \in D$ which implies $\sigma(x) = 1$ and so $x \in \Lambda_t(x)$, a contradiction). Thus H satisfies the statement of Theorem 2.1 and since (2.1) holds then H_1 has a fixed point $x \in U$. Thus $x \in H_1(x) = \Lambda_{\sigma(x)}(x)$ so $x \in D$ which implies $\sigma(x) = 1$ and so $x \in \Lambda_1(x)$.

Remark 2.4. By a space we mean a Hausdorff topological space. Let Q be a class of topological spaces. A space Y is an extension space for Q (written $Y \in ES(Q)$) if $\forall X \in Q$, $\forall K \subseteq X$ closed in X, any continuous function $f_0 : K \to Y$ extends to a continuous function $f : X \to Y$. Recall [1, 5] if $X \in ES(\text{compact})$ and $F \in \mathcal{U}_c^\kappa(X, X)$ a compact map, then F has a fixed point. Consequently we have the following result: if $Y \in ES(\text{compact})$ then any map $\Phi \in cAd(Y, Y)$ has a fixed point so trivially (2.1) holds (notice the condition that $\Phi \cong \{p\}$ in $cAd(Y, Y)$ in (2.1) does not play any role in this example). Condition (2.1) was discussed in [6] and we refer the reader to that paper.

Remark 2.5. Let Y be a metric space. By the Arens–Eells theorem we know we can embed Y can as a closed subset (again we call it Y) of a normed space E (recall the Arens–Eells theorem states that any metric space can be isometrically embedded as a closed subset in a normed linear space). Now if we assume

$$\begin{cases} \text{for any } \Phi \in cAd(Y, Y) \text{ and any } p \in Y \text{ with } \Phi \cong \{p\} \\ \text{in } cAd(Y, Y) \text{ there exists a } \Phi^* \in cAd(E, E) \text{ with a} \\ \text{fixed point, and suppose also any fixed point of } \Phi^* \\ \text{is a fixed point of } \Phi, \end{cases}$$

then (2.1) holds.

Definition 2.3. We say $F \in cP(Y, Y)$ if $F \in P(Y, Y)$ is a compact map.

Definition 2.4. If $F \in cP(Y, Y)$ and $p \in Y$ then we say $F \cong \{p\}$ in $cP(Y, Y)$ if there exists an upper semicontinuous compact map $\Omega : Y \times [0, 1] \to K(Y)$ with $\Omega \in cP(Y \times [0, 1], Y)$, $\Omega_1 = F$ and $\Omega_0 = \{p\}$ (here $\Omega_t(x) = \Omega(x, t)$).

The next two results follow the same argument above.

Theorem 2.2. *Let Y be a completely regular topological space, U an open subset of Y and $u_0 \in U$. Suppose there exists an upper semicontinuous compact map $H :$ $Y \times [0, 1] \to K(Y)$ with $H \in cP(Y \times [0, 1], Y)$, $H(x, 0) = \{u_0\}$ for each $x \in Y$ and*

$x \notin H_t(x)$ for $x \in \partial U$ and $t \in (0,1]$. In addition assume the following property holds:

$$(2.2) \quad \begin{cases} \text{for any } \Phi \in c\mathcal{P}(Y,Y) \text{ and any } p \in Y \text{ with } \Phi \cong \{p\} \\ \text{in } c\mathcal{P}(Y,Y) \text{ we have that } \Phi \text{ has a fixed point in } Y. \end{cases}$$

Then H_1 has a fixed point in U.

Corollary 2.2. Let Y be a completely regular topological space, U an open subset of Y and $u_0 \in U$. Suppose there exists a retraction (continuous) $r : Y \to \overline{U}$ and assume there exists an upper semicontinuous compact map $\Lambda : \overline{U} \times [0,1] \to K(Y)$ with $\Lambda \in c\mathcal{P}(\overline{U} \times [0,1], Y)$, $\Lambda(x,0) = \{u_0\}$ for each $x \in \overline{U}$ and $x \notin \Lambda_t(x)$ for $x \in \partial U$ and $t \in (0,1]$. In addition assume (2.2) holds. Then Λ_1 has a fixed point in U.

Remark 2.6. There is a similar remark as in Remark 2.2 for the $c\mathcal{P}$ maps.

Definition 2.5. We say $F \in c\mathcal{U}_c^k(Y,Y)$ if $F \in \mathcal{U}_c^k(Y,Y)$ is a compact map.

Definition 2.6. If $F \in c\mathcal{U}_c^k(Y,Y)$ and $p \in Y$ then we say $F \cong \{p\}$ in $c\mathcal{U}_c^k(Y,Y)$ if there exists an upper semicontinuous compact map $\Omega : Y \times [0,1] \to K(Y)$ with $\Omega \in c\mathcal{U}_c^k(Y \times [0,1], Y)$, $\Omega_1 = F$ and $\Omega_0 = \{p\}$ (here $\Omega_t(x) = \Omega(x,t)$).

The next two results follow the same argument above.

Theorem 2.3. Let Y be a completely regular topological space, U an open subset of Y and $u_0 \in U$. Suppose there exists an upper semicontinuous compact map $H : Y \times [0,1] \to K(Y)$ with $H \in c\mathcal{U}_c^k(Y \times [0,1], Y)$, $H(x,0) = \{u_0\}$ for each $x \in Y$ and $x \notin H_t(x)$ for $x \in \partial U$ and $t \in (0,1]$. In addition assume the following property holds:

$$(2.3) \quad \begin{cases} \text{for any } \Phi \in c\mathcal{U}_c^k(Y,Y) \text{ and any } p \in Y \text{ with } \Phi \cong \{p\} \\ \text{in } c\mathcal{U}_c^k(Y,Y) \text{ we have that } \Phi \text{ has a fixed point in } Y. \end{cases}$$

Then H_1 has a fixed point in U.

Corollary 2.3. Let Y be a completely regular topological space, U an open subset of Y and $u_0 \in U$. Suppose there exists a retraction (continuous) $r : Y \to \overline{U}$ and assume there exists an upper semicontinuous compact map $\Lambda : \overline{U} \times [0,1] \to K(Y)$ with $\Lambda \in c\mathcal{U}_c^k(\overline{U} \times [0,1], Y)$, $\Lambda(x,0) = \{u_0\}$ for each $x \in \overline{U}$ and $x \notin \Lambda_t(x)$ for $x \in \partial U$ and $t \in (0,1]$. In addition assume (2.3) holds. Then Λ_1 has a fixed point in U.

Remark 2.7. There is a similar remark as in Remark 2.2 for the $c\mathcal{U}_c^k$ maps.

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