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MEASURE OF NONCOMPACTNESS AND PARTIAL DIFFERENTIAL EQUATIONS INVOLVING RIEMANN-LIOUVILLE FRACTIONAL DERIVATIVE

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ABSTRACT. In this paper, we prove the existence of mild solutions for an initial value problem for a semilinear differential equation involving the Riemann-Liouville fractional derivative. The technique relies on the concept of measures of noncompactness and Mönch's fixed point theorem.

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1. INTRODUCTION

Recently, much attention has been paid to the existence of solutions for fractional differential equations due to its wide application in engineering, electrochemistry, economics and other fields, see for instance the monographs of Kilbas *et al* [31], Lakshmikantham *et al.* [33], Miller and Ross [37], Podlubny [41] and the papers of Agarwal *et al* [1], Benchohra *et al* [13, 14], Chang and Nieto [18], Delbosco and Rodino [19], Diethelm *et al* [20], Furati and Tatar [21, 22], Gaul *et al.* [23], Glockle and Nonnenmacher [24], Lakshmikantham and Devi [34], Mainardi [35], Metzler *et al.* [36], Yu and Gao [46] and the references therein. Jaradat *et al.* [30] studied the existence and uniqueness of mild solutions for a class of initial value problem for a semilinear integrodifferential equation involving the Caputo's fractional derivative.

Functional differential and partial differential equations arise in many areas of applied mathematics and such equations have received an increasing interest in recent years. A good guide to the literature for functional differential equations is the books by Hale [26] and Hale and Verduyn Lunel [27], Kolmanovskii and Myshkis [32] and Wu [45] and the references therein.

In a series of papers (see [9, 10, 15, 16]) the authors considered some classes of initial value problems for functional differential equations involving the Riemann-Liouville and Caputo fractional derivatives of order $0 < r \leq 1$. In [11, 12] some classes of semilinear functional differential equations involving the Riemann-Liouville have

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been considered. For more details on the geometric and physical interpretation for fractional derivatives of both the Riemann-Liouville and Caputo types see [28, 42].

In this paper we consider a semilinear functional differential equation of fractional order of the form

$$D^r y(t) = Ay(t) + f(t, y_t), \quad t \in J = [0, b], \quad 0 < r \leq 1 \tag{1.1}$$

$$y(t) = \phi(t), \quad t \in [-\rho, 0], \tag{1.2}$$

where D^r is the standard Riemann-Liouville fractional derivative, $f : J \times C([-\rho, 0], E) \to E$ is a given function, A is a closed linear operator (possibly unbounded), $\phi : [-\rho, 0] \to E$ a given continuous function with $\phi(0) = 0$ and $(E, |\cdot|)$ a Banach space. For any function y defined on $[-\rho, b]$ and any $t \in J$ we denote by y_t the element of $C([-\rho, 0], E)$ defined by

$$y_t(\theta) = y(t + \theta), \quad \theta \in [-\rho, 0].$$

Here $y_t(\cdot)$ represents the history of the state from time $t - \rho$, up to the present time t.

The reason for studying equation (1.1) is that it appear in mathematical models of viscoelasticity [43], and in other fields of science [29, 42]. Equation (1.1) is equivalent to solve an integral equation of convolution type. It is also of interest to explore the neighborhood of the diffusion ($r = 1$).

In this paper we use the technique of measures of noncompactness. It is well known that this method provides an excellent tool for obtaining existence of solutions of nonlinear differential equations. this technique works fruitfully for both integral and differential equations, details are found in Akhmerov *et al.* [3], Benchohra *et al.* [17], Alvàrez [4], Banaś *et al.* [5, 6, 7, 8], Guo *et al.* [25], Mönch [38], Mönch and Von Harten [39], and Szufla [44]. As far as we know there is a few number of papers related to fractional differential equations on Banach spaces (see [17, 34]). This paper initiates the application of the measure of noncompactness to semilinear fractional differential equations with finite delay.

2. PRELIMINARIES

For application in what follows, we first state the following definitions, lemmas and some notations. Denote by $C(J, E)$ the Banach space of continuous functions $J \to E$, with the usual supremum norm

$$\|y\|_\infty = \sup\{|y(t)|, \ t \in J\}.$$

For $\psi \in C([-\rho, 0], E)$ the norm of ψ is defined by

$$\|\psi\|_{\mathcal{C}} = \sup\{|\psi(\theta)|, \ \theta \in [-\rho, 0]\}.$$

$B(E)$ denotes the Banach space of bounded linear operators from E into E, with norm

$$\|N\|_{B(E)} = \sup\{|N(y)| : |y| = 1\}.$$

Let $L^1(J, E)$ be the Banach space of measurable functions $y : J \to E$ which are Bochner integrable, equipped with the norm

$$\|y\|_{L^1} = \int_J |y(t)|dt.$$

In all our paper we suppose that the operator $A : D(A) \subset E \to E$ is the infinitesimal generator of a C_0-semigroup $\{T(t)\}_{t\geq 0}$. Denote by

$$M = \sup\{\|T(t)\|_{B(E)} : t \in J\}.$$

For a given set V of functions $v : [-\rho, b] \to E$, let us denote by

$$V(t) = \{v(t) : v \in V\}, \quad t \in [-\rho, b]$$

and

$$V(J) = \{v(t) : v \in V,\ t \in [-\rho, b]\}.$$

Now let us recall some fundamental facts of the notion of Kuratowski measure of noncompactness.

Definition 2.1 ([6])**.** Let E be a Banach space and Ω_E the bounded subsets of E. The Kuratowski measure of noncompactness is the map $\alpha : \Omega_E \to [0, \infty]$ defined by

$$\alpha(B) = \inf\{\epsilon > 0 : B \subseteq \cup_{i=1}^{n} B_i \ \text{ and } \operatorname{diam}(B_i) \leq \epsilon\}; \quad \text{here} \quad B \in \Omega_E.$$

Properties: The Kuratowski measure of noncompactness satisfies some properties (for more details see [6])

(a) $\alpha(B) = 0 \Leftrightarrow \overline{B}$ is compact (B is relatively compact).
(b) $\alpha(B) = \alpha(\overline{B})$.
(c) $A \subset B \Rightarrow \alpha(A) \leq \alpha(B)$.
(d) $\alpha(A + B) \leq \alpha(A) + \alpha(B)$
(e) $\alpha(cB) = |c|\alpha(B)$; $c \in \mathbb{R}$.
(f) $\alpha(\text{conv } B) = \alpha(B)$.

For completeness we recall the definition of Riemann-Liouville fractional primitive and fractional derivative.

Definition 2.2 ([31, 41])**.** The Riemann-Liouville fractional primitive of order $r > 0$ of a function $h : (0, b] \to E$ is defined by

$$I_0^r h(t) = \frac{1}{\Gamma(r)} \int_0^t (t - s)^{r-1} h(s) ds,$$

provided the right side is pointwise defined on $(0, b]$, and where Γ is the gamma function.

Definition 2.3 ([31, 41])**.** The Riemann-Liouville fractional derivative of order $r \in (0,1]$ of a continuous function $h : (0,b] \to E$ is defined by

$$\begin{aligned} \frac{d^r h(t)}{dt^r} &= \frac{1}{\Gamma(1-r)} \frac{d}{dt} \int_0^t (t-s)^{-r} h(s)ds \\ &= \frac{d}{dt} I_0^{1-r} h(t). \end{aligned}$$

Definition 2.4. A map $f : J \times C([-\rho,0],E) \to E$ is said to be Carathéodory if

(i) $t \longmapsto f(t,u)$ is measurable for each $u \in C([-\rho,0],E)$;
(ii) $u \longmapsto f(t,u)$ is continuous for almost all $t \in J$.

For our purpose we will only need the following fixed point theorem, and the important Lemma.

Theorem 2.5 ([2, 38])**.** *Let D be a bounded, closed and convex subset of a Banach space such that $0 \in D$, and let N be a continuous mapping of D into itself. If the implication*

$$V = \overline{\mathrm{conv}} N(V) \quad or \quad V = N(V) \cup \{0\} \Rightarrow \alpha(V) = 0$$

holds for every subset V of D, then N has a fixed point.

Lemma 2.6 ([44])**.** *Let D be a bounded, closed and convex subset of the Banach space $C(J,E)$, G a continuous function on $J \times J$ and f a function from $J \times C([-\rho,0],E) \to E$ which satisfies the Carathéodory conditions and there exists $p \in L^1(J,\mathbb{R}_+)$ such that for each $t \in J$ and each bounded set $B \subset C([-\rho,0],E)$ we have*

$$\lim_{k\to 0^+} \alpha(f(J_{t,k} \times B)) \le p(t)\alpha(B); \quad here \quad J_{t,k} = [t-k,t] \cap J.$$

If V is an equicontinuous subset of D, then

$$\alpha\left(\left\{\int_J G(s,t) f(s,y_s)ds : y \in V\right\}\right) \le \int_J \|G(t,s)\| p(s)\alpha(V(s))ds.$$

3. THE MAIN RESULT

Before stating our main result in this section for problem (1.1)–(1.2) we give the definition of the mild solution.

Definition 3.1 ([30])**.** We say that a continuous function $y : [-\rho,b] \to E$ is a mild solution of problem (1.1)–(1.2) if $y(t) = \phi(t)$, $t \in [-\rho,0]$, and

$$y(t) = \frac{1}{\Gamma(r)} \int_0^t (t-s)^{r-1} T(t-s) f(s,y_s)ds, \quad t \in J.$$

Let us list some conditions on the functions involved in the IVP (1.1)–(1.2).

(H1) The semigroup $\{T(t)\}_{t\in J}$ is compact for $t > 0$.
(H2) $f : J \times C([-\rho,0],E) \to E$ satisfies the Carathéodory conditions.

(H3) There exists a function $p \in C(J, \mathbb{R}_+)$ such that

$$|f(t,u)| \leq p(t)(\|u\|_C + 1), \quad \text{for each} \quad t \in J, \quad \text{and each} \quad u \in C([-\rho, 0], E).$$

(H4) For each $t \in J$ and each bounded set $B \subset C([-\rho, 0], E)$ we have

$$\lim_{h \to 0^+} \alpha(f(J_{t,h} \times B)) \leq p(t)\alpha(B); \quad \text{here} \quad J_{t,h} = [t-h, t] \cap J.$$

Let $p^* = \sup_{t \in J} p(t)$. Our main result reads as follows

Theorem 3.2. *Assume that assumptions* (H1)–(H4) *hold. If*

$$\frac{Mp^* b^r}{\Gamma(r+1)} < 1, \tag{3.1}$$

then the the problem (1.1)–(1.2) *has at least one mild solution.*

Proof. We shall reduce the existence of solutions of (1.1)–(1.2) to a fixed point problem. Consider the operator $N : C([-\rho, b], E) \to C([-\rho, b], E)$ defined by

$$N(y)(t) = \begin{cases} \phi(t), & t \in [-\rho, 0], \\ \frac{1}{\Gamma(r)} \int_0^t (t-s)^{r-1} T(t-s) f(s, y_s) ds, & t \in [0, b]. \end{cases}$$

Clearly, the fixed points of the operator N are solution of the problem (1.1)–(1.2). Let $r_0 > 0$ be such that

$$r_0 \geq \frac{Mp^* b^r}{\Gamma(r+1) - Mp^* b^r},$$

and consider the set

$$D_{r_0} = \{y \in C([-\rho, b], E) : \|y\|_\infty \leq r_0\}$$

Clearly, the subset D_{r_0} is closed, bounded and convex. We shall show that N satisfies the assumptions of Theorem 2.5. The proof will be given in three steps.

Step 1: *N is continuous.*

Let us consider a sequence $\{y_n\}$ such that $y_n \to y$ in $C([-\rho, b], E)$. Then for each $t \in J$

$$\begin{aligned} |N(y_n)(t) - N(y)(t)| &\leq \left| \frac{1}{\Gamma(r)} \int_0^t (t-s)^{r-1} T(t-s)[f(s, y_{n_s}) - f(s, y_s)] ds \right| \\ &\leq \frac{Mb^r}{\Gamma(r+1)} \|f(., y_{n.}) - f(., y.)\|_\infty. \end{aligned}$$

Since f is of Carathéodory type, then by the Lebesgue dominated convergence theorem we have

$$\|N(y_n) - N(y)\|_\infty \leq \frac{Mb^r}{\Gamma(r+1)} \|f(., y_{n.}) - f(., y.)\|_\infty \to 0 \quad as \quad n \mapsto \infty.$$

Step 2: *N maps D_{r_0} into itself.*

For each $y \in D_{r_0}$, by $(H3)$ and (3.1) we have for each $t \in J$

$$\begin{aligned} |N(y)(t)| &= \left|\frac{1}{\Gamma(r)}\int_0^t (t-s)^{r-1}T(t-s)f(s,y_s)ds\right| \\ &\leq \frac{Mp^*(r_0+1)}{\Gamma(r)}\int_0^t (t-s)^{r-1}ds \\ &\leq \frac{Mb^r p^*(r_0+1)}{\Gamma(r+1)} \leq r_0. \end{aligned}$$

Step 3: *$N(D_{r_0})$ is bounded and equicontinuous.*

By Step 2, it is obvious that $N(D_{r_0}) \subset D_{r_0}$ is bounded. For the equicontinuity of $N(D_{r_0})$. Let $\tau_1, \tau_2 \in J$, $\tau_1 < \tau_2$. Thus if $\epsilon > 0$ and $\epsilon \leq \tau_1 \leq \tau_2$ we have for any $y \in D_{r_0}$

$$\begin{aligned} |N(y)(\tau_2) - N(y)(\tau_1)| \leq& \frac{1}{\Gamma(r)}\left|\int_0^{\tau_1-\epsilon} \left[(\tau_2-s)^{r-1}T(\tau_2-s)\right.\right. \\ &- \left.\left.(\tau_1-s)^{r-1}T(\tau_1-s)\right] f(s,y_s)ds\right| \\ &+ \frac{1}{\Gamma(r)}\left|\int_{\tau_1-\epsilon}^{\tau_1} \left[(\tau_2-s)^{r-1}T(\tau_2-s)\right.\right. \\ &- \left.\left.(\tau_1-s)^{r-1}T(\tau_1-s)\right] f(s,y_s)ds\right| \\ &+ \frac{1}{\Gamma(r)}\left|\int_{\tau_1}^{\tau_2} (\tau_2-s)^{r-1}T(\tau_2-s)f(s,y_s)ds\right| \\ \leq& \frac{Mp^*(r_0+1)}{\Gamma(r)}\left(\left|\int_0^{\tau_1-\epsilon} \left[(\tau_2-s)^{r-1} - (\tau_1-s)^{r-1}\right] ds\right|\right. \\ &+ \left|\int_0^{\tau_1-\epsilon} (\tau_2-s)^{r-1}T(\tau_1-\epsilon-s)\left(T(\tau_2-\tau_1-\epsilon) - T(\epsilon)\right) ds\right| \\ &+ \int_{\tau_1-\epsilon}^{\tau_1} \left((\tau_2-s)^{r-1} - (\tau_1-s)^{r-1}\right) ds \\ &+ \left.\int_{\tau_1}^{\tau_2} (\tau_2-s)^{r-1}ds\right) \\ \leq& \frac{Mp^*(r_0+1)}{\Gamma(r)}\left(\int_0^{\tau_1-\epsilon} \left[(\tau_2-s)^{r-1} - (\tau_1-s)^{r-1}\right] ds\right. \\ &+ \|T(\tau_2-\tau_1-\epsilon) - T(\epsilon)\|_{B(E)} \int_0^{\tau_1-\epsilon} (\tau_2-s)^{r-1}ds \\ &+ \int_{\tau_1-\epsilon}^{\tau_1} \left((\tau_2-s)^{r-1} - (\tau_1-s)^{r-1}\right) ds \\ &+ \left.\int_{\tau_1}^{\tau_2} (\tau_2-s)^{r-1}ds\right). \end{aligned}$$

As $\tau_1 \to \tau_2$ and ϵ sufficiently small, the right-hand side of the above inequality tends to zero, since $T(t)$ is a strongly continuous operator and the compactness of $T(t)$ for $t > 0$ implies the continuity in the uniform operator topology ([40]).

Now let V be a subset of D_{r_0} such that $V \subset \overline{conv}(N(V) \cup \{0\})$. V is bounded and equicontinuous and therefore the function $v \to v(t) = \alpha(V(t))$ is continuous on $[-\rho, b]$. By (H4), Lemma 2.6 and the properties of the measure α we have for each $t \in [-\rho, b]$

$$\begin{aligned} v(t) &\leq \alpha(N(V)(t) \cup \{0\}) \leq \alpha(N(V)(t)) \\ &\leq \frac{1}{\Gamma(r)} \int_0^t (t-s)^{r-1} \|T(t-s)\|_{B(E)} p(s) \alpha(V(s)) ds \\ &\leq \frac{M}{\Gamma(r)} \int_0^t (t-s)^{r-1} p(s) v(s) \leq \|v\|_\infty \frac{Mp^* b^r}{\Gamma(r+1)}. \end{aligned}$$

This means that

$$\|v\|_\infty \Big(1 - \frac{Mp^* b^r}{\Gamma(r+1)}\Big) \leq 0.$$

By (3.1) it follows that $\|v\|_\infty = 0$, that is $v(t) = 0$ for each $t \in [-\rho, b]$, and then $V(t)$ is relatively compact in E. In view of the Ascoli-Arzelà theorem, V is relatively compact in D_{r_0}. Applying now Theorem 2.5 we conclude that N has a fixed point which is a mild solution for the problem (1.1)–(1.2). $\square$

3.1. **AN EXAMPLE.** As an application of our results we consider the following partial functional differential equation of the form

$$\begin{cases} \frac{\partial^r}{\partial t^r} z(t,x) = \frac{\partial^2}{\partial x^2} z(t,x) + Q(t, z(t-\rho, x)), \\ x \in [0,\pi], \quad t \in [0,1], \quad r \in (0,1], \end{cases} \tag{3.2}$$

$$z(t,0) = z(t,\pi) = 0, \quad t \in [0,1] \tag{3.3}$$

$$z(t,x) = \phi(t,x), \quad t \in [-\rho, 0], \quad x \in [0,\pi], \tag{3.4}$$

where $\rho > 0$, $\phi : [-\rho, 0] \times [0,\pi] \to \mathbb{R}$ is continuous and $Q : [0,1] \times \mathbb{R} \to \mathbb{R}$ is a given function.

Let

$$y(t)(x) = z(t,x), \quad t \in [0,1], \quad x \in [0,\pi],$$

$$F(t,\phi)(x) = Q(t, \phi(\theta, x)), \quad \theta \in [-\rho, 0], \quad x \in [0,\pi],$$

$$\phi(\theta)(x) = \phi(\theta, x), \quad \theta \in [-\rho, 0], \quad x \in [0,\pi].$$

Take $E = L^2[0,\pi]$ and define $A : D(A) \subset E \to E$ by $Aw = w''$ with domain

$$D(A) = \{w \in E, w, w' \text{ are absolutely continuous, } w'' \in E, w(0) = w(\pi) = 0\}.$$

Then

$$Aw = \sum_{n=1}^{\infty} n^2 (w, w_n) w_n, \quad w \in D(A)$$

where (,) is the inner product in L^2 and $w_n(s) = \sqrt{\frac{2}{\pi}} \sin ns$, $n = 1, 2, \dots$ is the orthogonal set of eigenvectors in A. It is well known (see [40]) that A is the infinitesimal generator of an analytic semigroup $T(t)$, $t \in [0, 1]$ in E and is given by

$$T(t)w = \sum_{n=1}^{\infty} \exp(-n^2 t)(w, w_n) w_n, \quad w \in E.$$

Since the analytic semigroup $T(t)$ is compact, there exists a constant $M \geq 1$ such that

$$\|T(t)\|_{B(E)} \leq M.$$

Also assume that there exists $\sigma \in C[0, 1], \mathbb{R}^+)$ with $\frac{M\sigma^*}{\Gamma(r+1)} < 1$ such that

$$|Q(t, w(t - r, x))| \leq \sigma(t)(|w| + 1),$$

where $\sigma^* = \sup_{t\in[0,1]} \sigma(t)$.

We can show that problem (1.1)–(1.2) is an abstract formulation of problem (3.2)–(3.4). Since all the conditions of Theorem 3.2 are satisfied, the problem (3.2)–(3.4) has a solution z on $[-\rho, 1] \times [0, \pi]$.

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