SURFACE AREAS AND SURFACE INTEGRALS ON TIME SCALES

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ABSTRACT. We study surfaces parametrized by time scale parameters, obtain an integral fomula for computing the area of time scale surfaces, introduce delta integrals over time scale surfaces, and give sufficient conditions that ensure existence of these integrals.

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1. INTRODUCTION

In the original paper of S. Hilger [9] and further textbooks by M. Bohner and A. Peterson [7, 8], the single variable time scale calculus was developed in order to create a theory of, so-called, dynamic equations, that can unify and extend the theories of differential equations and difference equations. Next in [1, 2, 4, 3, 5, 6], the authors introduced and investigated some topics of multivariable time scale analysis to prepare an instrument for developing a theory of partial dynamic equations. The present paper continues those papers of the authors and introduces surface integration over time scale surfaces.

The paper is organized as follows. In Section 2, for the convenience of the reader, we present some essentials of the two-variable time scales calculus. In Section 3, we introduce the concept of a surface parametrized by time scale parameters and give an integral formula for computation of its area. In Section 4, we define a surface delta integral on time scales and give sufficient conditions for the existence of this integral and also offer a formula for its evaluation. Finally, in Section 5, we end with some concluding remarks.

2. TWO-VARIABLE TIME SCALES CALCULUS

A time scale is an arbitrary nonempty closed subset of the real numbers. For a general introduction to one-variable time scale calculus and notations we refer the

reader to [7, 8, 9]. In the present paper we deal with two-variable time scales calculus. Therefore, in this section, following [1, 2], we fix some notions and notation related to the two-variable time scales calculus.

Let $\mathbb{T}_1$ and $\mathbb{T}_2$ be two time scales and put $\mathbb{T}_1 \times \mathbb{T}_2 = \{(t, s) : t \in \mathbb{T}_1,\ s \in \mathbb{T}_2\}$, the Cartesian product of the time scales $\mathbb{T}_1$ and $\mathbb{T}_2$, which is a complete metric space with the Euclidean metric (distance) d defined by

$$d((t, s), (t', s')) = \sqrt{(t - t')^2 + (s - s')^2} \quad \text{for} \quad (t, s), (t', s') \in \mathbb{T}_1 \times \mathbb{T}_2.$$

Hence, we have for $\mathbb{T}_1 \times \mathbb{T}_2$ the usual concepts related to general metric spaces. For instance, given $\delta > 0$, the δ-neighborhood $U_\delta(t_0, s_0)$ of a point $(t_0, s_0) \in \mathbb{T}_1 \times \mathbb{T}_2$ is the set of all points $(t, s) \in \mathbb{T}_1 \times \mathbb{T}_2$ such that $d((t_0, s_0), (t, s)) < \delta$. Let σ_1 and σ_2 be the forward jump operators on $\mathbb{T}_1$ and $\mathbb{T}_2$, respectively. The first-order partial delta derivatives of a function $f : \mathbb{T}_1 \times \mathbb{T}_2 \to \mathbb{R}$ at a point $(t_0, s_0) \in \mathbb{T}_1^\kappa \times \mathbb{T}_2^\kappa$ are defined to be

$$\frac{\partial f(t_0, s_0)}{\Delta_1 t} = \lim_{\substack{t \to t_0 \\ t \neq \sigma_1(t_0)}} \frac{f(\sigma_1(t_0), s_0) - f(t, s_0)}{\sigma_1(t_0) - t}$$

and

$$\frac{\partial f(t_0, s_0)}{\Delta_2 s} = \lim_{\substack{s \to s_0 \\ s \neq \sigma_2(s_0)}} \frac{f(t_0, \sigma_2(s_0)) - f(t_0, s)}{\sigma_2(s_0) - s}.$$

These derivatives will be denoted also by $f^{\Delta^1}(t_0, s_0)$ and $f^{\Delta^2}(t_0, s_0)$, respectively.

Suppose $a < b$ are points in $\mathbb{T}_1$, $c < d$ are points in $\mathbb{T}_2$, $[a, b)$ is a half-closed bounded interval in $\mathbb{T}_1$, and $[c, d)$ is a half-closed bounded interval in $\mathbb{T}_2$,

$$[a, b) = \{t \in \mathbb{T}_1 : a \leq t < b\}, \quad [c, d) = \{s \in \mathbb{T}_2 : c \leq s < d\}.$$

In what follows, all ocurring intervals will be time scale intervals. Let us introduce a time scale "rectangle" (or "delta rectangle") in $\mathbb{T}_1 \times \mathbb{T}_2$ by

$$R = [a, b) \times [c, d) = \{(t, s) : t \in [a, b),\ s \in [c, d)\}.$$

The area (measure) of the rectangle R is defined to be

$$m(R) = (b - a)(d - c).$$

Let

$$\{t_0, t_1, \ldots, t_n\} \subset [a, b), \quad \text{where} \quad a = t_0 < t_1 < \ldots < t_n = b,$$

$$\{s_0, s_1, \ldots, s_l\} \subset [c, d), \quad \text{where} \quad c = s_0 < s_1 < \ldots < s_l = d.$$

The numbers n and l may be arbitrary positive integers. We call the collection of intervals

$$P_1 = \{[t_{i-1}, t_i) : 1 \leq i \leq n\}$$

a Δ-partition (or delta partition) of $[a, b)$ and denote the set of all Δ-partitions of $[a, b)$ by $\mathcal{P}([a, b))$. Similarly, the collection of intervals

$$P_2 = \{[s_{j-1}, s_j) : 1 \leq j \leq l\}$$

is called a Δ-partition of $[c, d)$, and the set of all Δ-partitions of $[c, d)$ is denoted by $\mathcal{P}([c, d))$. Let us set

(2.1) $R_{ij} = [t_{i-1}, t_i) \times [s_{j-1}, s_j), \quad$ where $\quad 1 \leq i \leq n, \ 1 \leq j \leq l.$

We call the collection

(2.2) $P = \{R_{ij} : 1 \leq i \leq n, \ 1 \leq j \leq l\}$

a Δ-partition of R, generated by the Δ-partitions P_1 and P_2 of $[a, b)$ and $[c, d)$, respectively, and write $P = P_1 \times P_2$. The rectangles R_{ij}, $1 \leq i \leq n$, $1 \leq j \leq l$, are called the subrectangles of the partition P. The set of all Δ-partitions of R is denoted by $\mathcal{P}(R)$.

We will need the following auxiliary result (see [8, Lemma 5.7] for the proof) for making a limit process in getting time scale integrals from time scale integral sums.

Lemma 2.1. *For any $\delta > 0$ there exists at least one $P_1 \in \mathcal{P}([a, b))$ generated by a set*

$$\{t_0, t_1, \ldots, t_n\} \subset [a, b), \quad \text{where} \quad a = t_0 < t_1 < \ldots < t_n = b,$$

so that for each $i \in \{1, 2, \ldots, n\}$,

$$\text{either} \quad t_i - t_{i-1} \leq \delta \quad \text{or} \quad t_i - t_{i-1} > \delta \text{ and } \sigma_1(t_{i-1}) = t_i.$$

We denote by $\mathcal{P}_\delta([a, b))$ the set of all $P_1 \in \mathcal{P}([a, b))$ that posses the property indicated in Lemma 2.1. Similarly we define $\mathcal{P}_\delta([c, d))$. Further, by $\mathcal{P}_\delta(R)$ we denote the set of all $P \in \mathcal{P}(R)$ such that

$$P = P_1 \times P_2, \quad \text{where} \quad P_1 \in \mathcal{P}_\delta([a, b)) \quad \text{and} \quad P_2 \in \mathcal{P}_\delta([c, d)).$$

Definition 2.2. Let $\Omega \subset \mathbb{T}_1 \times \mathbb{T}_2$. A point $u = (t, s) \in \mathbb{T}_1 \times \mathbb{T}_2$ is called a *boundary point* of Ω if every open (two-dimensional) ball $B(u, r) = \{v \in \mathbb{T}_1 \times \mathbb{T}_2 : d(u, v) < r\}$ of radius r and center u contains at least one point of Ω and at least one point of $(\mathbb{T}_1 \times \mathbb{T}_2) \setminus \Omega$. The set of all boundary points of Ω is called the *boundary* of Ω, and it is denoted by $\partial\Omega$.

Definition 2.3. Let $\Omega \subset \mathbb{T}_1 \times \mathbb{T}_2$. A point $u = (t, s) \in \mathbb{T}_1 \times \mathbb{T}_2$ is called a Δ-*boundary point* of Ω if every rectangle of the form $V = [t, t') \times [s, s') \subset \mathbb{T}_1 \times \mathbb{T}_2$ with $t' \in \mathbb{T}_1$, $t' > t$ and $s' \in \mathbb{T}_2$, $s' > s$, contains at least one point of Ω and at least one point of $(\mathbb{T}_1 \times \mathbb{T}_2) \setminus \Omega$. The set of all Δ-boundary points of Ω is called the Δ-*boundary* of Ω, and it is denoted by $\partial_\Delta\Omega$.

For $i = 1, 2$, let us introduce the set $\mathbb{T}_i^0$ as follows: If $\mathbb{T}_i$ has a finite maximum t^*, then $\mathbb{T}_i^0 = \mathbb{T}_i \setminus \{t^*\}$, otherwise $\mathbb{T}_i^0 = \mathbb{T}_i$. Briefly we will write $\mathbb{T}_i^0 = \mathbb{T}_i \setminus \{\max \mathbb{T}_i\}$. Evidently, for every point $t \in \mathbb{T}_i^0$ there exists an interval of the form $[\alpha, \beta) \subset \mathbb{T}_i$ (with $\alpha, \beta \in \mathbb{T}_i$ and $\alpha < \beta$) that contains the point t.

Obviously, each Δ-boundary point of Ω is a boundary point of Ω, but the converse is not necessarily true. Each Δ-boundary point of Ω must belong to $\mathbb{T}_1^0 \times \mathbb{T}_2^0$. Note also that any rectangle of the form $\Omega = [a, b) \times [c, d) \subset \mathbb{T}_1 \times \mathbb{T}_2$, where $a, b \in \mathbb{T}_1$, $a < b$ and $c, d \in \mathbb{T}_2$, $c < d$, has no Δ-boundary point, i.e., $\partial_\Delta \Omega = \emptyset$, the empty set. If $\mathbb{T}_1 = \mathbb{T}_2 = \mathbb{Z}$, then any set $\Omega \subset \mathbb{Z} \times \mathbb{Z}$ has no boundary as well as no Δ-boundary points. For other examples of the Δ-boundary see [2, 4].

Definition 2.4. Let $\Omega \subset \mathbb{T}_1^0 \times \mathbb{T}_2^0$ be a bounded set and let $\partial_\Delta \Omega$ be its Δ-boundary. Let $R = [a, b) \times [c, d)$ be a rectangle in $\mathbb{T}_1 \times \mathbb{T}_2$ such that $\Omega \cup \partial_\Delta \Omega \subset R$. Further, let $\mathcal{P}(R)$ denote the set of all Δ-partitions of R of type (2.1), (2.2). For every $P \in \mathcal{P}(R)$ define $J_*(\Omega, P)$ to be the sum of the areas of those subrectangles of P which are entirely contained in Ω, and let $J^*(\Omega, P)$ be the sum of the areas of those subrectangles of P each of which contains at least one point of $\Omega \cup \partial_\Delta \Omega$. The numbers

$$J_*(\Omega) = \sup\{J_*(\Omega, P) : P \in \mathcal{P}(R)\} \quad \text{and} \quad J^*(\Omega) = \inf\{J^*(\Omega, P) : P \in \mathcal{P}(R)\}$$

are called the (two-dimensional) *inner and outer Jordan Δ-measure* of Ω, respectively. The set Ω is called *Jordan Δ-measurable* if $J_*(\Omega) = J^*(\Omega)$, in which case this common value is called the *Jordan Δ-measure* of Ω, and it is denoted by $J(\Omega)$.

For any bounded set $\Omega \subset \mathbb{T}_1^0 \times \mathbb{T}_2^0$ we have

$$J^*(\partial_\Delta \Omega) = J^*(\Omega) - J_*(\Omega).$$

Hence Ω is Jordan Δ-measurable if and only if its Δ-boundary $\partial_\Delta \Omega$ has Jordan Δ-measure zero.

Note that every rectangle $R = [a, b) \times [c, d) \subset \mathbb{T}_1 \times \mathbb{T}_2$, where $a, b \in \mathbb{T}_1$, $a < b$ and $c, d \in \mathbb{T}_2$, $c < d$, is Jordan Δ-measurable with the Jordan Δ-measure $J(R) = (b - a)(d - c)$. The Δ-boundary of R is empty and therefore has Jordan Δ-measure zero.

Now we define double Δ-integrals over Jordan Δ-measurable sets. Let $\Omega \subset \mathbb{T}_1^0 \times \mathbb{T}_2^0$ be a bounded Jordan Δ-measurable set and $f : \Omega \to \mathbb{R}$ be a function. Further, let $R = [a, b) \times [c, d) \subset \mathbb{T}_1 \times \mathbb{T}_2$ be a rectangle such that $\Omega \subset R$. To define the double Δ-integral of f over Ω, we begin with a Δ-partition $P \in \mathcal{P}(R)$ of type (2.1), (2.2) and assume that $P = \{R_1, R_2, \ldots, R_N\}$ (every partition (2.2) can be labeled in this form, and the order in which those subrectangles are labeled makes no difference). Some of the subrectangles of P will lie entirely within Ω, some will be outside of Ω, and some will lie partly within and partly outside Ω. We consider the

collection $P' = \{R_1, R_2, \ldots, R_k\}$ of all those subrectangles in P that lie completely within the set Ω. This collection P' is called an inner Δ-partition of the set Ω. By choosing an arbitrary point (ξ_i, η_i) in the ith subrectangle R_i of P' for $i \in \{1, \ldots, k\}$, we obtain a selection for the inner Δ-partition P'. Let us denote by $m(R_i)$ the area of R_i: If $R_i = [t_i, t_i') \times [s_i, s_i')$, then $m(R_i) = (t_i' - t_i)(s_i' - s_i)$. Then this selection gives the sum

$$\Lambda = \sum_{i=1}^{k} f(\xi_i, \eta_i) m(R_i).$$

We call Λ a Riemann Δ-sum of f, corresponding to the inner Δ-partition P' of Ω, determined by the partition $P \in \mathcal{P}(R)$.

Definition 2.5. We say that f is *Riemann Δ-integrable* over $\Omega \subset \mathbb{T}_1^0 \times \mathbb{T}_2^0$ if there exists a number I with the property that for each $\varepsilon > 0$ there exists a number $\delta > 0$ such that $|\Lambda - I| < \varepsilon$ for every Riemann Δ-sum Λ of f corresponding to any inner Δ-partition $P' = \{R_1, R_2, \ldots, R_k\}$ of Ω, determined by a partition $P \in \mathcal{P}_\delta(R)$, independent of the choice of the points $(\xi_i, \eta_i) \in R_i$ for $1 \le i \le k$. The number I is called the *Riemann double Δ-integral* of f over Ω, and it is denoted by

$$\int\int_\Omega f(t, s) \Delta_1 t \Delta_2 s.$$

A function $f : \mathbb{T}_1 \times \mathbb{T}_2 \to \mathbb{R}$ is said to be continuous at $u \in \mathbb{T}_1 \times \mathbb{T}_2$ if for every $\varepsilon > 0$ there exists $\delta > 0$ such that $|f(u) - f(v)| < \varepsilon$ for all points $v \in \mathbb{T}_1 \times \mathbb{T}_2$ satisfying $d(u, v) < \delta$. If u is an isolated point of $\mathbb{T}_1 \times \mathbb{T}_2$, then our definition implies that every function $f : \mathbb{T}_1 \times \mathbb{T}_2 \to \mathbb{R}$ is continuous at $u \in \mathbb{T}_1 \times \mathbb{T}_2$. For, no matter which $\varepsilon > 0$ we choose, we can pick $\delta > 0$ so that the only point $v \in \mathbb{T}_1 \times \mathbb{T}_2$ for which $d(u, v) < \delta$ is $v = u$; then $|f(u) - f(v)| = 0 < \varepsilon$. In particular, every function $f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{R}$ is continuous at each point of $\mathbb{Z} \times \mathbb{Z}$.

The following theorem gives conditions sufficient for the existence of the Riemann double Δ-integral.

Theorem 2.6. *Suppose $\Omega \subset \mathbb{T}_1^0 \times \mathbb{T}_2^0$ is a bounded Jordan Δ-measurable set. Every function $f : \Omega \to \mathbb{R}$ continuous on the closure $\overline{\Omega}$ of Ω is Riemann Δ-integrable over Ω.*

3. SURFACE AREAS

Let $\Omega \subset \mathbb{T}_1 \times \mathbb{T}_2$ and let

$$\varphi : \Omega \to \mathbb{R}, \quad \psi : \Omega \to \mathbb{R}, \quad \chi : \Omega \to \mathbb{R}$$

be continuous (in the time scale topology) functions on Ω. Consider the xyz-space, i.e., the set of all ordered triples (x, y, z) of real numbers x, y, and z. Each such triple

determines a point of the space, and the numbers x, y, and z are the coordinates of that point.

Definition 3.1. The triple of functions

(3.1) $x = \varphi(t, s), \quad y = \psi(t, s), \quad z = \chi(t, s), \quad (t, s) \in \Omega \subset \mathbb{T}_1 \times \mathbb{T}_2$

is said to define a (time scale continuous) *surface S*. The points (x, y, z) with the coordinates x, y, and z defined by (3.1) are called the *points* of the surface, and the set of all points of the surface, i.e., the range of the mapping (3.1), is referred to as simply the *surface* (when no ambiguity can arise).

Let S be a (time scale continuous) surface with equation (3.1). It is convenient to use vector notation, and we write the system (3.1) in the form

(3.2) $\overrightarrow{r} = \overrightarrow{r}(t, s) = \varphi(t, s)\overrightarrow{e_1} + \psi(t, s)\overrightarrow{e_2} + \chi(t, s)\overrightarrow{e_3},$

where

$$\overrightarrow{e_1} = (1, 0, 0), \quad \overrightarrow{e_2} = (0, 1, 0), \quad \overrightarrow{e_3} = (0, 0, 1).$$

Let $P' = \{R_1, R_2, \ldots, R_k\}$ be an arbitrary inner Δ-partition of Ω, described above in Section 2, and let us set

$$R_i = [t_i, t'_i) \times [s_i, s'_i), \quad i \in \{1, 2, \ldots, k\}.$$

Further, for each $i \in \{1, 2, \ldots, k\}$, we set

$$\overrightarrow{u_i} = \overrightarrow{r}(t'_i, s_i) - \overrightarrow{r}(t_i, s_i) \quad \text{and} \quad \overrightarrow{v_i} = \overrightarrow{r}(t_i, s'_i) - \overrightarrow{r}(t_i, s_i)$$

so that

$$\overrightarrow{u_i} = u_i^{(1)}\overrightarrow{e_1} + u_i^{(2)}\overrightarrow{e_2} + u_i^{(3)}\overrightarrow{e_3} \quad \text{and} \quad \overrightarrow{v_i} = v_i^{(1)}\overrightarrow{e_1} + v_i^{(2)}\overrightarrow{e_2} + v_i^{(3)}\overrightarrow{e_3}$$

with

$$u_i^{(1)} = \varphi(t'_i, s_i) - \varphi(t_i, s_i), \; u_i^{(2)} = \psi(t'_i, s_i) - \psi(t_i, s_i), \; u_i^{(3)} = \chi(t'_i, s_i) - \chi(t_i, s_i)$$

and

$$v_i^{(1)} = \varphi(t_i, s'_i) - \varphi(t_i, s_i), \; v_i^{(2)} = \psi(t_i, s'_i) - \psi(t_i, s_i), \; v_i^{(3)} = \chi(t_i, s'_i) - \chi(t_i, s_i).$$

The area of the parallelogram spanned by the vectors $\overrightarrow{u_i}$ and $\overrightarrow{v_i}$ is equal to $|\overrightarrow{u_i} \times \overrightarrow{v_i}|$, where

$$\overrightarrow{u_i} \times \overrightarrow{v_i} = \begin{vmatrix} \overrightarrow{e_1} & \overrightarrow{e_2} & \overrightarrow{e_3} \\ u_i^{(1)} & u_i^{(2)} & u_i^{(3)} \\ v_i^{(1)} & v_i^{(2)} & v_i^{(3)} \end{vmatrix}$$

is the vector product of the vectors $\overrightarrow{u_i}$ and $\overrightarrow{v_i}$. Let us set

(3.3) $A(S, P') = \sum_{i=1}^{k} |\overrightarrow{u_i} \times \overrightarrow{v_i}|.$

Definition 3.2. The surface S is said to be *squarable* (or *rectifiable*) if

$$\sup\{A(S, P') : P' \text{ is an inner } \Delta\text{-partition of } \Omega\} =: A(S) < \infty,$$

where the least upper bound is taken over all possible inner Δ-partitions P' of Ω. The nonnegative real number $A = A(S)$ is called the area of the surface S. If the supremum (finite) does not exist, then the surface is said to be nonsquarable (or nonrectifiable).

Let $P = [a, b) \times [c, d) \subset \mathbb{T}_1 \times \mathbb{T}_2$ be a time scale rectangle that contains Ω. Let $P, Q \in \mathcal{P}(R)$ and $P = P_1 \times P_2$, $Q = Q_1 \times Q_2$, where

$$P_1, P_2 \in \mathcal{P}([a, b)) \quad \text{and} \quad Q_1, Q_2 \in \mathcal{P}([c, d)).$$

We say that Q is a *refinement* of P if Q_1 is a refinement of P_1 and Q_2 is a refinement of P_2.

Lemma 3.3. *Let $P, Q \in \mathcal{P}(R)$ and Q be a refinement of P. Suppose that P' and Q' are inner Δ-partitions of Ω, determined by P and Q, respectively. Then $A(S, P') \leq A(S, Q')$.*

Proof. An induction argument shows that we may assume that Q' has only one more element than P'. If P' is given by

$$P' = \{R_1, R_2, \ldots, R_k\},$$

then there is some $j \in \{1, 2, \ldots, k\}$ such that Q' is given by

$$Q' = \{R_1, \ldots, R_{j-1}, R_j^{(1)}, R_j^{(2)}, R_{j+1}, \ldots, R_k\},$$

where $R_j^{(1)} \cup R_j^{(2)} = R_j$. Assume that

$$R_j = [t_j, t_j') \times [s_j, s_j'), \quad R_j^{(1)} = [t_j, \tau_j) \times [s_j, s_j'), \quad R_j^{(2)} = [\tau_j, t_j') \times [s_j, s_j'),$$

where $\tau_j \in \mathbb{T}_1$ and $t_j < \tau_j < t_j'$. Let us put

$$\overrightarrow{u_j} = \overrightarrow{r}(t_j', s_j) - \overrightarrow{r}(t_j, s_j), \quad \overrightarrow{v_j} = \overrightarrow{r}(t_j, s_j') - \overrightarrow{r}(t_j, s_j),$$

$$\overrightarrow{u_{j1}} = \overrightarrow{r}(\tau_j, s_j) - \overrightarrow{r}(t_j, s_j), \quad \overrightarrow{v_{j1}} = \overrightarrow{r}(\tau_j, s_j') - \overrightarrow{r}(\tau_j, s_j),$$

$$\overrightarrow{u_{j2}} = \overrightarrow{r}(t_j', s_j) - \overrightarrow{r}(\tau_j, s_j).$$

Then

$$A(S, Q') - A(S, P') = |\overrightarrow{u_{j1}} \times \overrightarrow{v_j}| + |\overrightarrow{u_{j2}} \times \overrightarrow{v_{j1}}| - |\overrightarrow{u_j} \times \overrightarrow{v_j}|.$$

Next, we have

$$\overrightarrow{v_{j1}} = \overrightarrow{v_j} \quad \text{and} \quad \overrightarrow{u_{j1}} + \overrightarrow{u_{j2}} = \overrightarrow{u_j}.$$

Therefore

$$|\overrightarrow{u_j} \times \overrightarrow{v_j}| = |\overrightarrow{u_{j1}} \times \overrightarrow{v_j} + \overrightarrow{u_{j2}} \times \overrightarrow{v_j}|$$

$$\leq |\overrightarrow{u_{j1}} \times \overrightarrow{v_j}| + |\overrightarrow{u_{j2}} \times \overrightarrow{v_j}| = |\overrightarrow{u_{j1}} \times \overrightarrow{v_j}| + |\overrightarrow{u_{j2}} \times \overrightarrow{v_{j1}}|,$$

and we get that $A(S, Q') - A(S, P') \geq 0$. $\square$

Now we present sufficient conditions for squarability of surfaces and give a formula for evaluating their areas.

Theorem 3.4. *Let the functions φ, ψ, and χ be continuous and have continuous first partial delta derivatives in the closure of the region $\Omega \subset \mathbb{T}_1^0 \times \mathbb{T}_2^0$. Suppose that the region Ω is bounded and Jordan Δ-measurable. Then the surface S defined by the parametric equations in (3.1) is squarable and its area $A(S)$ can be evaluated by the formula*

(3.4) $$A(S) = \int \int_\Omega \left| \overrightarrow{r}^{\Delta^1} \times \overrightarrow{r}^{\Delta^2} \right| \Delta_1 t \Delta_2 s = \int \int_\Omega \sqrt{EG - F^2} \Delta_1 t \Delta_2 s,$$

where

$$E = \left(\frac{\partial \overrightarrow{r}}{\Delta_1 t} \right)^2 = \left| \frac{\partial \overrightarrow{r}}{\Delta_1 t} \right|^2 = \left(\frac{\partial \varphi}{\Delta_1 t} \right)^2 + \left(\frac{\partial \psi}{\Delta_1 t} \right)^2 + \left(\frac{\partial \chi}{\Delta_1 t} \right)^2,$$

(3.5) $$G = \left(\frac{\partial \overrightarrow{r}}{\Delta_2 s} \right)^2 = \left| \frac{\partial \overrightarrow{r}}{\Delta_2 s} \right|^2 = \left(\frac{\partial \varphi}{\Delta_2 s} \right)^2 + \left(\frac{\partial \psi}{\Delta_2 s} \right)^2 + \left(\frac{\partial \chi}{\Delta_2 s} \right)^2,$$

$$F = \frac{\partial \overrightarrow{r}}{\Delta_1 t} \cdot \frac{\partial \overrightarrow{r}}{\Delta_2 s} = \frac{\partial \varphi}{\Delta_1 t} \frac{\partial \varphi}{\Delta_2 s} + \frac{\partial \psi}{\Delta_1 t} \frac{\partial \psi}{\Delta_2 s} + \frac{\partial \chi}{\Delta_1 t} \frac{\partial \chi}{\Delta_2 s}.$$

Proof. First we show that the surface S is squarable. Let $P' = \{R_1, R_2, \ldots, R_k\}$ be an arbitrary inner Δ-partition of Ω and let

$$R_i = [t_i, t_i') \times [s_i, s_i') \quad \text{for} \quad i \in \{1, 2, \ldots, k\}.$$

In order to make further calculations shorter, let us put

$$\theta_1(t, s) = \varphi(t, s), \quad \theta_2(t, s) = \psi(t, s), \quad \theta_3(t, s) = \chi(t, s).$$

Then we have, for $p = 1, 2, 3$,

$$u_i^{(p)} = \theta_p(t_i', s_i) - \theta_p(t_i, s_i), \quad v_i^{(p)} = \theta_p(t_i, s_i') - \theta_p(t_i, s_i),$$

and

$$|\overrightarrow{u_i} \times \overrightarrow{v_i}|^2 \leq |\overrightarrow{u_i}|^2 |\overrightarrow{v_i}|^2 = \sum_{p=1}^3 \left| u_i^{(p)} \right|^2 \sum_{p=1}^3 \left| v_i^{(p)} \right|^2$$

(3.6) $$= \sum_{p=1}^3 |\theta_p(t_i', s_i) - \theta_p(t_i, s_i)|^2 \sum_{p=1}^3 |\theta_p(t_i, s_i') - \theta_p(t_i, s_i)|^2.$$

For each $p = 1, 2, 3$, applying to the function $\theta_p(t, s)$ the mean value theorem (see [1, Theorem 4.6]), we get that there exist points ξ_{ip}, ξ_{ip}' in $[t_i, t_i')$ and points η_{ip}, η_{ip}' in $[s_i, s_i')$ such that

(3.7) $$\theta_p^{\Delta^1}(\xi_{ip}, s_i)(t_i' - t_i) \leq \theta_p(t_i', s_i) - \theta_p(t_i, s_i) \leq \theta_p^{\Delta^1}(\xi_{ip}', s_i)(t_i' - t_i),$$

(3.8) $$\theta_p^{\Delta^2}(t_i, \eta_{ip})(s_i' - s_i) \leq \theta_p(t_i, s_i') - \theta_p(t_i, s_i) \leq \theta_p^{\Delta^2}(t_i, \eta_{ip}')(s_i' - s_i).$$

Further, by the assumption of the theorem, the derivatives $\theta_p^{\Delta^1}$ and $\theta_p^{\Delta^2}$ are bounded on Ω for $p = 1, 2, 3$. Then there is a finite positive constant C such that

$$(3.9) \quad \left|\theta_p^{\Delta^1}(t, s)\right| \leq C \quad \text{and} \quad \left|\theta_p^{\Delta^2}(t, s)\right| \leq C \quad \text{for all} \quad (t, s) \in \Omega \text{ and } p = 1, 2, 3.$$

Consequently, we get from (3.7) and (3.8) that

$$|\theta_p(t_i', s_i) - \theta_p(t_i, s_i)| \leq C(t_i' - t_i), \quad |\theta_p(t_i, s_i') - \theta_p(t_i, s_i)| \leq C(s_i' - s_i)$$

for all $i \in \{1, 2, \ldots, k\}$ and $p = 1, 2, 3$, and we find from (3.3), using (3.6) and assuming $\Omega \subset R = [a, b) \times [c, d) \subset \mathbb{T}_1 \times \mathbb{T}_2$,

$$A(S, P') \leq 3C^2 \sum_{i=1}^{k} (t_i' - t_i)(s_i' - s_i) \leq 3C^2(b - a)(d - c).$$

This shows that the set

$$\{A(S, P') : P' \text{ is an inner } \Delta\text{-partition of } \Omega\}$$

is bounded, and hence by Definition 3.2, the surface S is squarable.

Now we prove that the area $A(S)$ of the surface S can be evaluated by the formula (3.4). Let us put

$$(3.10) \qquad I = \int\int_\Omega \left|\overrightarrow{r}^{\Delta^1}(t, s) \times \overrightarrow{r}^{\Delta^2}(t, s)\right| \Delta_1 t \Delta_2 s$$

and consider the Riemann Δ-sum

$$(3.11) \qquad \Lambda = \sum_{i=1}^{k} \left|\overrightarrow{r}^{\Delta^1}(\xi_i, \eta_i) \times \overrightarrow{r}^{\Delta^2}(\xi_i, \eta_i)\right| m(R_i)$$

of the Δ-integrable function $\left|\overrightarrow{r}^{\Delta^1}(t, s) \times \overrightarrow{r}^{\Delta^2}(t, s)\right|$, corresponding to the inner Δ-partition $P' = \{R_1, R_2, \ldots, R_k\}$ of Ω and any choice of the points (ξ_i, η_i) in R_i. Take an arbitrary $\varepsilon > 0$. Let us show that there is $\delta > 0$ such that

$$(3.12) \qquad |A(S, P') - \Lambda| < \frac{\varepsilon}{4}$$

for every inner Δ-partition $P' = \{R_1, R_2, \ldots, R_k\}$ of Ω, determined by a partition $P \in \mathcal{P}_\delta(R)$, where $R = [a, b) \times [c, d) \subset \mathbb{T}_1 \times \mathbb{T}_2$ is a time scale rectangle that contains Ω.

We have, by (3.3) and (3.11),

$$A(S, P') - \Lambda = \sum_{i=1}^{k} \left\{ |\overrightarrow{u_i} \times \overrightarrow{v_i}| - \left|\overrightarrow{r}^{\Delta^1}(\xi_i, \eta_i) \times \overrightarrow{r}^{\Delta^2}(\xi_i, \eta_i)\right| m(R_i) \right\},$$

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where

$$\vec{u_i} = \sum_{p=1}^{3} u_i^{(p)}\vec{e_p}, \quad \vec{v_i} = \sum_{p=1}^{3} v_i^{(p)}\vec{e_p},$$

$$\vec{r}^{\Delta^1}(\xi_i, \eta_i) = \sum_{p=1}^{3} U_i^{(p)}\vec{e_p}, \quad \vec{r}^{\Delta^2}(\xi_i, \eta_i) = \sum_{p=1}^{3} V_i^{(p)}\vec{e_p},$$

with

$$u_i^{(p)} = \theta_p(t_i', s_i) - \theta_p(t_i, s_i), \quad v_i^{(p)} = \theta_p(t_i, s_i') - \theta_p(t_i, s_i),$$
$$U_i^{(p)} = \theta_p^{\Delta^1}(\xi_i, \eta_i), \quad V_i^{(p)} = \theta_p^{\Delta^2}(\xi_i, \eta_i).$$

From (3.7) and (3.8), we have

$$0 \le \theta_p(t_i', s_i) - \theta_p(t_i, s_i) - \theta_p^{\Delta^1}(\xi_{ip}, s_i)(t_i' - t_i)$$
$$\le \left[\theta_p^{\Delta^1}(\xi_{ip}', s_i) - \theta_p^{\Delta^1}(\xi_{ip}, s_i)\right](t_i' - t_i),$$
$$0 \le \theta_p(t_i, s_i') - \theta_p(t_i, s_i) - \theta_p^{\Delta^2}(t_i, \eta_{ip})(s_i' - s_i)$$
$$\le \left[\theta_p^{\Delta^2}(t_i, \eta_{ip}') - \theta_p^{\Delta^2}(t_i, \eta_{ip})\right](s_i' - s_i),$$

and consequently

$$u_i^{(p)} = \theta_p(t_i', s_i) - \theta_p(t_i, s_i) = [\theta_p^{\Delta^1}(\xi_{ip}, s_i) + \alpha_{ip}](t_i' - t_i) = \overline{u}_i^{(p)}(t_i' - t_i),$$
$$v_i^{(p)} = \theta_p(t_i, s_i') - \theta_p(t_i, s_i) = [\theta_p^{\Delta^2}(t_i, \eta_{ip}) + \beta_{ip}](s_i' - s_i) = \overline{v}_i^{(p)}(s_i' - s_i),$$

where

$$\overline{u}_i^{(p)} = \theta_p^{\Delta^1}(\xi_{ip}, s_i) + \alpha_{ip}, \quad \overline{v}_i^{(p)} = \theta_p^{\Delta^2}(t_i, \eta_{ip}) + \beta_{ip},$$

(3.13) $$0 \le \alpha_{ip} \le \theta_p^{\Delta^1}(\xi_{ip}', s_i) - \theta_p^{\Delta^1}(\xi_{ip}, s_i) \le M_{ip} - m_{ip},$$

(3.14) $$0 \le \beta_{ip} \le \theta_p^{\Delta^2}(t_i, \eta_{ip}') - \theta_p^{\Delta^2}(t_i, \eta_{ip}) \le N_{ip} - n_{ip},$$

in which M_{ip} and m_{ip} are the supremum and infimum of $\theta_p^{\Delta^1}$ on R_i, respectively, and N_{ip} and n_{ip} are the corresponding numbers for $\theta_p^{\Delta^2}$. Thus,

$$|\vec{u_i} \times \vec{v_i}| = m(R_i)\left|\overline{\vec{u}}_i \times \overline{\vec{v}}_i\right| = m(R_i)\sqrt{x_i^2 + y_i^2 + z_i^2},$$
$$\left|\vec{r}^{\Delta^1}(\xi_i, \eta_i) \times \vec{r}^{\Delta^2}(\xi_i, \eta_i)\right| = \sqrt{X_i^2 + Y_i^2 + Z_i^2},$$

where

$$x_i = \overline{u}_i^{(2)}\overline{v}_i^{(3)} - \overline{u}_i^{(3)}\overline{v}_i^{(2)}, \ y_i = \overline{u}_i^{(3)}\overline{v}_i^{(1)} - \overline{u}_i^{(1)}\overline{v}_i^{(3)}, \ z_i = \overline{u}_i^{(1)}\overline{v}_i^{(2)} - \overline{u}_i^{(2)}\overline{v}_i^{(1)},$$
$$X_i = U_i^{(2)}V_i^{(3)} - U_i^{(3)}V_i^{(2)}, \ Y_i = U_i^{(3)}V_i^{(1)} - U_i^{(1)}V_i^{(3)}, \ Z_i = U_i^{(1)}V_i^{(2)} - U_i^{(2)}V_i^{(1)}.$$

Therefore, using the inequality (for arbitrary real numbers x, y, z, X, Y, Z)

$$\left|\sqrt{x^2 + y^2 + z^2} - \sqrt{X^2 + Y^2 + Z^2}\right| \le |x - X| + |y - Y| + |z - Z|,$$

we get

$$|A(S, P') - \Lambda| = \left| \sum_{i=1}^{k} \left\{ \sqrt{x_i^2 + y_i^2 + z_i^2} - \sqrt{X_i^2 + Y_i^2 + Z_i^2} \right\} m(R_i) \right|$$

(3.15)
$$\leq \sum_{i=1}^{k} \left\{ |x_i - X_i| + |y_i - Y_i| + |z_i - Z_i| \right\} m(R_i).$$

Next,

$$
\begin{aligned}
x_i - X_i &= \overline{u}_i^{(2)} \overline{v}_i^{(3)} - \overline{u}_i^{(3)} \overline{v}_i^{(2)} - U_i^{(2)} V_i^{(3)} + U_i^{(3)} V_i^{(2)} \\
&= \left[\theta_2^{\Delta^1}(\xi_{i2}, s_i) + \alpha_{i2} \right] \left[\theta_3^{\Delta^2}(t_i, \eta_{i3}) + \beta_{i3} \right] \\
&\quad - \left[\theta_3^{\Delta^1}(\xi_{i3}, s_i) + \alpha_{i3} \right] \left[\theta_2^{\Delta^2}(t_i, \eta_{i2}) + \beta_{i2} \right] \\
&\quad - \theta_2^{\Delta^1}(\xi_i, \eta_i) \theta_3^{\Delta^2}(\xi_i, \eta_i) + \theta_3^{\Delta^1}(\xi_i, \eta_i) \theta_2^{\Delta^2}(\xi_i, \eta_i) \\
&= \theta_2^{\Delta^1}(\xi_{i2}, s_i) \theta_3^{\Delta^2}(t_i, \eta_{i3}) - \theta_2^{\Delta^1}(\xi_i, \eta_i) \theta_3^{\Delta^2}(\xi_i, \eta_i) \\
&\quad + \theta_3^{\Delta^1}(\xi_i, \eta_i) \theta_2^{\Delta^2}(\xi_i, \eta_i) - \theta_3^{\Delta^1}(\xi_{i3}, s_i) \theta_2^{\Delta^2}(t_i, \eta_{i2}) \\
&\quad + \alpha_{i2} \theta_3^{\Delta^2}(t_i, \eta_{i3}) + \left[\theta_2^{\Delta^1}(\xi_{i2}, s_i) + \alpha_{i2} \right] \beta_{i3} \\
&\quad - \alpha_{i3} \theta_2^{\Delta^2}(t_i, \eta_{i2}) - \left[\theta_3^{\Delta^1}(\xi_{i3}, s_i) + \alpha_{i3} \right] \beta_{i2}.
\end{aligned}
$$

Hence, taking into account (3.9), (3.13), and (3.14), we get

$$
\begin{aligned}
|x_i - X_i| &\leq \left| \theta_2^{\Delta^1}(\xi_{i2}, s_i) \theta_3^{\Delta^2}(t_i, \eta_{i3}) - \theta_2^{\Delta^1}(\xi_i, \eta_i) \theta_3^{\Delta^2}(\xi_i, \eta_i) \right| \\
&\quad + \left| \theta_3^{\Delta^1}(\xi_i, \eta_i) \theta_2^{\Delta^2}(\xi_i, \eta_i) - \theta_3^{\Delta^1}(\xi_{i3}, s_i) \theta_2^{\Delta^2}(t_i, \eta_{i2}) \right|
\end{aligned}
$$

(3.16)
$$+ C(\alpha_{i2} + 3\beta_{i3} + \alpha_{i3} + 3\beta_{i2}).$$

Further,

$$
\begin{aligned}
&\theta_2^{\Delta^1}(\xi_{i2}, s_i) \theta_3^{\Delta^2}(t_i, \eta_{i3}) - \theta_2^{\Delta^1}(\xi_i, \eta_i) \theta_3^{\Delta^2}(\xi_i, \eta_i) \\
&\quad = \theta_2^{\Delta^1}(\xi_{i2}, s_i) \left[\theta_3^{\Delta^2}(t_i, \eta_{i3}) - \theta_3^{\Delta^2}(\xi_i, \eta_i) \right] + \left[\theta_2^{\Delta^1}(\xi_{i2}, s_i) - \theta_2^{\Delta^1}(\xi_i, \eta_i) \right] \theta_3^{\Delta^2}(\xi_i, \eta_i)
\end{aligned}
$$

and

$$\left| \theta_3^{\Delta^2}(t_i, \eta_{i3}) - \theta_3^{\Delta^2}(\xi_i, \eta_i) \right| \leq N_{i3} - n_{i3}, \quad \left| \theta_2^{\Delta^1}(\xi_{i2}, s_i) - \theta_2^{\Delta^1}(\xi_i, \eta_i) \right| \leq M_{i2} - m_{i2}.$$

Therefore

$$\left| \theta_2^{\Delta^1}(\xi_{i2}, s_i) \theta_3^{\Delta^2}(t_i, \eta_{i3}) - \theta_2^{\Delta^1}(\xi_i, \eta_i) \theta_3^{\Delta^2}(\xi_i, \eta_i) \right| \leq C \left(N_{i3} - n_{i3} + M_{i2} - m_{i2} \right).$$

Similarly, from

$$
\begin{aligned}
&\theta_3^{\Delta^1}(\xi_i, \eta_i) \theta_2^{\Delta^2}(\xi_i, \eta_i) - \theta_3^{\Delta^1}(\xi_{i3}, s_i) \theta_2^{\Delta^2}(t_i, \eta_{i2}) \\
&\quad = \theta_3^{\Delta^1}(\xi_i, \eta_i) \left[\theta_2^{\Delta^2}(\xi_i, \eta_i) - \theta_2^{\Delta^2}(t_i, \eta_{i2}) \right] + \left[\theta_3^{\Delta^1}(\xi_i, \eta_i) - \theta_3^{\Delta^1}(\xi_{i3}, s_i) \right] \theta_2^{\Delta^2}(t_i, \eta_{i2}),
\end{aligned}
$$

we find that

$$\left| \theta_3^{\Delta^1}(\xi_i, \eta_i) \theta_2^{\Delta^2}(\xi_i, \eta_i) - \theta_3^{\Delta^1}(\xi_{i3}, s_i) \theta_2^{\Delta^2}(t_i, \eta_{i2}) \right| \le C \left(N_{i2} - n_{i2} + M_{i3} - m_{i3} \right).$$

Therefore, taking into account (3.13) and (3.14), we get from (3.16),

$$|x_i - X_i| \le 2C \left[M_{i2} - m_{i2} + M_{i3} - m_{i3} + 2(N_{i2} - n_{i2} + N_{i3} - n_{i3}) \right].$$

Hence

(3.17)

$$\sum_{i=1}^{k} |x_i - X_i| \, m(R_i) \le 2C \left\{ U(\theta_2^{\Delta^1}, P') - L(\theta_2^{\Delta^1}, P') + U(\theta_3^{\Delta^1}, P') - L(\theta_3^{\Delta^1}, P') \right.$$

$$\left. + 2 \left[U(\theta_2^{\Delta^2}, P') - L(\theta_2^{\Delta^2}, P') + U(\theta_3^{\Delta^2}, P') - L(\theta_3^{\Delta^2}, P') \right] \right\},$$

where U and L denote the upper and lower Darboux Δ-sums, respectively. Since the functions $\theta_p^{\Delta^1}, \theta_p^{\Delta^2}$ $(p = 1, 2, 3)$ are Δ-integrable over Ω, it follows from (3.17) that for given $\varepsilon > 0$ there is $\delta > 0$ such that

$$\sum_{i=1}^{k} |x_i - X_i| \, m(R_i) < \frac{\varepsilon}{12}$$

for every inner Δ-partition $P' = \{R_1, R_2, \ldots, R_k\}$ of Ω, determined by a partition $P \in \mathcal{P}_\delta(R)$.

Similarly we can show (diminishing δ if necessary) that

$$\sum_{i=1}^{k} |y_i - Y_i| \, m(R_i) < \frac{\varepsilon}{12} \quad \text{and} \quad \sum_{i=1}^{k} |z_i - Z_i| \, m(R_i) < \frac{\varepsilon}{12}.$$

Now (3.12) follows from (3.15).

By definition of the Δ-integral, diminishing δ if necessary, we may assume that for the same partitions P' for which (3.12) is satisfied we have

(3.18)
$$|\Lambda - I| < \frac{\varepsilon}{4}.$$

On the other hand, among the partitions P' for which (3.12) and (3.18) are satisfied, we can find a partition P' such that

(3.19)
$$|A(S, P') - A(S)| < \frac{\varepsilon}{2}.$$

Indeed, from Definition 3.2 it follows that there is an inner Δ-partition P_0' of Ω such that

(3.20)
$$0 \le A(S) - A(S, P_0') < \frac{\varepsilon}{2}.$$

Next, we refine the partition P_0' so that we get an inner Δ-partition P' which is determined by a partition $P \in \mathcal{P}_\delta(R)$. Then, by Lemma 3.3, $A(S, P') \ge A(S, P_0')$, and (3.20) yields

$$0 \le A(S) - A(S, P') < \frac{\varepsilon}{2},$$

so that (3.19) is shown.

Now using (3.12), (3.18), and (3.19), we get that

$$
\begin{aligned}
|A(S) - I| &= |A(S) - A(S, P') + A(S, P') - \Lambda + \Lambda - I| \\
&\leq |A(S) - A(S, P')| + |A(S, P') - \Lambda| + |\Lambda - I| \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \varepsilon.
\end{aligned}
$$

Hence, since $\varepsilon > 0$ is arbitrary, we obtain $A(S) = I$. This concludes the proof. $\square$

Remark 3.5. Let the surface S be given as the graph of a function $z = f(x, y)$, $(x, y) \in \Omega$, where Ω is a bounded and Jordan Δ-measurable subset of $\mathbb{T}_1^0 \times \mathbb{T}_2^0$ and f is continuous and has continuous first partial delta derivatives in the closure of Ω. Then Theorem 3.4 implies (by taking $x = t$, $y = s$, and $z = f(t, s)$) that the surface S is squarable, and its area $A(S)$ can be evaluated by the formula

$$
A(S) = \int \int_\Omega \sqrt{1 + [f^{\Delta^1}(x, y)]^2 + [f^{\Delta^2}(x, y)]^2} \Delta_1 x \Delta_2 y.
$$

4. SURFACE DELTA INTEGRALS

Let $\mathbb{T}_1$ and $\mathbb{T}_2$ be two time scales. For $i = 1, 2$, let σ_i and Δ_i denote the forward jump operator and the delta differentiation operator, respectively, on $\mathbb{T}_i$. Let $\Omega \subset \mathbb{T}_1^0 \times \mathbb{T}_2^0$ be a bounded and Jordan Δ-measurable set and S a time scale continuous surface defined by the parametric equations

(4.1) $x = \varphi(t, s), \quad y = \psi(t, s), \quad z = \chi(t, s), \quad (t, s) \in \Omega \subset \mathbb{T}_1^0 \times \mathbb{T}_2^0,$

where the functions φ, ψ, and χ are continuous and have continuous first partial delta derivatives in the closure of the region Ω. Next, let $h(x, y, z)$ be a function that is defined and continuous on the closure $\overline{S}$ of the suface S. This means that for each $(x_0, y_0, z_0) \in \overline{S}$ and each $\varepsilon > 0$ there exists $\delta > 0$ such that

$$
|h(x, y, z) - h(x_0, y_0, z_0)| < \varepsilon
$$

whenever $(x, y, z) \in \overline{S}$ and

$$
\sqrt{(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2} < \delta.
$$

Let $P' = \{R_1, R_2, \ldots, R_k\}$ be an arbitrary inner Δ-partition of Ω. Denote by A_i the area of the piece of the surface S corresponding to the piece R_i of Ω. By Theorem 3.4, the formula

(4.2) $A_i = \int \int_{R_i} \sqrt{EG - F^2} \Delta_1 t \Delta_2 s$

holds, where E, G, and F are defined by (3.5). Take any $(\xi_i, \eta_i) \in R_i$ for $i \in \{1, 2, \ldots, k\}$ and introduce the integral sum (Δ-integral sum)

$$(4.3) \qquad \Sigma = \sum_{i=1}^{k} h(\varphi(\xi_i, \eta_i), \psi(\xi_i, \eta_i), \chi(\xi_i, \eta_i)) A_i.$$

Definition 4.1. We say that a number I is the *surface Δ-integral* of the function h over the surface S if for each $\varepsilon > 0$ there exists $\delta > 0$ such that $|\Sigma - I| < \varepsilon$ for every integral sum Σ of h corresponding to any inner Δ-partition $P' = \{R_1, R_2, \ldots, R_k\}$ of Ω, determined by a partition $P \in \mathcal{P}_\delta(R)$, independent of the choice of the points $(\xi_i, \eta_i) \in R_i$ for $1 \le i \le k$, where $R = [a, b) \times [c, d) \subset \mathbb{T}_1 \times \mathbb{T}_2$ is a rectangle that contains Ω. We denote the number I, symbolically, by

$$(4.4) \qquad \int\!\!\int_S h(x, y, z) \Delta A.$$

The following theorem gives conditions sufficient for the existence of the surface Δ-integral.

Theorem 4.2. *Suppose that the surface S is given by the parametric equations in (4.1), where the region $\Omega \subset \mathbb{T}_1^0 \times \mathbb{T}_2^0$ is bounded and Jordan Δ-measurable, the functions φ, ψ, and χ are continuous and have continuous first partial delta derivatives in the closure $\overline{\Omega}$ of Ω, and the function h is continuous on the closure $\overline{S}$ of the surface S. Then the surface integral (4.4) exists and can be computed by*

$$(4.5) \qquad \int\!\!\int_S h(x, y, z) \Delta A = \int\!\!\int_\Omega h(\varphi(t, s), \psi(t, s), \chi(t, s)) \sqrt{EG - F^2} \Delta_1 t \Delta_2 s,$$

where E, G, and F are defined by (3.5).

Proof. First of all, note that the double Δ-integral on the right-hand side of formula (4.5) exists by virtue of Theorem 2.6 and continuity of the integrand. Let $R = [a, b) \times [c, d) \subset \mathbb{T}_1 \times \mathbb{T}_2$ be a rectangle that contains $\Omega \cup \partial_\Delta \Omega$. Take an arbitrary $\varepsilon > 0$. Since Ω is Jordan Δ-mesurable, there exists $\delta > 0$ such that for every partition $P \in \mathcal{P}_\delta(R)$ the sum of areas of subrectangles of P which have a common point with $\partial_\Delta \Omega$ is less than ε (see [2, Lemma 4.18]). Let $R_1, R_2, \ldots, R_k$ be all the subrectangles of the partition P that are entirely within Ω and $R_{k+1}, R_{k+2}, \ldots, R_N$ be all the subrectangles of P that are not entirely within Ω and each of which has a common point with $\partial_\Delta \Omega$ (refining the partition P, if necessary, we can assume that each subrectangle of P that has a common point with Ω belongs to one of these two types). The collection $P' = \{R_1, R_2, \ldots, R_k\}$ forms an inner Δ-partition of Ω and

$$\sum_{i=k+1}^{N} m(R_i) < \varepsilon.$$

Taking into account (4.2), we represent the integral sum (4.3) corresponding to this P' in the form

$$\Sigma = \sum_{i=1}^{k} H(\xi_i, \eta_i) \int\int_{R_i} \Phi(t, s) \Delta_1 t \Delta_2 s,$$

where

$$H(t, s) = h(\varphi(t, s), \psi(t, s), \chi(t, s)),$$
$$\Phi(t, s) = \sqrt{E(t, s)G(t, s) - F^2(t, s)},$$

for $(t, s) \in \Omega$, and $H(t, s) = 0$, $\Phi(t, s) = 0$, for $(t, s) \in R \setminus \Omega$.

Denote the double Δ-integral on the right-hand side of (4.5) by I and represent it in the form

$$I = \int\int_{\Omega} H(t, s)\Phi(t, s)\Delta_1 t \Delta_2 s = \int\int_{R} H(t, s)\Phi(t, s)\Delta_1 t \Delta_2 s$$

$$= \sum_{i=1}^{k} \int\int_{R_i} H(t, s)\Phi(t, s)\Delta_1 t \Delta_2 s + \sum_{i=k+1}^{N} \int\int_{R_i} H(t, s)\Phi(t, s)\Delta_1 t \Delta_2 s.$$

Let us estimate the difference

$$\Sigma - I = \sum_{i=1}^{k} \int\int_{R_i} \left[H(\xi_i, \eta_i) - H(t, s) \right] \Phi(t, s)\Delta_1 t \Delta_2 s$$

$$- \sum_{i=k+1}^{N} \int\int_{R_i} H(t, s)\Phi(t, s)\Delta_1 t \Delta_2 s.$$

We have

$$|\Sigma - I| \leq \sum_{i=1}^{k} \int\int_{R_i} |H(\xi_i, \eta_i) - H(t, s)| \Phi(t, s)\Delta_1 t \Delta_2 s$$

(4.6)
$$+ \sum_{i=k+1}^{N} \int\int_{R_i} |H(t, s)| \Phi(t, s)\Delta_1 t \Delta_2 s.$$

Under the conditions imposed on h and φ, ψ, χ, the functions $H(t, s)$ and $\Phi(t, s)$ are continuous on $\overline{\Omega}$. Consequently, these functions are bounded and uniformly continuous on $\overline{\Omega}$. Therefore

$$\sup \left\{ |H(t, s)| \, \Phi(t, s) : \ (t, s) \in \Omega \right\} =: M < \infty,$$

and for the above given $\varepsilon > 0$ and $\delta > 0$ (we can diminish δ if necessary),

(4.7)
$$(t, s), (t', s') \in \overline{\Omega} \quad \text{and} \quad |t - t'| \leq \delta, \ |s - s'| \leq \delta$$

imply

(4.8)
$$|H(t, s) - H(t', s')| < \varepsilon.$$

Therefore

$$(4.9) \qquad \sum_{i=k+1}^{N} \int\!\!\int_{R_i} |H(t,s)|\,\Phi(t,s)\Delta_1 t\Delta_2 s \le M \sum_{i=k+1}^{N} m(R_i) < M\varepsilon.$$

Further, let $R_i = [t_i, t_i') \times [s_i, s_i')$ for $i \in \{1, 2, \ldots, k\}$. We can write

$$\sum_{i=1}^{k} \int\!\!\int_{R_i} |H(\xi_i, \eta_i) - H(t,s)|\,\Phi(t,s)\Delta_1 t\Delta_2 s$$

$$= \sum_{\substack{t_i'-t_i\le\delta \\ s_i'-s_i\le\delta}} \int\!\!\int_{R_i} |H(\xi_i, \eta_i) - H(t,s)|\,\Phi(t,s)\Delta_1 t\Delta_2 s$$

$$+ \sum_{\substack{t_i'-t_i\le\delta \\ s_i'-s_i>\delta}} \int\!\!\int_{R_i} |H(\xi_i, \eta_i) - H(t,s)|\,\Phi(t,s)\Delta_1 t\Delta_2 s$$

$$+ \sum_{\substack{t_i'-t_i>\delta \\ s_i'-s_i\le\delta}} \int\!\!\int_{R_i} |H(\xi_i, \eta_i) - H(t,s)|\,\Phi(t,s)\Delta_1 t\Delta_2 s$$

$$+ \sum_{\substack{t_i'-t_i>\delta \\ s_i'-s_i>\delta}} \int\!\!\int_{R_i} |H(\xi_i, \eta_i) - H(t,s)|\,\Phi(t,s)\Delta_1 t\Delta_2 s.$$

Next, if $t_i' - t_i \le \delta$ and $s_i' - s_i \le \delta$, then taking into account (4.7), (4.8) and that $(\xi_i, \eta_i) \in R_i$, we have

$$\sum_{\substack{t_i'-t_i\le\delta \\ s_i'-s_i\le\delta}} \int\!\!\int_{R_i} |H(\xi_i, \eta_i) - H(t,s)|\,\Phi(t,s)\Delta_1 t\Delta_2 s$$

$$< \varepsilon \sum_{\substack{t_i'-t_i\le\delta \\ s_i'-s_i\le\delta}} \int\!\!\int_{R_i} \Phi(t,s)\Delta_1 t\Delta_2 s$$

$$\le \varepsilon \sum_{i=1}^{k} \int\!\!\int_{R_i} \Phi(t,s)\Delta_1 t\Delta_2 s \le \varepsilon A(S).$$

If $t_i' - t_i \le \delta$ and $s_i' - s_i > \delta$, then $s_i' = \sigma_2(s_i)$, $\eta_i = s_i$ and hence, using also (4.7), (4.8),

$$\int\!\!\int_{R_i} |H(\xi_i, \eta_i) - H(t,s)|\,\Phi(t,s)\Delta_1 t\Delta_2 s$$

$$= \int_{s_i}^{\sigma_2(s_i)} \int_{t_i}^{t_i'} |H(\xi_i, s_i) - H(t,s)|\,\Phi(t,s)\Delta_1 t\Delta_2 s$$

$$= \int_{t_i}^{t_i'} |H(\xi_i, s_i) - H(t, s_i)|\,\Phi(t, s_i)\,[\sigma_2(s_i) - s_i]\,\Delta_1 t$$

$$< \varepsilon \int_{t_i}^{t_i'} \Phi(t, s_i)\,[\sigma_2(s_i) - s_i]\,\Delta_1 t = \varepsilon \int\!\!\int_{R_i} \Phi(t,s)\Delta_1 t\Delta_2 s$$

and therefore,

$$\sum_{\substack{t_i'-t_i\leq\delta \\ s_i'-s_i>\delta}} \int\int_{R_i} |H(\xi_i,\eta_i) - H(t,s)|\,\Phi(t,s)\Delta_1 t\Delta_2 s$$

$$< \varepsilon \sum_{\substack{t_i'-t_i\leq\delta \\ s_i'-s_i>\delta}} \int\int_{R_i} \Phi(t,s)\Delta_1 t\Delta_2 s$$

$$\leq \varepsilon \sum_{i=1}^{k} \int\int_{R_i} \Phi(t,s)\Delta_1 t\Delta_2 s \leq \varepsilon A(S).$$

If $t_i' - t_i > \delta$ and $s_i' - s_i \leq \delta$, then $t_i' = \sigma_1(t_i)$, $\xi_i = t_i$ and hence, using also (4.7), (4.8),

$$\int\int_{R_i} |H(\xi_i,\eta_i) - H(t,s)|\,\Phi(t,s)\Delta_1 t\Delta_2 s$$

$$= \int_{s_i}^{s_i'} \int_{t_i}^{\sigma_1(t_i)} |H(t_i,\eta_i) - H(t,s)|\,\Phi(t,s)\Delta_1 t\Delta_2 s$$

$$= \int_{s_i}^{s_i'} |H(t_i,\eta_i) - H(t_i,s)|\,\Phi(t_i,s)\left[\sigma_1(t_i) - t_i\right]\Delta_2 s$$

$$< \varepsilon \int_{s_i}^{s_i'} \Phi(t_i,s)\left[\sigma_1(t_i) - t_i\right]\Delta_2 s = \varepsilon \int\int_{R_i} \Phi(t,s)\Delta_1 t\Delta_2 s$$

and therefore,

$$\sum_{\substack{t_i'-t_i>\delta \\ s_i'-s_i\leq\delta}} \int\int_{R_i} |H(\xi_i,\eta_i) - H(t,s)|\,\Phi(t,s)\Delta_1 t\Delta_2 s$$

$$< \varepsilon \sum_{\substack{t_i'-t_i>\delta \\ s_i'-s_i\leq\delta}} \int\int_{R_i} \Phi(t,s)\Delta_1 t\Delta_2 s$$

$$\leq \varepsilon \sum_{i=1}^{k} \int\int_{R_i} \Phi(t,s)\Delta_1 t\Delta_2 s \leq \varepsilon A(S).$$

Finally, if $t_i' - t_i > \delta$ and $s_i' - s_i > \delta$, then $t_i' = \sigma_1(t_i)$, $s_i' = \sigma_2(s_i)$, $\xi_i = t_i$, $\eta_i = s_i$, and hence

$$\int\int_{R_i} |H(\xi_i,\eta_i) - H(t,s)|\,\Phi(t,s)\Delta_1 t\Delta_2 s$$

$$= |H(t_i,s_i) - H(t_i,s_i)|\,\Phi(t_i,s_i)\left[\sigma_1(t_i) - t_i\right]\left[\sigma_2(s_i) - s_i\right] = 0$$

and therefore,

$$\sum_{\substack{t_i'-t_i>\delta \\ s_i'-s_i>\delta}} \int\int_{R_i} |H(\xi_i,\eta_i) - H(t,s)|\,\Phi(t,s)\Delta_1 t\Delta_2 s = 0.$$

Thus

$$(4.10) \qquad \sum_{i=1}^{k} \int \int_{R_i} |H(\xi_i, \eta_i) - H(t, s)| \, \Phi(t, s) \Delta_1 t \Delta_2 s < 3\varepsilon A(S).$$

Substituting (4.9) and (4.10) in (4.6), we get

$$|\Sigma - I| < [3A(S) + M]\, \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, this completes the proof. $\qquad\qquad\square$

5. CONCLUDING REMARKS

1. As is known (see [2, 4]), there are four kinds of time scale double integrals. Accordingly, four kinds of time scale surface integrals can be defined:

(i) surface $\Delta\Delta$-integral, which is defined by using partitions consisting of subrectangles of the form $[\alpha, \beta) \times [\gamma, \delta)$;

(ii) surface $\nabla\nabla$-integral, which is defined by using partitions consisting of subrectangles of the form $(\alpha, \beta] \times (\gamma, \delta]$;

(iii) surface $\Delta\nabla$-integral, which is defined by using partitions consisting of subrectangles of the form $[\alpha, \beta) \times (\gamma, \delta]$;

(iv) surface $\nabla\Delta$-integral, which is defined by using partitions consisting of subrectangles of the form $(\alpha, \beta] \times [\gamma, \delta)$.

For brevity, we call the first surface integral simply the surface Δ-integral, and in this paper we have dealt solely with such surface Δ-integrals. Note that for the same function and the same surface, the above four kinds of surface integrals are in general different from each other.

2. The area of the same time scale surface S given by (3.1) can be computed by using any one of the four integrals

$$A(S) = \int \int_{\Omega} \left| \overrightarrow{r}^{\Delta_1} \times \overrightarrow{r}^{\Delta_2} \right| \Delta_1 t \Delta_2 s = \int \int_{\Omega} \left| \overrightarrow{r}^{\nabla_1} \times \overrightarrow{r}^{\nabla_2} \right| \nabla_1 t \nabla_2 s$$

$$= \int \int_{\Omega} \left| \overrightarrow{r}^{\Delta_1} \times \overrightarrow{r}^{\nabla_2} \right| \Delta_1 t \nabla_2 s = \int \int_{\Omega} \left| \overrightarrow{r}^{\nabla_1} \times \overrightarrow{r}^{\Delta_2} \right| \nabla_1 t \Delta_2 s$$

provided that the conditions of Theorem 3.4 are satisfied accordingly.

3. The problem of independence of surface area from the parametrization of the surface is proved in the classical case by using the change of variable formula for double integrals (see, for example, [10, 11]). However, for time scale double integrals, no change of variable formula has been worked out yet. Further, above in Section 4, we have considered only the so-called Δ-surface integrals of the first type. The Δ-surface integrals of the second type as well as time scale analogues of the classical Gauss's divergence theorem and Stokes's theorem await their investigation.

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