A FITE TYPE RESULT FOR SEQUENTIAL FRACTIONAL DIFFERENTIAL EQUATIONS

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ABSTRACT. Given the solution f of the sequential fractional differential equation $_aD_t^\alpha(_aD_t^\alpha f) + P(t)f = 0$, $t \in [b,c]$, where $-\infty < a < b < c < +\infty$, $\alpha \in \left(\frac{1}{2}, 1\right)$ and $P : [a, +\infty) \to [0, P_\infty]$, $P_\infty < +\infty$, is continuous. Assume that there exist $t_1, t_2 \in [b,c]$ such that $f(t_1) = (_aD_t^\alpha f)(t_2) = 0$. Then, we establish here a positive lower bound for $c - a$ which depends solely on α, P_∞. Such a result might be useful in discussing disconjugate fractional differential equations and fractional interpolation, similarly to the case of (integer order) ordinary differential equations.

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1. INTRODUCTION

The fractional calculus and its plethora of techniques for the study of fractional mathematical models in natural and social disciplines constitute already a vast part of modern science [9, 3, 8, 5, 6]. The oscillation/disconjugacy theory of the fractional differential equations, however, is still at the very beginning [5, 6, 3]. To give an example, there are no fractional counterparts of the fundamental comparison and separation theorems of Sturm ([10, 7]).

Complications such as the Faà di Bruno formula for the fractional derivative of composite functions or the Leibniz series of the fractional derivative of a product [6, pp. 96, 98] make such enterprises extremely difficult.

In 1918, the following striking result of W. B. Fite [2, Theorem VII] has opened the way for the disconjugacy analysis [1] of (integer order) linear ordinary differential

equations: given the real numbers $a < b < c$ and the differential equation

$$x'' + P(t)x = 0, \quad t \in [b, c],$$

where $P : [a, +\infty) \to [0, P_\infty]$, $P_\infty < +\infty$, is a continuous coefficient. If $x(t_1) = x'(t_2) = 0$ for some (non-trivial) solution x. Then,

$$(c - b) \cdot \max\{1, P_\infty\} \geq 1.$$

In this article, we shall establish a variant of Fite's theorem for the case of linear sequential fractional (ordinary) differential equations (SFDE's). In our result, the estimate of $c - b$ is replaced with an estimate of $c - a$.

To set the hypotheses, fix $\zeta \in (0, 1)$. Introduce the Riemann-Liouville (fractional) derivative [6, p. 68]

$$({}_aD_t^\zeta f)(t) = \frac{1}{\Gamma(1 - \zeta)} \cdot \frac{d}{dt} \left[\int_a^t \frac{f(s)}{(t - s)^\zeta} ds \right], \quad t > a,$$

and the Riemann-Liouville (fractional) integral [6, p. 65] (sometimes denoted with ${}_aI_t^{1-\zeta}$, see [3])

$$({}_aD_t^{\zeta-1} f)(t) = \frac{1}{\Gamma(1 - \zeta)} \cdot \int_a^t \frac{f(s)}{(t - s)^\zeta} ds, \quad t > a.$$

Obviously, ${}_aD_t^0 f = f$ (a limit case) and ${}_aD_t^\zeta f = \frac{d}{dt} \left({}_aD_t^{\zeta-1} f \right)$. We shall discuss in the next section the regularity assumptions regarding f that are needed in our investigation.

Consider the following linear sequential differential equation (SFDE)

(1.1) $$\qquad {}_aD_t^\alpha({}_aD_t^\alpha f) + P(t)f = 0, \quad t \in [b, c],$$

where $\alpha \in \left(\frac{1}{2}, 1 \right)$ and the coefficient $P : [a, +\infty) \to [0, P_\infty]$, $P_\infty < +\infty$, is assumed continuous. The equation (1.1) is equivalent to the next linear SFDE system

(1.2) $$\qquad \begin{cases} {}_aD_t^\alpha f = g, \\ {}_aD_t^\alpha g = -P(t)f, \end{cases} \quad t \in [b, c],$$

which is a particular case of the linear SFDE system

(1.3) $$\qquad \begin{cases} ({}_aD_t^\alpha f)(t) = G(t)g(t) + Q(t), \\ ({}_aD_t^\alpha g)(t) = R(t)f(t) + V(t), \end{cases} \quad t \in [b, c],$$

where the coefficients $G, Q, R, V : [a, +\infty) \to \mathbb{R}$ are continuous and G, R are bounded: $\|G\|_\infty + \|R\|_\infty < +\infty$.

Assume that, for the solutions (f_i, g_i), with $i = 1, 2$, of the system (1.3), there exist points $t_i \in [b, c]$ such that

(1.4) $$\qquad f_1(t_1) = f_2(t_1), \quad g_1(t_2) = g_2(t_2).$$

Our main contribution in the present article, is to establish that there exists a positive lower bound for $c - a$. It is clear that, in the particular case of (1.2), by taking

(1.5) $$(f_1, g_1) = (f, {}_aD_t^\alpha f) \quad \text{and} \quad (f_2, g_2) = (0, 0),$$

where f is a non-trivial solution of (1.1), we obtain the fractional version of Fite's theorem.

2. PRELIMINARIES

To make the presentation self-contained, we shall discuss here the regularity of the solutions (f, g) and the transformation of (1.3) into an integral system.

2.1. **Regularity of solutions.** For $\gamma \in (0, 1)$, consider the integral quantity

$$(Q_{\gamma, A} f)(t) = \int_a^t \frac{A(s) f(s)}{(t - s)^\gamma} ds, \quad t > a, \quad \gamma \in (0, 1)$$

where $A : [a, +\infty) \to \mathbb{R}$, and $f : (a, +\infty) \to \mathbb{R}$ are continuous and f satisfies the restriction

(2.1) $$\lim_{t \searrow a}(t - a)^\gamma f(t) = f_a \in \mathbb{R}..$$

Observe that f may be infinite at $t = a$.

In the following, for such functions f, we shall use the notations

$$\|f\|_{L^\infty(a, \gamma; b, c)} = \max_{t \in [b, c]}[(t - a)^\gamma |f(t)|]$$

and

$$\|f\|_{L^\infty(a, \gamma; c)} = \sup_{t \in (a, c]}[(t - a)^\gamma |f(t)|]$$

for all $c > b > a$.

Obviously, $\|f\|_{L^\infty(a, \gamma; b, c)} \le \|f\|_{L^\infty(a, \gamma; c)}$. It is easy to see that

(2.2) $$|f_a| \le \sup_{b \in (a, c)} \|f\|_{L^\infty(a, \gamma; b, c)} = \|f\|_{L^\infty(a, \gamma; c)}.$$

Notice that for $a < t \le c$ and $\beta \in (0, 1)$

$$\int_a^t \frac{|A(s) f(s)|}{(t - s)^\beta} ds = \int_a^t \frac{|A(s)| \cdot (s - a)^\gamma |f(s)|}{(t - s)^\beta (s - a)^\gamma} ds$$

$$\le \int_a^t \frac{\|A\|_{L^\infty(a, c)} \|f\|_{L^\infty(a, \gamma; c)}}{(t - s)^\beta (s - a)^\gamma} ds$$

$$= (t - a)^{1 - \beta - \gamma} B(1 - \gamma, 1 - \beta)$$

(2.3) $$\times \|A\|_{L^\infty(a, c)} \|f\|_{L^\infty(a, \gamma; c)}.$$

We have used the change of variables $s = a + \lambda(t - a)$ for $\lambda \in [0, 1]$ and the *Beta* function B, see [6, p. 6]. Remark also that

$$\lim_{t \searrow a}(t - a)^\gamma (Q_{\beta,A}f)(t) = \lim_{t \searrow a}(t - a)^\beta (Q_{\beta,A}f)(t) = 0.$$

In particular, we have the estimate

$$|(Q_{1-\gamma,A}f)(t)| \leq \int_a^t \frac{|A(s)f(s)|}{(t - s)^{1-\gamma}}ds$$

$$\leq B(1 - \gamma, \gamma) \cdot \|A\|_{L^\infty(a,c)}\|f\|_{L^\infty(a,\gamma;c)}.$$

Let us denote with $C_\gamma((a, c], \mathbb{R})$ the set of functions $f \in C((a, c], \mathbb{R})$ which satisfy (2.1) endowed with the usual function operations. Then, it is easy to check that $X_\gamma = (C_\gamma((a, c], \mathbb{R}), \| \star \|_{L^\infty(a,\gamma;c)})$ is a Banach space.

So, given $f \in X_\gamma$ and $\beta \in (0, 1)$, the function $s \longmapsto \frac{A(s)f(s)}{(t-s)^\beta}$ belongs to $L^1((a, t), \mathbb{R})$ for every $t \in (a, c]$.

The next estimates are crucial for establishing the Fite-type result in this note.

On assuming that $a < b \leq t_1 < t_2 \leq c$ and making the change of variables $s = a + \lambda(t_2 - a)$ for $\lambda \in [0, 1]$, we get

$$\int_{t_1}^{t_2} \frac{ds}{(t_2 - s)^\beta(s - a)^\gamma} = (t_2 - a)^{1-\beta-\gamma} \cdot \int_{\frac{t_1-a}{t_2-a}}^1 \frac{d\lambda}{(1 - \lambda)^\beta \lambda^\gamma}$$

$$= (t_2 - a)^{1-\beta-\gamma} \cdot \int_0^1 \frac{1}{(1 - \lambda)^\beta \lambda^\gamma} \cdot \chi_{\left[\frac{t_1-a}{t_2-a},1\right]}(\lambda)d\lambda$$

$$= (t_2 - a)^{1-\beta-\gamma}$$

$$\times \left[\int_0^{\frac{1}{2}} \frac{1}{(1 - \lambda)^\beta \lambda^\gamma}\chi_{\left[\frac{t_1-a}{t_2-a},1\right]}(\lambda)d\lambda + \int_{\frac{1}{2}}^1 \frac{1}{(1 - \lambda)^\beta \lambda^\gamma}\chi_{\left[\frac{t_1-a}{t_2-a},1\right]}(\lambda)d\lambda\right]$$

(2.4) $= (t_2 - a)^{1-\beta-\gamma} \cdot (I_1 + I_2),$

where χ_S denotes the characteristic function of a Lebesgue integrable subset S of $\mathbb{R}$. For simplicity, we shall write χ instead of χ_S from now on.

Introduce $p, q, v, w > 1$ such that $\frac{1}{p} + \frac{1}{q} = \frac{1}{v} + \frac{1}{w} = 1$ and $\gamma p, \beta v < 1$.

In the following, we shall estimate the integrals I_1, I_2 by means of Hölder inequality:

$$I_1 \leq \int_0^{\frac{1}{2}} \frac{1}{\left(\frac{1}{2}\right)^\beta \lambda^\gamma}\chi(\lambda)d\lambda = 2^\beta \int_0^{\frac{1}{2}} \left(\frac{1}{\lambda}\right)^\gamma \chi(\lambda)d\lambda$$

$$\leq 2^\beta \left[\int_0^{\frac{1}{2}} \left(\frac{1}{\lambda}\right)^{\gamma p} d\lambda\right]^{\frac{1}{p}} \cdot \left\{\int_0^{\frac{1}{2}} [\chi(\lambda)]^q d\lambda\right\}^{\frac{1}{q}}$$

$$= 2^\beta \cdot \frac{1}{(1 - \gamma p)^{\frac{1}{p}} \cdot 2^{\frac{1}{p}-\gamma}} \cdot \left\{\int_0^{\frac{1}{2}} [\chi(\lambda)]d\lambda\right\}^{\frac{1}{q}}$$

$$= \frac{2^{\beta+\gamma-\frac{1}{p}}}{(1-\gamma p)^{\frac{1}{p}}} \cdot \left\{ \int_0^{\frac{1}{2}} [\chi(\lambda)] d\lambda \right\}^{\frac{1}{q}} \leq \frac{2^{\beta+\gamma-\frac{1}{p}}}{(1-\gamma p)^{\frac{1}{p}}} \cdot \left\{ \int_0^1 [\chi(\lambda)] d\lambda \right\}^{\frac{1}{q}}$$

$$(2.5) \qquad = \frac{2^{\beta+\gamma-\frac{1}{p}}}{(1-\gamma p)^{\frac{1}{p}}} \cdot \left(1 - \frac{t_1-a}{t_2-a} \right)^{\frac{1}{q}} \leq c(p,\beta,\gamma) \cdot \left(\frac{t_2-t_1}{t_2-a} \right)^{\frac{1}{q}}$$

and

$$I_2 \leq 2^\gamma \int_{\frac{1}{2}}^1 \left(\frac{1}{1-\lambda} \right)^\beta \chi(\lambda) d\lambda$$

$$\leq 2^\gamma \left[\int_{\frac{1}{2}}^1 \frac{d\lambda}{(1-\lambda)^{\beta v}} \right]^{\frac{1}{v}} \cdot \left\{ \int_0^1 [\chi(\lambda)]^w d\lambda \right\}^{\frac{1}{w}}$$

$$= \frac{2^\gamma}{(1-\beta v)^{\frac{1}{v}} \cdot 2^{\frac{1}{v}-\beta}} \cdot \left(\frac{t_2-t_1}{t_2-a} \right)^{\frac{1}{w}}$$

$$(2.6) \qquad = c(v,\gamma,\beta) \cdot \left(\frac{t_2-t_1}{t_2-a} \right)^{\frac{1}{w}}.$$

By taking into account (2.5), (2.6) and (2.4), we deduce that

$$\int_{t_1}^{t_2} \frac{ds}{(t_2-s)^\beta (s-a)^\gamma} \leq (t_2-a)^{1-\beta-\gamma} [c(p,\beta,\gamma) + c(v,\gamma,\beta)]$$

$$\times \max \left\{ \left(\frac{t_2-t_1}{t_2-a} \right)^{\frac{1}{q}}, \left(\frac{t_2-t_1}{t_2-a} \right)^{\frac{1}{w}} \right\}$$

$$(2.7) \qquad = (t_2-a)^{1-\beta-\gamma} [c(p,\beta,\gamma) + c(v,\gamma,\beta)] \cdot \left(\frac{t_2-t_1}{t_2-a} \right)^{\frac{1}{\max\{q,w\}}}.$$

Next, we would like to estimate the quantity

$$\int_a^{t_1} \frac{1}{(s-a)^\gamma} \cdot \left[\frac{1}{(t_1-s)^\beta} - \frac{1}{(t_2-s)^\beta} \right] ds,$$

where $\beta, \gamma \in (0,1)$ and $\beta + \gamma \leq 1$.

Observing that

$$B(1-\gamma, 1-\beta) = \frac{1}{(t_1-a)^{1-\beta-\gamma}} \cdot \int_a^{t_1} \frac{ds}{(s-a)^\gamma (t_1-s)^\beta}$$

$$= \frac{1}{(t_2-a)^{1-\beta-\gamma}} \cdot \int_a^{t_2} \frac{ds}{(s-a)^\gamma (t_2-s)^\beta},$$

we get that

$$\int_a^{t_1} \frac{1}{(s-a)^\gamma} \cdot \left[\frac{1}{(t_1-s)^\beta} - \frac{1}{(t_2-s)^\beta} \right] ds$$

$$= \int_a^{t_1} \frac{ds}{(s-a)^\gamma (t_1-s)^\beta} - \int_a^{t_2} \frac{ds}{(s-a)^\gamma (t_2-s)^\beta}$$

$$+ \int_{t_1}^{t_2} \frac{ds}{(t_2-s)^\beta (s-a)^\gamma} = I_3 + I_4.$$

The integral I_4 was already evaluated, see (2.7).

Recalling that $1 - \beta - \gamma \geq 0$, we have

$$I_3 = B(1 - \gamma, 1 - \beta) \cdot [(t_1 - a)^{1-\beta-\gamma} - (t_2 - a)^{1-\beta-\gamma}] \leq 0.$$

So,

$$\int_a^{t_1} \frac{1}{(s - a)^\gamma} \cdot \left[\frac{1}{(t_1 - s)^{1-\gamma}} - \frac{1}{(t_2 - s)^{1-\gamma}} \right] ds \leq I_4$$

(2.8)
$$\leq (t_2 - a)^{1-\beta-\gamma}[c(p, \beta, \gamma) + c(v, \gamma, \beta)] \cdot \left(\frac{t_2 - t_1}{t_2 - a} \right)^{\frac{1}{\max\{q,w\}}}.$$

Finally, take $f \in X_\gamma$ and $\gamma, \beta \in (0, 1)$ such that $\beta + \gamma \leq 1$, and observe that

$$|(Q_{\beta,A}f)(t_1) - (Q_{\beta,A}f)(t_2)|$$

$$= \left| \int_a^{t_1} [A(s) \cdot (s - a)^\gamma f(s)] \cdot \frac{1}{(s - a)^\gamma} \left[\frac{1}{(t_1 - s)^\beta} - \frac{1}{(t_2 - s)^\beta} \right] ds \right.$$

$$\left. - \int_{t_1}^{t_2} [A(s) \cdot (s - a)^\gamma f(s)] \cdot \frac{ds}{(t_2 - s)^\beta (s - a)^\gamma} \right|$$

$$\leq \|A\|_{L^\infty(a,c)} \|f\|_{L^\infty(a,\gamma;b,c)} \cdot \int_{t_1}^{t_2} \frac{ds}{(t_2 - s)^\beta (s - a)^\gamma}$$

$$+ \|A\|_{L^\infty(a,c)} \|f\|_{L^\infty(a,\gamma;c)} \int_a^{t_1} \frac{1}{(s - a)^\gamma} \cdot \left[\frac{1}{(t_1 - s)^\beta} - \frac{1}{(t_2 - s)^\beta} \right] ds$$

$$\leq \|A\|_{L^\infty(a,c)} \|f\|_{L^\infty(a,\gamma;c)} \cdot 2(t_2 - a)^{1-\beta-\gamma}[c(p, \beta, \gamma) + c(v, \gamma, \beta)]$$

$$\times \left(\frac{t_2 - t_1}{t_2 - a} \right)^{\frac{1}{\max\{q,w\}}}$$

$$= 2(t_2 - a)^{1-\beta-\gamma-\frac{1}{\max\{q,w\}}} \cdot [c(p, \beta, \gamma) + c(v, \gamma, \beta)]$$

$$\times (t_2 - t_1)^{\frac{1}{\max\{q,w\}}} \cdot \|A\|_{L^\infty(a,c)} \|f\|_{L^\infty(a,\gamma;c)}$$

$$= C \cdot (t_2 - a)^{1-\beta-\gamma-\frac{1}{\max\{q,w\}}}$$

(2.9)
$$\times (t_2 - t_1)^{\frac{1}{\max\{q,w\}}} \cdot \|A\|_{L^\infty(a,c)} \|f\|_{L^\infty(a,\gamma;c)},$$

where $C = C(p, v, \beta, \gamma) = 2[c(p, \beta, \gamma) + c(v, \gamma, \beta)]$.

Before going further, recall the following inequality: given the numbers $A, B \geq 0$ and $\beta \in (0, 1)$, one has

(2.10)
$$(A + B)^\beta \leq A^\beta + B^\beta.$$

For $A = t_1 - a$ and $B = t_2 - t_1$, the inequality (2.10) leads to

$$0 \leq (t_2 - a)^\beta - (t_1 - a)^\beta \leq (t_2 - t_1)^\beta,$$

an estimate that will be used in the next series of computations.

Now,

$$|(t_1 - a)^\beta (Q_{\beta,A}f)(t_1) - (t_2 - a)^\beta (Q_{\beta,A}f)(t_2)|$$

$$= |(t_1 - a)^\beta [(Q_{\beta,A}f)(t_1) - (Q_{\beta,A}f)(t_2)]$$

$$- (Q_{\beta,A}f)(t_2)[(t_2 - a)^\beta - (t_1 - a)^\beta]|$$

$$\leq (t_1 - a)^\beta \cdot |(Q_{\beta,A}f)(t_1) - (Q_{\beta,A}f)(t_2)|$$

$$+ |(Q_{\beta,A}f)(t_2)| \cdot (t_2 - t_1)^\beta$$

$$\leq (t_1 - a)^\beta \cdot C(t_2 - a)^{1 - \beta - \gamma - \frac{1}{\max\{q,w\}}} (t_2 - t_1)^{\frac{1}{\max\{q,w\}}}$$

$$\times \|A\|_{L^\infty(a,c)} \|f\|_{L^\infty(a,\gamma;c)}$$

$$+ (t_2 - a)^{1-\beta-\gamma} B(1 - \gamma, 1 - \beta) \cdot \|A\|_{L^\infty(a,c)} \|f\|_{L^\infty(a,\gamma;c)}$$

$$\times (t_2 - t_1)^\beta$$

$$(2.11) \qquad \leq \left[C(c - a)^\beta \cdot (t_2 - a)^{1 - \beta - \gamma - \frac{1}{\max\{q,w\}}} \right.$$

$$\left. + (c - a)^{1-\beta-\gamma} \cdot B(1 - \gamma, 1 - \beta) \right] \|A\|_{L^\infty(a,c)} \|f\|_{L^\infty(a,\gamma;c)}$$

$$\times \max \left\{ (t_2 - t_1)^{\frac{1}{\max\{q,w\}}}, (t_2 - t_1)^\beta \right\}$$

$$(2.12) \qquad \leq D\|A\|_{L^\infty(a,c)} \|f\|_{L^\infty(a,\gamma;c)} \max \left\{ (t_2 - t_1)^{\frac{1}{\max\{q,w\}}}, (t_2 - t_1)^\beta \right\},$$

where $D = D(p, v, \beta, \gamma, b - a, c - a)$. We took into account the estimates (2.3), (2.9).

Let us make a comment regarding D and (2.11). If we don't know the sign of the quantity $1 - \beta - \gamma - \frac{1}{\max\{q,w\}}$ then we are forced to use the raw estimate

$$(t_2 - a)^{1-\beta-\gamma-\frac{1}{\max\{q,w\}}} \leq (b - a)^{1-\beta-\gamma-\frac{1}{\max\{q,w\}}} + (c - a)^{1-\beta-\gamma-\frac{1}{\max\{q,w\}}}$$

which shall reflect on the size of D. If, on the other hand, we have $1-\beta-\gamma-\frac{1}{\max\{q,w\}} > 0$ then

$$(t_2 - a)^{1-\beta-\gamma-\frac{1}{\max\{q,w\}}} \leq (c - a)^{1-\beta-\gamma-\frac{1}{\max\{q,w\}}}, \quad t_2 \in [b, c],$$

and so

$$(2.13) \qquad D = C \cdot (c - a)^{1-\gamma-\frac{1}{\max\{q,w\}}} + (c - a)^{1-\beta-\gamma} \cdot B(1 - \gamma, 1 - \beta).$$

Observe that D is *independent* of b, a fact of crucial importance for our investigation.

In conclusion, given $f \in X_\gamma$, the functions $t \longmapsto (Q_{\beta,A}f)(t)$ and $t \longmapsto (t - a)^\beta (Q_{\beta,A}f)(t)$, with $0 < \beta \leq 1 - \gamma$, are continuous in $(a, c]$ by being uniformly continuous on each interval $[b, c]$, with $b \in (a, c)$.

In this note, by a *solution* of the fractional differential system (1.3) we mean any pair (f, g) of functions from $X_{1-\alpha}$ which verify the equations of the system.

2.2. **Integral representation of (1.3).** Consider now the first of equations (1.3), namely

$$(2.14) \qquad ({}_aD^\alpha_t f)(t) = G(t)g(t) + Q(t), \quad t \in [b, c],$$

where $\alpha \in \left(\frac{1}{2}, 1\right)$.

To use the estimates from the preceding subsection, introduce the quantities

$$0 < \gamma = \beta = 1 - \alpha < \frac{1}{2}.$$

There exist $p = v > 1$ such that

$$\gamma p = \beta v < \frac{1}{2}.$$

Also, $q = w$.

An important consequence of this choice of constants is that

$$1 - \beta - \gamma - \frac{1}{\max\{q, w\}} = 1 - 2\gamma - \frac{1}{q} = \frac{1}{p} - 2\gamma > 0,$$

which leads to a quantity D in (2.13) which is independent of $b \in (a, c)$.

Also, since $c(p, \gamma, \gamma) = \frac{2^{2\gamma - \frac{1}{p}}}{(1 - \gamma p)^{\frac{1}{p}}} < \frac{2^{2\gamma - \frac{1}{p}}}{\left(\frac{1}{2}\right)^{\frac{1}{p}}} = 2^{2\gamma}$, we have

$$C = 2[c(p, \beta, \gamma) + c(v, \gamma, \beta)] = 4c(p, \gamma, \gamma) < 2^{2(1+\gamma)} = 2^{2(2-\alpha)}$$

and respectively

$$D < 2^{2(2-\alpha)}(c - a)^{\alpha - \frac{1}{q}} + B(\alpha, \alpha)(c - a)^{2\alpha - 1}$$

$$(2.15) \qquad \leq [2^{2(2-\alpha)} + B(\alpha, \alpha)] \cdot \frac{(c - a)^\alpha}{\min\{(c - a)^{\frac{1}{q}}, (c - a)^{1-\alpha}\}}.$$

Now, recasting (2.14) as

$$\frac{d}{dt}\left[\int_a^t \frac{f(s)}{(t - s)^\alpha}ds\right] = \Gamma(1 - \alpha)[G(t)g(t) + Q(t)],$$

we deduce that

$$\int_a^t \frac{f(s)}{(t - s)^\alpha}ds - \lim_{t \searrow a}\left[\int_a^t \frac{f(s)}{(t - s)^\alpha}ds\right]$$

$$(2.16) \qquad = \Gamma(1 - \alpha)\int_a^t [G(s)g(s) + Q(s)]ds.$$

By taking into account the simple estimate

$$\left|\int_a^t \frac{1}{(t - s)^\alpha(s - a)^{1-\alpha}} \cdot [(s - a)^{1-\alpha}f(s) - f_a]ds\right|$$

$$\leq B(\alpha, 1 - \alpha) \cdot \sup_{s \in (a,t]} |(s - a)^{1-\alpha}f(s) - f_a|,$$

the identity (2.16) yields

$$\int_a^t \frac{f(s)}{(t-s)^\alpha} ds - f_a B(\alpha, 1-\alpha) = \Gamma(1-\alpha) \int_a^t [G(s)g(s) + Q(s)] ds.$$

Further, we have

$$\int_a^x \frac{1}{(x-t)^{1-\alpha}} \int_a^t \frac{f(s)}{(t-s)^\alpha} ds \, dt - f_a B(\alpha, 1-\alpha) \cdot \frac{(x-a)^\alpha}{\alpha}$$
$$= \Gamma(1-\alpha) \int_a^x \frac{1}{(x-t)^{1-\alpha}} \int_a^t [G(s)g(s) + Q(s)] ds \, dt.$$

Next, the left hand member of the preceding identity becomes

$$\int_a^x f(s) \int_s^x \frac{dt}{(x-t)^{1-\alpha}(t-s)^\alpha} ds - f_a B(\alpha, 1-\alpha) \cdot \frac{(x-a)^\alpha}{\alpha}$$
$$= B(\alpha, 1-\alpha) \cdot \int_a^x f(s) ds - f_a B(\alpha, 1-\alpha) \cdot \frac{(x-a)^\alpha}{\alpha},$$

while the right hand member reads as, by means of the Abel computations [9, p. 32, eq. (2.13)],

$$\Gamma(1-\alpha) \int_a^x \int_a^t \frac{G(s)g(s) + Q(s)}{(t-s)^{1-\alpha}} ds \, dt.$$

By differentiating with respect to x, where $x > a$, the identity

$$B(\alpha, 1-\alpha) \cdot \int_a^x f(s) ds - f_a B(\alpha, 1-\alpha) \cdot \frac{(x-a)^\alpha}{\alpha}$$
$$= \Gamma(1-\alpha) \int_a^x \int_a^t \frac{G(s)g(s) + Q(s)}{(t-s)^{1-\alpha}} ds \, dt,$$

we get

$$(2.17) \quad f(x) = \frac{f_a}{(x-a)^{1-\alpha}} + \frac{1}{\Gamma(\alpha)} \int_a^x \frac{G(s)g(s) + Q(s)}{(x-s)^{1-\alpha}} ds, \quad x \in (a, c].$$

Notice that, since $g \in X_{1-\alpha}$, the function $s \longmapsto G(s)g(s) + Q(s)$, denoted H, is also in $X_{1-\alpha}$. We have $H_a = G(a) \cdot g_a$. As $\beta + \gamma = (1-\alpha) + (1-\alpha) < 1$, the integral from the right side of (2.17) is $(Q_{\beta,1}H)(x)$ and both $(Q_{\beta,1}H)(x)$, $(x-a)^\beta (Q_{\beta,1}H)(x)$ are continuous in $(a, c]$.

Similarly,

$$(2.18) \quad g(x) = \frac{g_a}{(x-a)^{1-\alpha}} + \frac{1}{\Gamma(\alpha)} \int_a^x \frac{R(s)f(s) + V(s)}{(x-s)^{1-\alpha}} ds, \quad x \in (a, c].$$

For other derivations of formulas (2.17), (2.18), see [9, p. 837 and following] and [6, pp. 122–123]

3. **MAIN RESULT**

Theorem 3.1. *Set $p > 1$ such that $(1 - \alpha)p < \frac{1}{2}$ and take $q = \frac{p-1}{p}$. Assume that there exist the solutions (f_i, g_i) of the SFDE system (1.3) such that (1.4) holds for some $t_1, t_2 \in [b, c]$. Then,*

$$m \cdot (c - a)^\alpha \frac{\max\left\{(c - a)^{\frac{1}{q}}, (c - a)^{1-\alpha}\right\}}{\min\{(c - a)^{\frac{1}{q}}, (c - a)^{1-\alpha}\}} \geq \frac{\Gamma(\alpha)}{2^{2(2-\alpha)} + B(\alpha, \alpha)},$$

where $m = \max\{\|G\|_\infty, \|R\|_\infty\}$.

Proof. We have the relations

$$
\begin{cases}
f_i(t) = \dfrac{(f_i)_a}{(t - a)^{1-\alpha}} + \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_a^t \dfrac{G(s)g_i(s) + Q(s)}{(t - s)^{1-\alpha}} ds, \\[4mm]
g_i(t) = \dfrac{(g_i)_a}{(t - a)^{1-\alpha}} + \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_a^t \dfrac{R(s)f_i(s) + V(s)}{(t - s)^{1-\alpha}} ds,
\end{cases}
\quad t \in [b, c], \quad i \in \{1, 2\}.
$$

Take $b \leq t \leq t_1 \leq c$. By means of (2.12), we deduce that

$$
\begin{aligned}
&|(t - a)^{1-\alpha}(f_2 - f_1)(t)| \\
&= |(t_1 - a)^{1-\alpha}(f_2 - f_1)(t_1) - (t - a)^{1-\alpha}(f_2 - f_1)(t)| \\
&= \left| \frac{1}{\Gamma(\alpha)} \cdot [(t - a)^{1-\alpha}(Q_{1-\alpha, G}(g_2 - g_1))(t) \right. \\
&\quad \left. - (t_1 - a)^{1-\alpha}(Q_{1-\alpha, G}(g_2 - g_1))(t_1)] \right| \\
&\leq \frac{D}{\Gamma(\alpha)} \cdot \|G\|_{L^\infty(a,c)} \|g_2 - g_1\|_{L^\infty(a, 1-\alpha; c)} \\
&\quad \times \max\left\{(t_1 - t)^{\frac{1}{q}}, (t_1 - t)^{1-\alpha}\right\} \\
&\leq \frac{D\|G\|_\infty}{\Gamma(\alpha)} \max\left\{(c - b)^{\frac{1}{q}}, (c - b)^{1-\alpha}\right\} \cdot \|g_2 - g_1\|_{L^\infty(a, 1-\alpha; c)} \\
&\leq \frac{D\|G\|_\infty}{\Gamma(\alpha)} \max\left\{(c - a)^{\frac{1}{q}}, (c - a)^{1-\alpha}\right\} \cdot \|g_2 - g_1\|_{L^\infty(a, 1-\alpha; c)}.
\end{aligned}
$$

The case $b \leq t_1 \leq t \leq c$ leads to the same conclusion. So,

$$
\begin{aligned}
\|f_2 - f_1\|_{L^\infty(a, 1-\alpha; b, c)} &\leq E \cdot \|G\|_\infty \cdot \|g_2 - g_1\|_{L^\infty(a, 1-\alpha; c)} \\
&\leq E \cdot \max\{\|G\|_\infty, \|R\|_\infty\} \cdot \|g_2 - g_1\|_{L^\infty(a, 1-\alpha; c)},
\end{aligned}
\tag{3.1}
$$

where, by using (2.15), we have

$$
\begin{aligned}
E &= \frac{D}{\Gamma(\alpha)} \max\left\{(c - a)^{\frac{1}{q}}, (c - a)^{1-\alpha}\right\} \\
&\leq \frac{2^{2(2-\alpha)} + B(\alpha, \alpha)}{\Gamma(\alpha)} \cdot (c - a)^\alpha \frac{\max\left\{(c - a)^{\frac{1}{q}}, (c - a)^{1-\alpha}\right\}}{\min\{(c - a)^{\frac{1}{q}}, (c - a)^{1-\alpha}\}}.
\end{aligned}
\tag{3.2}
$$

Similarly,

$$\|g_2 - g_1\|_{L^\infty(a,1-\alpha;b,c)} \leq E \cdot \|R\|_\infty \cdot \|f_2 - f_1\|_{L^\infty(a,1-\alpha;c)}$$

$$(3.3) \qquad\qquad \leq E \cdot \max\{\|G\|_\infty, \|R\|_\infty\} \cdot \|f_2 - f_1\|_{L^\infty(a,1-\alpha;c)}.$$

In conclusion, as E is independent of b, by taking into account (3.1), (3.3) and (2.2), we get

$$X \leq E \cdot \max\{\|G\|_\infty, \|R\|_\infty\} \cdot X,$$

where $X = \max\{\|f_2 - f_1\|_{L^\infty(a,1-\alpha;c)}, \|g_2 - g_1\|_{L^\infty(a,1-\alpha;c)}\}$.

The proof is complete. $\square$

4. **PARTICULAR CASES**

In the particular case of (1.1), (1.5), Theorem 3.1 provides an upper bound to the length of the open interval (a, c) in which at least one of f and ${}_aD_t^\alpha f$ has no zero. Such an interval can be called a (fractional) *disconjugacy* interval of SFDE (1.1). This type of investigation might be useful for problems related to the localization of zeros of various fractional variants of sine and cosine type of functions, cf. [4], [6, p. 19].

Another particular case is that of sequential relaxation-oscillation equations, namely

$$ {}_aD_t^\alpha({}_aD_t^\alpha f) + P \cdot f = V(t), \quad t \in [b, c], $$

where $\alpha \in \left(\frac{1}{2}, 1\right)$, the coefficient $P > 0$ is constant and the free term $V : [a, +\infty) \to \mathbb{R}$ is continuous. As before, we are interested in the solutions f such that f, ${}_aD_t^\alpha f \in X_{1-\alpha}$. Such solutions exhibit several zeros for various terms V — see the numerical investigations from [6, pp. 224–226, 238–242].

5. **ACKNOWLEDEMENT**

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