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IMPLICIT DIFFERENCE METHODS FOR PARABOLIC FDE ON CYLINDRICAL DOMAINS

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ABSTRACT. Implicit difference schemes for quasilinear parabolic functional differential equations are presented. Ben- efits of implicit methods are pointed. The attention is focused here on cylindrical domains. Operators approximating mixed derivatives on irregular grids are introduced. A complete convergence analysis for methods is presented. Nonlinear estimates of the Perron type for given functions with respect to functional variables are used. Results obtained in the paper can be applied to differential integral problems and to equations with deviated variables. Numerical examples display the results of our investigations.

Key words: functional differential equations, implicit difference

1. INTRODUCTION

Parabolic functional differential equations have the following property: difference schemes for suitable initial boundary value problems are obtained by replacing partial derivatives with difference expressions. Moreover, because differential equations contain functional variables, some interpolating operators are needed. Then we obtain difference functional initial boundary value problems which satisfy consistency conditions on classical solutions of original problems. Methods of difference inequalities and simple theorems on recurrent inequalities are used in the investigation of the stability of nonlinear difference functional equations generated by parabolic problems.

The papers [7]–[9] initiated investigations of implicit difference schemes for nonlinear parabolic equations. Classical solutions of initial boundary value problems of the Dirichlet type for nonlinear equations without mixed derivatives are approximated in [7], [8] by solutions of difference schemes which are implicit with respect to the time variable. The paper [9] deals with initial boundary value problems of the Neumann type for nonlinear equation with mixed derivatives.

Semilinear parabolic equations with initial boundary conditions of the Dirichlet type were considered in [18]. It is shown that there are implicit difference schemes

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which are convergent. Classical solutions of quasilinear parabolic differential functional equations and implicit difference methods on rectangular domains are investigated in [3].

High order implicit difference methods for parabolic differential equations without mixed derivatives are considered in [11]–[13]. In all those papers authors consider rectangular domains with regular cubic or square space grids (space variable are two- or three- dimensional). Rectangular domain with non-uniform space grids is considered in one-dimensional case in [10].

Implicit difference functional inequalities generated by nonlinear parabolic differential functional equations were investigated in [6].

Various monotone iterative methods and finite difference schemes for computing of numerical solutions of reaction diffusion equations with time delay were presented in [14]–[16]. The present paper is motivated by those articles.

That following system of parabolic equations with time delays was investigated in [15]. Let us suppose that $\Omega \in \mathbb{R}^n$ is a bounded domain with the boundary $\partial\Omega$ which is of class C^1. The paper concerns the system of nonlinear parabolic equations with time delay:

$$\partial_t u^{(i)} - L^{(i)} u^{(i)} = f^{(i)}(t, x, \mathbf{u}, \mathbf{u}_\tau), \quad x \in \Omega, \quad t \in (0, T], \quad i = 1, \ldots, N$$

and with the initial boundary condition

$$B^{(i)} u^{(i)} = g^{(i)}(t, x), \quad x \in \partial\Omega, \quad t \in (0, T], \quad i = 1, \ldots, N,$$

$$u^{(i)}(t, x) = \psi^{(i)}(t, x), \quad x \in \Omega, -\tau_i \le t \le 0, \quad t \in (0, T], \quad i = 1, \ldots, N$$

where $\mathbf{u} = (u^{(1)}(t, x), \ldots, u^{(N)}(t, x))$, $\mathbf{u}_\tau = (u^{(1)}(t - \tau_1, x), \ldots, u^{(N)}(t - \tau_N, x))$. The operators $L^{(i)}$ and $B^{(i)}$ are given by

$$L^{(i)} u^{(i)} = \nabla \cdot (D^{(i)} \nabla u^{(i)}) + \mathbf{v}^{(i)} \cdot \nabla u^{(i)}, \quad i = 1, \ldots, N,$$

$$B^{(i)} u^{(i)} = \alpha^{(i)} \partial u_\nu^{(i)} + \beta^{(i)} u^{(i)}, \quad i = 1, \ldots, N.$$

In the above problem the constants $(\tau_1, \ldots, \tau_N)$ represent time delays in vector function $\mathbf{u}_\tau$ and are positive, ν denotes the outward normal vector on $\partial\Omega$. It is also assumed that coefficients $D^{(i)} = D^{(i)}(t, x)$ and $\mathbf{v}^{(i)} = (v_1^{(i)}, \ldots, v_n^{(i)})$, where $v_\nu^{(i)} = v_\nu^{(i)}(t, x)$ are continuous on $[0, T] \times \bar{\Omega}$ and $D^{(i)}$ is strictly positive on its domain for every $T > 0$. Coefficients $\alpha^{(i)} = \alpha^{(i)}(t, x)$ and $\beta^{(i)} = \beta^{(i)}(t, x)$ are continuous and such that $\alpha^{(i)} + \beta^{(i)} > 0$ on $[0, T] \times \partial\Omega$. Functions $f^{(i)}$, $g^{(i)}$ and $\psi^{(i)}$ are known.

Discretizing that system by the finite implicit (with respect to the time variable) difference method author obtains coupled systems of nonlinear algebraic equations. Obtained system is analyzed by a method of lower and upper solutions and associated monotone iterations. Author presents three monotone iterative schemes and shows that each one of these iterative schemes converges monotonically to a unique solution of the finite difference system.

Our aim is to extend that result. We will consider more general differential equation in which mixed derivatives appear and in which all coefficients depend on functional variable. In particular we will introduce difference operators approximating mixed derivatives on irregular grids. Moreover we will propose such implicit difference scheme, that leads us to linear system of equations which are easily solvable and no iterative schemes are required. Assumptions on the quasi monotonicity (or mixed quasi monotonicity) are needed in [14]–[16] for the construction of monotone iterative processes for finite difference systems. In our implicit difference schemes we omit the above requirements on the quasi monotonicity. We consider a general class of quasilinear functional differential systems. It is also important in our considerations that the Lipschitz condition for given functions is replaced by nonlinear estimates of the Perron type.

In that article, for reader's convenience, we will consider one equation instead of system of N equations, however it will be easily visible that our result is also valid for system of equations.

We formulate our functional differential problem. For any two metric spaces X and Y we denote by $\mathbb{C}(X, Y)$ the class of all continuous functions defined on X and taking values in Y. Let $M[n]$ denote the set of all $n \times n$ real and symmetric matrices. We will use vectorial inequalities, understanding that the same inequalities hold between their corresponding components. Let $Q \subset \mathbb{R}^n$ be a bounded, open and convex domain with the boundary ∂Q and closure $\bar{Q}$. Write

$$E = [0, a] \times \bar{Q}, \quad E_0 = [-a_0, 0] \times \bar{Q}, \quad \partial_0 E = [0, a] \times \partial Q$$

where $a > 0$, $a_0 \in \mathbb{R}_+$, $\mathbb{R}_+ = [0, +\infty)$. Write $\Sigma = E \times \mathbb{C}(E_0 \cup E, \mathbb{R})$ and suppose that

$$f : \Sigma \to M[n], \quad f = [f_{ij}]_{i,j=1,\cdots,n}, \quad g : \Sigma \to \mathbb{R}^n, \quad g = (g_1, \ldots, g_n),$$

$$G : \Sigma \to \mathbb{R}, \ \varphi : E_0 \cup \partial_0 E \to \mathbb{R},$$

are given functions. We consider the functional differential equation

$$\partial_t z(t, x) = \sum_{i,j=1}^{n} f_{ij}(t, x, z) \partial_{x_i x_j} z(t, x) + \sum_{i=1}^{n} g_i(t, x, z) \partial_{x_i} z(t, x) + G(t, x, z), \tag{1.1}$$

with initial boundary condition

$$z(t, x) = \varphi(t, x) \quad \text{for} \quad (t, x) \in E_0 \cup \partial_0 E. \tag{1.2}$$

For $t \in [0, a]$, we write $E_t = [-a_0, t] \times \bar{Q}$. The function f is said to satisfy the Volterra condition if for each $(t, x) \in E$ and $z, \bar{z} \in C(E_0 \cup E, \mathbb{R})$ such that $z(\tau, y) = \bar{z}(\tau, y)$ for $(\tau, y) \in E_t$ there is $f(t, x, z) = f(t, x, \bar{z})$. Note that the Volterra condition means that the value of f at a point (t,x,z) in the space Σ depends on (t,x) and on the restriction of z to the set E_t only.

In a similar way, we define the Volterra condition for functions g ang G. We assume that f, g and G satisfy the Volterra condition and we consider classical solutions of (1.1), (1.2).

We are interested in establishing a method of numerical approximation of classical solutions of problem (1.1), (1.2) by means of solutions of associated difference functional equations and in estimating of the difference between exact and approximate solutions. We consider implicit difference schemes for (1.1), (1.2).

It is clear that the results presented in [3], [6]–[9], [18], are not applicable to problems (1.1), (1.2).

In this paper we analyze differential system with initial boundary condition od Dirichlet type, however we are convinced that our results can be also extended in the case of mixed initial boundary conditions.

The paper is organized as follows. In Section 2 we construct a class of implicit difference schemes for (1.1), (1.2). The existence and uniqueness of approximate solutions, which are not so obvious as in the case of the explicit methods, are proved in Section 3. In Section 4, which is the main part of the paper, we give sufficient conditions for the convergence of implicit difference schemes. Finally, numerical examples are presented in the last part of the paper.

For the bibliography on the existence of solutions of parabolic functional differential problems and applications see the papers [2], [17] and the monograph [19].

We give examples of differential functional equation which can be derived from (1.1) by specializing the function f, g and G.

Example 1.1. Assume that $\tilde{f} : E\times\mathbb{R}\to M[n]$, $\tilde{f} = \{\tilde{f}_{ij}\}_{i,j=1,\dots,n}$, $\tilde{g} : E\times\mathbb{R}\to\mathbb{R}^n$, $\tilde{g} = (\tilde{g}_1,\dots,\tilde{g}_n)$, $\tilde{G} : E\times\mathbb{R}\to\mathbb{R}$ and $\psi = (\psi_0,\psi_1,\dots,\psi_n) : E\to\mathbb{R}^{1+n}$ are given functions and $\psi_0(t,x)\le t$, $\psi(t,x)\in E_0\cup E$ for $(t,x)\in E$. Write

$$f(t,x,w) = \tilde{f}(t,x,w(\psi(t,x))),\quad g(t,x,w) = \tilde{g}(t,x,w(\psi(t,x))),$$

$$G(t,x,w) = \tilde{G}(t,x,w(\psi(t,x))).$$

Then (1.1) reduces to the equation with deviated variables

$$\begin{aligned}\partial_t z(t,x) = &\sum_{i,j=1}^{n}\tilde{f}_{ij}(t,x,z(\psi(t,x)))\partial_{x_ix_j}z(t,x)\\ &+\sum_{i=1}^{n}\tilde{g}_i(t,x,z(\psi(t,x)))\partial_{x_i}z(t,x)+\tilde{G}(t,x,z(\psi(t,x))).\end{aligned}$$

Example 1.2. For the above $\tilde{f}$, $\tilde{g}$ and $\tilde{G}$ and given set $E_{t,x} \subset [-a_0, t] \times \bar{Q}$ we put

$$f(t,x,w) = \tilde{f}(t,x,\int_{E_{t,x}} z(\tau,s)d\tau ds), \quad g(t,x,w) = \tilde{g}\left(t,x,\int_{E_{t,x}} z(\tau,s)d\tau ds\right),$$

$$G(t,x,w) = \tilde{G}\left(t,x,\int_{E_{t,x}} z(\tau,s)d\tau ds\right).$$

Then (1.1) is the integral differential system

$$\partial_t z(t,x) = \sum_{i,j=1}^{n} \tilde{f}_{ij}\left(t,x,\int_{E_{t,x}} z(\tau,s)d\tau ds\right)\partial_{x_i x_j} z(t,x)$$
$$+\sum_{i=1}^{n} \tilde{g}_i\left(t,x,\int_{E_{t,x}} z(\tau,s)d\tau ds\right)\partial_{x_i} z(t,x) + \tilde{G}\left(t,x,\int_{E_{t,x}} z(\tau,s)d\tau ds\right).$$

It is clear that more complicated equations with deviated variables and differential integral equations can be obtained from (1.1) by suitable definitions f, g and G.

2. DISCRETIZATION OF MIXED PROBLEMS

We will denote by $\mathbb{F}(X,Y)$ the class of all functions defined on X and taking values in Y, where X and Y are arbitrary sets. We will denote by $\mathbb{N}$ and $\mathbb{Z}$ the set of natural numbers and the set of integers, respectively. For $x, y \in \mathbb{R}^n$, $U \in M[n]$ where $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n)$, $U = [u_{ij}]_{i,j=1,\ldots,n}$ we write

$$||x|| = \sum_{i=1}^{n} |x_i|, \quad ||U|| = \sum_{i,j=1}^{n} |u_{ij}|.$$

We define a mesh on $\bar{Q}$ in the following way. Suppose that $h = (h_1, \ldots, h_n)$, $h_i > 0$ for $1 \le i \le n$, stand for steps of the mesh for spatial variables. For $m = (m_1, \ldots, m_n) \in \mathbb{Z}^n$ we write $x^{(m)} = (m_1 h_1, \ldots, m_n h_n)$ and (see Fig. 1)

$$\mathbb{R}^n_h = \{x^{(m)} : \ m \in \mathbb{Z}^n\}, \quad Q_h = Q \cap \mathbb{R}^n_h, \quad \bar{Q}_h = \bar{Q} \cap \mathbb{R}^n_h,$$

Let h_0 stand for steps of the mesh for the time variable and put $h' = (h_0, h)$. For $r \in \mathbb{Z}$ we write $t^{(r)} = rh_0$ and

$$I_{h_0} = \{t^{(r)} : 0 \le r \le N\}, \quad I_{0.h_0} = \{t^{(r)} : -N_0 \le r \le 0\},$$

where N and N_0 are such constants that $Nh_0 \le a < (N+1)h_0$ and $N_0 h_0 = a_0$. Set $e_i = (0, \ldots, 0, 1, 0, \ldots, 0) \in \mathbb{R}^n$ with 1 standing on the $i-th$ place and $i = 1, \ldots, n$. Write $J = \{(i,j) : i,j = 1, \ldots, n, i \ne j\}$. For $x^{(m)} \in \mathbb{R}^n_h$ we put

$$A_1^{(m)} = \{x^{(m+e_i)} : i = 1, \ldots, n\} \cup \{x^{(m-e_i)} : i = 1, \ldots, n\},$$

$$A_2^{(m)} = \{x^{(m+e_i+e_j)} : (i,j) \in J\} \cup \{x^{(m-e_i-e_j)} : (i,j) \in J\} \cup \{x^{(m-e_i+e_j)} : (i,j) \in J\}$$

and $A^{(m)} = A_1^{(m)} \cup A_2^{(m)}$. The following introduced sets are illustrated at Fig. 2

$$\text{Int } Q_h = \{x^{(m)} \in Q_h : A^{(m)} \subset \bar{Q}_h\}, \quad \partial_0 Q_h = Q_h \setminus \text{Int } Q_h.$$

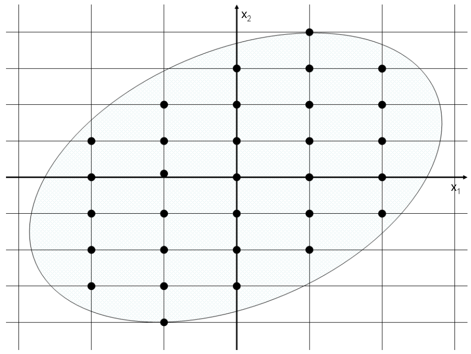

Fig. 1

$\bullet \in \bar{Q}_h$

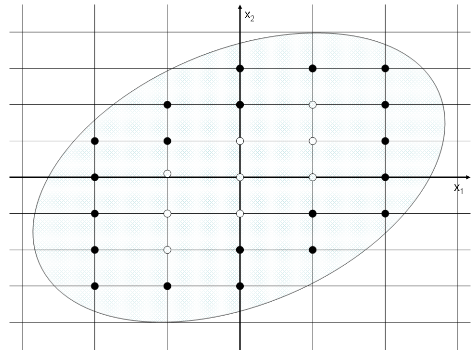

Fig. 2

$\circ \in \text{Int } Q_h$, $\bullet \in \partial_0 \, Q_h$

We will approximate partial derivatives $\partial_x = (\partial_{x_1}, \ldots, \partial_{x_n})$ and $\partial_{xx} = [\partial_{x_i x_j}]_{i,j=1,\ldots,n}$ with difference operators $\delta = (\delta_1, \ldots, \delta_n)$ and $\delta^{(2)} = [\delta_{ij}]_{i,j=1,\ldots,n}$. We will calculate the difference expressions $\delta z(t^{(r)}, x^{(m)})$ and $\delta^{(2)} z(t^{(r)}, x^{(m)})$ for each point $(t^{(r)}, x^{(m)}) \in I_{h_0} \times Q_h$. Then we need additional mesh points on the set ∂Q. For $x^{(m)} \in Q_h$ we define illustrated at Fig. 3 coefficients

$$\begin{aligned}
\theta_{i_+}^{(m)} &= \max\{\tau \in (0,1] : x^{(m)} + \tau h_i e_i \in \bar{Q}\}, \quad i = 1, \ldots, n, \\
\theta_{i_-}^{(m)} &= \max\{\tau \in (0,1] : x^{(m)} - \tau h_i e_i \in \bar{Q}\}, \quad i = 1, \ldots, n, \\
\theta_{i_+ j_-}^{(m)} &= \max\{\tau \in (0,1] : x^{(m)} + \tau h_i e_i - \tau h_j e_j \in \bar{Q}\}, \quad (i,j) \in J, \\
\theta_{i_- j_+}^{(m)} &= \max\{\tau \in (0,1] : x^{(m)} - \tau h_i e_i + \tau h_j e_j \in \bar{Q}\}, \quad (i,j) \in J, \\
\theta_{i_+ j_+}^{(m)} &= \max\{\tau \in (0,1] : x^{(m)} + \tau h_i e_i + \tau h_j e_j \in \bar{Q}\}, \quad (i,j) \in J, \\
\theta_{i_- j_-}^{(m)} &= \max\{\tau \in (0,1] : x^{(m)} - \tau h_i e_i - \tau h_j e_j \in \bar{Q}\}, \quad (i,j) \in J.
\end{aligned}$$

For simplicity of notation we write θ_{i_+}, θ_{i_-}, $\theta_{i_+ j_-}$, $\theta_{i_+ j_+}$, $\theta_{i_- j_+}$, $\theta_{i_- j_-}$ instead of $\theta_{i_+}^{(m)}$, $\theta_{i_-}^{(m)}$, $\theta_{i_+ j_-}^{(m)}$, $\theta_{i_+ j_+}^{(m)}$, $\theta_{i_- j_+}^{(m)}$, $\theta_{i_- j_-}^{(m)}$. The following sets are illustrated at Fig. 4

$$S_h^{(1)} = \{x \in \partial Q : \text{ there are } x^{(m)} \in Q_h \text{ and } i \in \{1, \ldots, n\}$$

$$\text{such that } x = x^{(m)} + \theta_{i_+} h_i e_i \text{ or } x = x^{(m)} - \theta_{i_-} h_i e_i\},$$

$$S_h^{(2)} = \{x \in \partial Q : \text{ there are } x^{(m)} \in Q_h \text{ and } (i,j) \in J \text{ such that } x = x^{(m)} + \theta_{i_+ j_+}(h_i e_i + h_j e_j)$$

$$\text{or } x = x^{(m)} - \theta_{i_- j_-}(h_i e_i + h_j e_j)) \text{ or } x = x^{(m)} + \theta_{i_+ j_-}(h_i e_i - h_j e_j)$$

and $S_h = S_h^{(1)} \cup S_h^{(2)}$. Write $X_h = Q_h \cup S_h$, and

$$E_{h'} = I_{h_0} \times X_h, \; E_{0.h'} = I_{0.h_0} \times X_h, \; \partial_0 E_{h'} = I_{h_0} \times S_h, \; E_{r.h'} = E_{t^{(r)}} \cap (E_{0.h'} \cup E_{h'}).$$

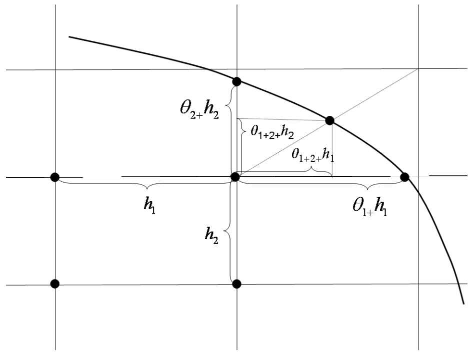

Fig. 3
Coefficients θ_{1+}, θ_{2+} and θ_{1+2+}

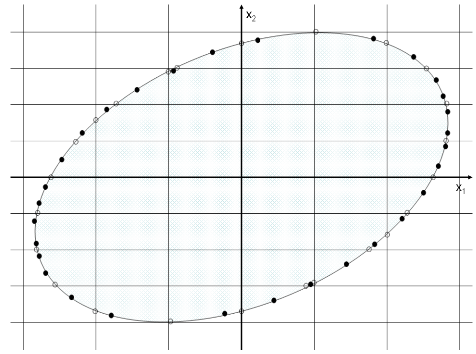

Fig. 4
$\circ \in S_h^{(1)}$, $\bullet \in S_h^{(2)}$

For $z : E_{0.h'} \cup E_{h'} \to \mathbb{R}$, $\mu, \xi \in [-1, 1]$ and for $x^{(m)} \in Q_h$ we put $z^{(r,m+\mu e_i+\xi e_j)} = z(t^{(r)}, x^{(m+\mu e_i+\xi e_j)})$, $(i, j) \in J$.

We formulate now a difference initial boundary value problem corresponding to (1.1), (1.2). We first observe that solutions of difference equation are defined on the set $E_{0.h'} \cup E_{h'}$ and equation (1.1) contains the functional variable z which is an element of the space $\mathbb{C}(E_0 \cup E, \mathbb{R})$. Then we need an interpolating operator $T_{h'} : \mathbb{F}(E_{0.h'} \cup E_{h'}, \mathbb{R}) \to \mathbb{C}(E_0 \cup E, \mathbb{R})$. In the next part of the paper we formulate additional assumptions on T_h. Let us denote by

$$\delta_0, \quad \delta = (\delta_1, \ldots, \delta_n), \quad \delta^{(2)} = [\delta_{ij}]_{i,j=1,\ldots,n}$$

difference operators corresponding to the partial derivatives

$$\partial_t,\ \partial_x = (\partial_{x_1}, \ldots, \partial_{x_n}), \quad \partial_{xx} = [\partial_{x_i x_j}]_{i,j=1,\ldots,n}.$$

Write

$$\begin{aligned} F_{h'}[z]^{(r,m)} &= \sum_{i,j=1}^{n} f_{ij}(t^{(r)}, x^{(m)}, T_{h'}[z])\delta_{ij} z^{(r+1,m)} \\ &\quad + \sum_{i=1}^{n} g_i(t^{(r)}, x^{(m)}, T_{h'}[z])\delta_i z^{(r+1,m)} + G(t^{(r)}, x^{(m)}, T_{h'}[z]). \end{aligned}$$

We approximate classical solutions of (1.1), (1.2) with solutions of the implicit difference equation

$$\delta_0 z^{(r,m)} = F_{h'}[z]^{(r,m)}, \tag{2.1}$$

with initial boundary condition

$$z^{(r,m)} = \varphi_{h'}^{(r,m)} \text{ on } E_{0.h'} \cup \partial_0 E_{h'} \tag{2.2}$$

where $\varphi_{h'} : E_{0.h'} \cup \partial_0 E_{h'} \to \mathbb{R}$ is a given function. It is important that the numbers $\delta_{ij} z^{(r+1,m)}$, $\delta_i z^{(r+1,m)}$, $r \le i, j \le n$, appear in (2.1). Set

$$\delta_0 z^{(r,m)} = \frac{1}{h_0}(z^{(r+1,m)} - z^{(r,m)}).$$

Suppose that $z : E_{0.h'} \cup E_{h'} \to \mathbb{R}$ is a solution of (2.1), (2.2) on set $E_{r.h'}$. We will calculate the numbers $z(t^{(r+1)}, x)$ for $x \in Q_h$ in the following way. Let $J^{(r,m)}_{h'.+}[z], J^{(r,m)}_{h'.-}[z] \in J$ be defined by

$$J^{(r,m)}_{h'.+}[z] = \{(i,j) \in J : \ f_{ij}(t^{(r)}, x^{(m)}, T_{h'}[z]) \ge 0\}, \ J^{(r,m)}_{h'.-}[z] = J \setminus J^{(r,m)}_{h'.+}[z].$$

The definitions of the difference operators δ, $\delta^{(2)}$ falls naturally into two parts. In the first part we assume that $(t^{(r+1)}, x^{(m)}) \in I_{h_0} \times \operatorname{Int} Q_h$. In the second case we define $\delta z^{(r+1,m)}$ and $\delta^{(2)}$ for $(t^{(r+1)}, x^{(m)}) \in I_{h_0} \times \partial_0 Q_h$.

I. For $(t^{(r+1)}, x^{(m)}) \in I_{h_0} \times \operatorname{Int} Q_h$ we put

$$\delta_i^+ z^{(r+1,m)} = \frac{1}{h_i}\left(z^{(r+1,m+e_i)} - z^{(r+1,m)}\right), \delta_i^- z^{(r+1,m)} = \frac{1}{h_i}\left(z^{(r+1,m)} - z^{(r+1,m-e_i)}\right),$$

and

$$\delta_i z^{(r+1,m)} = \frac{1}{2}\left(\delta_i^+ z^{(r+1,m)} + \delta_i^- z^{(r+1,m)}\right), \tag{2.3}$$

$$\delta_{ii} z^{(r+1,m)} = \delta_i^+ \delta_i^- z^{(r+1,m)}, \tag{2.4}$$

where $1 \le i \le n$. The difference expressions $\delta_{ij} z^{(r+1,m)}$ are defined in the following way:

$$\delta_{ij} z^{(r+1,m)} = \frac{1}{2}\left(\delta_i^+ \delta_j^+ z^{(r+1,m)} + \delta_i^- \delta_j^- z^{(r+1,m)}\right) \text{ for } (i,j) \in J^{(r,m)}_{h'.+}[z], \tag{2.5}$$

$$\delta_{ij} z^{(r+1,m)} = \frac{1}{2}\left(\delta_i^+ \delta_j^- z^{(r+1,m)} + \delta_i^- \delta_j^+ z^{(r+1,m)}\right) \text{ for } (i,j) \in J^{(r,m)}_{h'.-}[z]. \tag{2.6}$$

II. We define difference operators δ, $\delta^{(2)}$ for $(t^{(r+1)}, x^{(m)}) \in I_{h_0} \times \partial_0 Q_h$. Write

$$\delta_i z^{(r+1,m)} = \frac{1}{2h_i}\left(z^{(r+1,m+\theta_{i_+} h_i)} \frac{1}{\theta_{i_+}} + z^{(r+1,m)} \frac{(\theta_{i_+} - \theta_{i_-})}{\theta_{i_+}\theta_{i_-}} - z^{(r+1,m-\theta_{i_-} h_i)} \frac{1}{\theta_{i_-}}\right), \tag{2.7}$$

(2.8)

$$\delta_{ii} z^{(r+1,m)} = \frac{2}{h_i^2}\left(z^{(r+1,m+\theta_{i_+} h_i)} \frac{1}{\theta_{i_+}(\theta_{i_+} + \theta_{i_-})} - z^{(r+1,m)} \frac{1}{\theta_{i_+}\theta_{i_-}} + z^{(r+1,m-\theta_{i_-} h_i)} \frac{1}{\theta_{i_-}(\theta_{i_+} + \theta_{i_-})}\right),$$

where $1 \le i \le n$. If $(i,j) \in J^{(r,m)}_{h'.+}[z]$ then:

$$\begin{aligned} \delta_{ij} z^{(r+1,m)} = {} & A_+^{(m)} z^{(r+1,m)} + F_+^{(m)} z^{(r+1,m-\theta_{i_-j_-}(h_i+h_j))} \\ & + G_+^{(m)} z^{(r+1,m+\theta_{i_+j_+}(h_i+h_j))} + B_+^{(m)} z^{(r+1,m-\theta_{i_-} h_i)} \\ & + C_+^{(m)} z^{(r+1,m+\theta_{i_+} h_i)} + D_+^{(m)} z^{(r+1,m-\theta_{j_-} h_j)} + E_+^{(m)} z^{(r+1,m+\theta_{j_+} h_j)} \end{aligned} \tag{2.9}$$

where

$$A_+^{(m)} = \frac{1}{h_i h_j}\left(\frac{1}{\theta_{i_+}\theta_{i_-}} + \frac{-1}{\theta_{i_-j_-}\theta_{i_+j_+}} + \frac{1}{\theta_{j_+}\theta_{j_-}}\right),$$

$$B_+^{(m)} = \frac{-1}{h_i h_j \theta_{i_-}(\theta_{i_-} + \theta_{i_+})}, \qquad C_+^{(m)} = \frac{-1}{h_i h_j \theta_{i_+}(\theta_{i_+} + \theta_{i_-})},$$

$$D_+^{(m)} = \frac{-1}{h_i h_j \theta_{j_-}(\theta_{j_-} + \theta_{j_+})}, \qquad E_+^{(m)} = \frac{-1}{h_i h_j \theta_{j_+}(\theta_{j_+} + \theta_{j_-})},$$

$$F_+^{(m)} = \frac{1}{h_i h_j \theta_{i_-j_-}(\theta_{i_-j_-} + \theta_{i_+j_+})}, \quad G_+^{(m)} = \frac{1}{h_i h_j \theta_{i_+j_+}(\theta_{i_-j_-}^{(m)} + \theta_{i_+j_+})}.$$

If $(i,j) \in J_{h'.-}^{(r,m)}[z]$ then:

(2.10)

$$\begin{aligned}\delta_{ij} z^{(r+1,m)} &= A_-^{(m)} z^{(r+1,m)} + F_-^{(m)} z^{(r+1,m+\theta_{i_+j_-}(h_i-h_j))} \\ &\quad + G_-^{(m)} z^{(r+1,m+\theta_{i_-j_+}(-h_i+h_j))} + B_-^{(m)} z^{(r+1,m-\theta_{i_-}h_i)} + C_-^{(m)} z^{(r+1,m+\theta_{i_+}h_i)} \\ &\quad + D_-^{(m)} z^{(r+1,m-\theta_{j_-}^{(m)}h_j)} + E_-^{(m)} z^{(r+1,m+\theta_{j_+}^{(m)}h_j)},\end{aligned}$$

where

$$A_-^{(m)} = \frac{1}{h_i h_j}\left(\frac{-1}{\theta_{i_+}\theta_{i_-}} + \frac{1}{\theta_{i_-j_+}\theta_{i_+j_-}} + \frac{-1}{\theta_{j_+}\theta_{j_-}}\right),$$

$$B_-^{(m)} = \frac{1}{h_i h_j \theta_{i_-}(\theta_{i_-} + \theta_{i_+})}, \qquad C_-^{(m)} = \frac{1}{h_i h_j \theta_{i_+}(\theta_{i_+} + \theta_{i_-})},$$

$$D_-^{(m)} = \frac{1}{h_i h_j \theta_{j_-}(\theta_{j_-} + \theta_{j_+})}, \qquad E_-^{(m)} = \frac{1}{h_i h_j \theta_{j_+}(\theta_{j_+} + \theta_{j_-})},$$

$$F_-^{(m)} = \frac{-1}{h_i h_j \theta_{i_+j_-}(\theta_{i_-j_+} + \theta_{i_+j_-})}, \qquad G_-^{(m)} = \frac{-1}{h_i h_j \theta_{i_-j_+}(\theta_{i_-j_+} + \theta_{i_+j_-})},$$

Definitions (2.3)–(2.6) and (2.7)–(2.10) have the following properties:

(i) if we put $\theta_{i_-} = \theta_{i_+} = 1$ in (2.7) and (2.8) then we obtain (2.3) and (2.4) respectively,

(ii) if we put $\theta_{i_-} = \theta_{i_+} = \theta_{i_-j_+} = \theta_{i_+j_-} = 1$ in (2.9) then we obtain (2.5),

(iii) if we put $\theta_{i_-} = \theta_{i_-} = \theta_{i_-j_-} = \theta_{i_+j_+} = 1$ in (2.10) then we obtain (2.6)

It follows from the above observation that it is sufficient to use only formulas (2.7)–(2.10) in next considerations.

Remark 2.1. It follows from (2.5), (2.6) and (2.9), (2.10) that the method of discretization of the mixed derivatives $\partial_{x_i x_j} z$, $(i,j) \in J$, at the point $(t^{(r+1)}, x^{(m)})$ depends on the sign of the number $f_{ij}(t^{(r)}, x^{(m)}, T_{h'}[z])$. There are the following consequences of our approach. Consider the nonlinear functional differential equation

$$\partial_t z(t,x) = \tilde{F}(t, x, z, \partial_x z(t,x), \partial_{xx} z(t,x)) \tag{2.11}$$

with the initial boundary condition (1.2) where $\tilde{F} : \Sigma \times \mathbb{R}^n \times M[n] \to \mathbb{R}$ is a given function of the variables (t, x, z, r, q), $r = (r_1, \ldots, r_n)$, $q = [g_{ij}]_{i,j=1,\ldots,n}$. In theorems concerning difference methods for (2.11), (1.2) it is assumed that the functions

$$sign\ \partial_{q_{ij}} \tilde{F}(t, x, z, r, q), \quad (i, j) \in J,$$

are constant on $\Sigma \times \mathbb{R}^n \times M[n]$, see [5], [7]–[9]. It is important in our considerations that we have omitted the above assumptions for equation (1.1).

Difference functional problem (2.1), (2.2) is considered as an implicit numerical method to problem (1.1), (1.2). The corresponding explicit difference scheme has the form

$$\begin{aligned} \delta_0 z(t, x) = & \sum_{i,j=1}^{n} f_{ij}(t^{(r)}, x^{(m)}, T_h[z]) \delta_{ij} z^{(r,m)} \\ & + \sum_{i=1}^{n} g_i(t^{(r)}, x^{(m)}, T_{h'}[z]) \delta_i z^{(r,m)} + G(t^{(r)}, x^{(m)}, T_{h'}[z]), \end{aligned} \tag{2.12}$$

$$z^{(r,m)} = \varphi^{(r,m)} \text{ for on } E_{0.h'} \cup \partial_0 E_{h'}. \tag{2.13}$$

It is clear that there exists exactly one solution $\tilde{u}_{h'} : E_{0.h'} \cup E_{h'} \to \mathbb{R}$ of problem (2.12), (2.13). We prove that under natural assumptions on given functions and on the mesh there exists exactly one solution $u_{h'} : E_{0.h'} \cup E_{h'} \to \mathbb{R}$ of the implicit difference scheme (2.1), (2.2). Solutions of (2.1), (2.2) are considered as approximate solutions of (1.1), (1.2). We give sufficient conditions for the convergence of sequences of approximate solutions to a classical solution of (1.1), (1.2).

3. SOLVABILITY OF IMPLICIT DIFFERENCE FUNCTIONAL PROBLEMS

For $(t^{(r)}, x^{(m)}, z) \in \Sigma_{h'} = E_{h'} \times \mathbb{F}(E_{0.h'} \cup E_{h'}, \mathbb{R})$ write $P^{(r,m)}[z] = (t^{(r)}, x^{(m)}, T_{h'}[z])$.

Assumption H$[f, g]$**.** Suppose that the functions $f : \Sigma \to M[n]$ and $g : \Sigma \to \mathbb{R}^n$ are such that the following condition is satisfied for $(t^{(r)}, x^{(m)}, z) \in \Sigma_{h'}$:

$$-\frac{1}{2h_i} \left| g_i(P^{(r,m)}[z]) \right| + \frac{1}{h_i^2} f_{ii}(P^{(r,m)}[z]) - \sum_{\substack{j=1 \\ j \neq i}}^{n} \frac{1}{h_i h_j} \left| f_{ij}(P^{(r,m)}[z]) \right| \geq 0, \ i = 1, \ldots, n, \tag{3.1}$$

Remark 3.1. Suppose that the functions $f : \Sigma \to M[n]$ and $g : \Sigma \to \mathbb{R}^n$ are bounded on Σ and that the following condition is satisfied

$$f_{ii}(P^{(r,m)}[z]) - \sum_{\substack{j=1 \\ j \neq i}}^{n} \left| f_{ij}(P^{(r,m)}[z]) \right| \geq \varepsilon, \ \varepsilon > 0, \text{ on } \Sigma_{h'}, \quad i = 1, \ldots, n.$$

If $h_1 = h_2 = \ldots = h_n$ then there exists such $\varepsilon_0 > 0$ that for $\|h\| < \varepsilon_0$ condition (3.1) is satisfied.

We prove a maximum principle for implicit parabolic difference functional inequalities. The difference functional equation

$$z^{(r+1,m)} = h_0 \sum_{i,j=1}^{n} f_{ij}(P^{(r,m)}[z])\delta_{ij} z^{(r+1,m)} + h_0 \sum_{i=1}^{n} g_i(P^{(r,m)}[z])\delta_i z^{(r+1,m)} \tag{3.2}$$

is a principal part of (2.1), (2.2). The maximum principle asserts that a solution of difference functional inequalities corresponding to (3.2) cannot have a positive maximum (or negative minimum) on the set $\{t^{(r+1)}\} \times Q_h$.

Theorem 3.2. *Suppose that Assumption $H[f,g]$ is satisfied and $0 \le r \le N-1$ is fixed. If $z_{h'} : E_{r+1.h'} \to \mathbb{R}$ satisfies the implicit difference inequality*

$$z_{h'}^{(r+1,m)} \le h_0 \sum_{i,j=1}^{n} f_{ij}(P^{(r,m)}[z_{h'}])\delta_{ij} z_{h'}^{(r+1,m)} + h_0 \sum_{i=1}^{n} g_i(P^{(r,m)}[z_{h'}])\delta_i z_{h'}^{(r+1,m)} \tag{3.3}$$

for $(t^{(r+1)}, x^{(m)}) \in \{t^{(r+1)}\} \times Q_h$ and $x \in X_h$ is such that

$$z_{h'}(t^{(r+1)}, x) = \max\left\{ z_{h'}(t^{(r+1)}, y),\ y \in X_h \right\}$$

and $z_{h'}(t^{(r+1)}, x) > 0$ then $x \in S_h$.

Proof. Write

$$\begin{aligned}
B^{(r,m)} &= \sum_{i=1}^{n} \left[f_{ii}(P^{(r,m)}[z_{h'}]) \frac{-2}{\theta_{i_-}\theta_{i_+} h_i^2} + g_i(P^{(r,m)}[z_{h'}]) \frac{\theta_{i_+} - \theta_{i_-}}{2\theta_{i_+}\theta_{i_-} h_i} \right] \\
&\quad + \sum_{(i,j)\in J} |f_{ij}(P^{(r,m)}[z_{h'}])| \left[\frac{1}{\theta_{i_+}\theta_{i_-}} - \frac{1}{\theta_{i_-j_-}\theta_{i_+j_+}} + \frac{1}{\theta_{j_+}\theta_{j_-}} \right] \frac{1}{h_i h_j} \\
B_{(i,+)}^{(r,m)} &= g_i(P^{(r,m)}[z_{h'}]) \frac{1}{2\theta_{i+} h_i} + \frac{2}{\theta_{i_+}(\theta_{i_+} + \theta_{i_-}) h_i^2} f_{ii}(P^{(r,m)}[z_{h'}]) \\
&\quad - \sum_{\substack{j=1 \\ j\neq i}}^{n} \left| f_{ij}(P^{(r,m)}[z_{h'}]) \right| \frac{2}{h_i h_j \theta_{i_+}(\theta_{i_+} + \theta_{i_-})} \\
B_{(i,-)}^{(r,m)} &= -g_i(P^{(r,m)}[z_{h'}]) \frac{1}{2\theta_{i-} h_i} + \frac{2}{\theta_{i_-}(\theta_{i_+} + \theta_{i_-}) h_i^2} f_{ii}(P^{(r,m)}[z_{h'}]) \\
&\quad - \sum_{\substack{j=1 \\ j\neq i}}^{n} \left| f_{ij}(P^{(r,m)}[z_{h'}]) \right| \frac{2}{h_i h_j \theta_{i_-}(\theta_{i_-} + \theta_{i_+})}
\end{aligned}$$

Let us suppose that $(t^{(r+1)}, x) \in \{t^{(r+1)}\} \times Q_h$. We conclude from (3.3) that

$$z_{h'}(t^{(r+1)}, x) \le h_0 z_{h'}(t^{(r+1)}, x)\Big\{ B^{(r,\mu)} + \sum_{i=1}^{n} B^{(r,\mu)}_{(i,+)} + \sum_{i=1}^{n} B^{(r,\mu)}_{(i,-)}$$
$$+ h_0 \sum_{(i,j)\in J_+^{(r,m)}[z]} f_{ij}(P^{(r,m)}[z_{h'}]) \left[\frac{1}{\theta_{i_+j_+}(\theta_{i_-j_-} + \theta_{i_+j_+})} + \frac{1}{\theta_{i_-j_-}(\theta_{i_-j_-} + \theta_{i_+j_+})} \right] \frac{1}{h_i h_j}$$
$$+ h_0 \sum_{(i,j)\in J_-^{(r,m)}[z]} f_{ij}(P^{(r,m)}[z_{h'}]) \left[\frac{-1}{\theta_{i_+j_-}(\theta_{i_+j_-} + \theta_{i_-j_+})} + \frac{-1}{\theta_{i_-j_+}(\theta_{i_+j_-} + \theta_{i_-j_+})} \right] \frac{1}{h_i h_j} \Big\} = 0,$$

what contradicts our assumption that $z(t^{(r+1)}, x) > 0$. This proves the theorem. □

Remark 3.3. Suppose that Assumption H$[f,g]$ is satisfied. Then Theorem 3.2 asserts that solutions of the implicit difference inequality (3.3) cannot have a positive maximum on $E_{h'}$. It is clear that solutions of inverse implicit difference inequalities cannot have a negative minimum on $E_{h'}$.

Lemma 3.4. *Suppose that $f : \Sigma \to M[n]$, $g : \Sigma \to \mathbb{R}^n$, $G : \Sigma \to \mathbb{R}$, $\varphi_{h'} : E_{0.h'} \cup \partial_0 E_{h'} \to \mathbb{R}$ and Assumption H[f, g] is satisfied. Then there is exactly one solution $u_{h'} : E_{0.h'} \cup E_{h'} \to \mathbb{R}$ of problem* (2.1), (2.2).

Proof. Suppose that $0 \le r \le N_0 - 1$ is fixed and that $u_{h'}$ is a solution of problem (2.1), (2.2) on $E_{r.h'}$. Consider the difference problem

$$\delta_0 z^{(r,m)}_{h'} = \sum_{i,j=1}^{n} f_{ij}(t^{(r)}, x^{(m)}, T_{h'}[u_{h'}])\delta_{ij} z^{(r+1,m)}_{h'} + \sum_{i=1}^{n} g_i(t^{(r)}, x^{(m)}, T_{h'}[u_{h'}])\delta_i z^{(r+1,m)}_{h'} + G(t^{(r)}, x^{(m)}, T_{h'}[u_{h'}]), \tag{3.4}$$

$$z^{(r,m)}_{h'} = \varphi^{(r,m)}_{h'} \text{ on } E_{0.h'} \cup \partial_0 E_{h'}. \tag{3.5}$$

It follows from Theorem 3.2 that the problem consisting of difference equation (3.2) with $P^{(r,m)}[u_{h'}]$ instead of $P^{(r,m)}[z]$ and boundary condition

$$z^{(r+1,m)} = 0 \text{ for } (t^{(r+1)}, x^{(m)}) \in S_h$$

has exactly one solution $\tilde{z}^{(r+1,m)}_{h'} = 0$ on $\{t^{(r+1)}\} \times Q_h$. Then there is exactly one solution $u^{(r+1,m)}_{h'}$ on $\{t^{(r+1)}\} \times Q_h$, of (3.4), (3.5) and $u_{h'}$ is defined on $E_{r.h'}$. Since $u_{h'}$ is given by (2.1) on $E_{0.h'}$ then we obtain the Lemma by induction with respect to r, $0 \le r \le N_0$. □

4. CONVERGENCE OF IMPLICIT DIFFERENCE METHODS

Let us introduce seminorms $\|\cdot\|_t$ and $\|\cdot\|_{r.h'}$ for functions $z : E_0 \cup E \to \mathbb{R}$ and $z_{h'} : E_{0.h'} \cup E \to \mathbb{R}$ respectively in the following way:

$$\|z\|_t = \max\{|z(\tau, x)| : (\tau, x) \in E_t\}, \quad 0 \le t \le a,$$

$$\|z_{h'}\|_{r.h'} = \max\{|z(\tau, x)| : (\tau, x) \in E_{r.h'}\}, \quad 0 \le r \le N.$$

Assumption H$[T_{h'}]$**.** The operator $T_{h'}$ satisfies the conditions

1) if $w \in \mathbb{F}(E_{0.h'} \cup E_{h'}, \mathbb{R})$, then $T_{h'}[w] \in C(E_0 \cup E, \mathbb{R})$,
2) for any functions $w, \bar{w} \in \mathbb{F}(E_{0.h'} \cup E_{h'}, \mathbb{R})$ we have

$$\|T_{h'}[w] - T_{h'}[\bar{w}]\|_t \le \|w - \bar{w}\|_{r.h'},$$

3) if the function $w : E_0 \cup E \to \mathbb{R}$ is of class C^2 then there is a function $\tilde{\gamma} : H \to \mathbb{R}_+$ such that

$$\|w - T_{h'}[w_{h'}]\|_{t^{(r)}} \le \tilde{\gamma}(h')$$

and $\lim_{h \to 0} \tilde{\gamma}(h') = 0$ where $w_{h'}$ is the restriction of w to the set $E_{0.h'} \cup E_{h'}$.

Remark 4.1. The above condition 2) states that $T_{h'}$ satisfies the Lipschitz condition with the constant $L = 1$. The meaning of the condition 3) is that $T_{h'}[w_{h'}]$ is an approximation of w and the error of the approximation is estimated by $\tilde{\gamma}(h')$.

Assumption H$[f, g, G]$**.** Estimates (3.1) are satisfied and

1) $v : E_0 \cup E \to \mathbb{R}$ is the solution of (1.1), (1.2) and v is of class C^2 on $E_0 \cup E$ and the numbers $c_1,\ c_2 \in \mathbb{R}_+$ are obtained by the relations

$$|\partial_{x_i} v(t, x)| \le c_1, \quad |\partial_{x_i x_j} v(t, x)| \le c_2 \ \text{ on } E, \quad i, j = 1, \ldots, n$$

2) there are $\sigma_0, \sigma_1, \sigma_2 \in \mathbb{C}([0, a] \times \mathbb{R}_+, \mathbb{R}_+)$ such that
   (i) they are nondecreasing with respect to both variables,
   (ii) the function $\sigma(t, p) = \sigma_0(t, p) + c_1\sigma_1(t, p) + c_2\sigma_2(t, p)$, $(t, p) \in [0, a] \times \mathbb{R}_+$, satisfies the condition: the maximal solution of the Cauchy problem

$$\zeta'(t) = \sigma(t, \zeta(t)), \quad \zeta(0) = 0, \tag{4.1}$$

   is $\tilde{\zeta(t)} = 0$ for $t \in [0, a]$,
4) the estimates

$$\big\|f(t, x, z) - f(t, x, \bar{z})\big\| \le \sigma_2(t, ||z - \bar{z}||_t), \ \big\|g(t, x, z) - g(t, x, \bar{z})\big\| \le \sigma_1(t, ||z - \bar{z}||_t),$$

$$\big|G(t, x, z) - G(t, x, \bar{z})\big| \le \sigma_0(t, ||z - \bar{z}||_t)$$

are hold on Σ.

Remark 4.2. Let $F : \Sigma \times \mathbb{R}^n \times M[n] \to \mathbb{R}$ be defined by

$$F(t,x,z,r,q) = \sum_{i,j=1}^{n} f_{ij}(t,x,z)q_{ij} + \sum_{i=1}^{n} g_i(t,x,z)r_i + G(t,x,z)$$

where $r = (r_1,\ldots,r_n)$, $q = [q_{ij}]_{i,j=1,\ldots,n}$. If Assumption $H[f,g,G]$ is satisfied then we have the estimate

$$|F(t,x,z,r,q) - F(t,x,\bar{z},r,q)| \le \sigma(t, \|z-\bar{z}\|_t)$$

where $(t,x,z) \in \Sigma$, $\bar{z} \in (E_0 \cup E, \mathbb{R})$ and $\|r\| \le c_1$, $\|g\| \le c_2$ and σ is a comparison function of the Perron type. The paper [1] contains results on comparison functions.

Now we prove a theorem on the convergence of method (2.1), (2.2).

Theorem 4.3. *Suppose that Assumptions $H[f,g,G]$ and $H[T_{h'}]$ are satisfied and*

1) *there is $c_0 > 0$ such that $h_i h_j^{-1} \le c_0$, $i,j = 1,\ldots,n$,*
2) *the function $u_{h'} : E_{h'} \cup E_{0.h'} \cup \partial_0 E_{h'} \to \mathbb{R}$ is a solution of (2.1), (2.2) and there is $\alpha_0 : \mathbb{R}^{1+n} \to \mathbb{R}_+$ such that*

$$|v^{(r,m)} - u_{h'}^{(r,m)}| \le \alpha_0(h') \text{ on } E_{0.h'} \cup \partial_0 E_{h'} \text{ and } \lim_{h'\to 0} \alpha_0(h') = 0. \tag{4.2}$$

Then there exists a function $\alpha : \mathbb{R}^{1+n} \to \mathbb{R}_+$ such that we have

$$|u_{h'}^{(r,m)} - v_{h'}^{(r,m)}| \le \alpha(h') \text{ on } E_{h'} \text{ and } \lim_{h'\to 0} \alpha(h') = 0 \tag{4.3}$$

where $v_{h'}$ is the restriction of v to the set $E_{h'}$.

Proof. Note that the existence of $u_{h'}$ follows from Lemma 3.4. Let $z_{h'} = u_{h'} - v_{h'}$. We construct a difference equation for $z_{h'}$. Let $\Gamma_{h'} : E_{h'} \to \mathbb{R}$, $\Gamma_{0.h'} : \partial_0 E_{h'} \cup E_{0.h'} \to \mathbb{R}$ be defined by the relations

$$\delta_0 v_{h'}^{(r,m)} = F_{h'}[v_{h'}]^{(r,m)} + \Gamma_{h'}^{(r,m)} \text{ on } E_{h'},$$

$$v_{h'}^{(r,m)} = \varphi_{h'}^{(r,m)} + \Gamma_{0.h'}^{(r,m)} \text{ on } \partial_0 E_{h'} \cup E_{0.h'}.$$

One can observe that there is $\gamma : \mathbb{R}^{1+n} \to \mathbb{R}$, such that

$$|\Gamma_{h'}^{(r,m)}| \le \gamma(h') \text{ on } E_{h'}, \quad \lim_{h'\to 0} \gamma(h') = 0.$$

Then we have

$$\begin{aligned}\delta_0 z_{h'}^{(r,m)} = &\sum_{i,j=1}^{n} f_{ij}(t^{(r)}, x^{(m)}, T_{h'}[u_{h'}])\delta_{ij} z_{h'}^{(r+1,m)} \\ &+ \sum_{i=1}^{n} g_i(t^{(r)}, x^{(m)}, T_{h'}[u_{h'}])\delta_i z_{h'}^{(r+1,m)} + \Lambda_{h'}^{(r,m)} + \Gamma_{h'}^{(r,m)},\end{aligned}$$

where

$$\Lambda_{h'}^{(r,m)} = \sum_{i,j=1}^{n} [f_{ij}(t^{(r)}, x^{(m)}, T_{h'}[u_{h'}]) - f_{ij}(t^{(r)}, x^{(m)}, T_{h'}[v_{h'}])]\delta_{ij} v_{h'}^{(r+1,m)}\|$$
$$+ \sum_{i=1}^{n} [g_i(t^{(r)}, x^{(m)}, T_{h'}[u_{h'}]) - g_i(t^{(r)}, x^{(m)}, T_{h'}[v_{h'}])]\delta_i v_{h'}^{(r+1,m)}$$
$$+ G(t^{(r)}, x^{(m)}, T_{h'}[u_{h'}]) - G(t^{(r)}, x^{(m)}, T_{h'}[v_{h'}])$$

and consequently

$$z_{h'}^{(r+1,m)}(1 - h_0 B^{(r,m)}) = z_{h'}^{(r,m)} + h_0 \sum_{i=1}^{n} z_{h'}^{(r+1,m+\theta_i h_i e_i)} B_{(i,+)}^{(r,m)} + h_0 \sum_{i=1}^{n} z_{h'}^{(r+1,m-\theta_i h_i e_i)} B_{(i,-)}^{(r,m)}$$
$$+ h_0 \sum_{(i,j)\in J_{h',+}^{(r,m)}[z_{h'}]} f_{ij}(P^{(r,m)}[u_{h'}]) z_{h'}^{(r+1,m+\theta_{i_+j_+}(h_i e_i + h_j e_j))} \frac{1}{h_i h_j \theta_{i_+j_+}(\theta_{i_-j_-} + \theta_{i_+j_+})}$$
$$+ h_0 \sum_{(i,j)\in J_{h',+}^{(r,m)}[z_{h'}]} f_{ij}(P^{(r,m)}[u_{h'}]) z_{h'}^{(r+1,m-\theta_{i_-j_-}(h_i e_i + h_j e_j))} \frac{1}{h_i h_j \theta_{i_-j_-}(\theta_{i_-j_-} + \theta_{i_+j_+})}$$
$$+ h_0 \sum_{(i,j)\in J_{h',-}^{(r,m)}[z_{h'}]} f_{ij}(P^{(r,m)}[u_{h'}]) z_{h'}^{(r+1,m+\theta_{i_-j_+}(-h_i e_i + h_j e_j))} \frac{-1}{h_i h_j \theta_{i_+j_-}(\theta_{i_+j_-} + \theta_{i_-j_+})}$$
$$+ h_0 \sum_{(i,j)\in J_{h',-}^{(r,m)}[z_{h'}]} f_{ij}(P^{(r,m)}[u_{h'}]) z_{h'}^{(r+1,m+\theta_{i_+j_-}(h_i e_i - h_j e_j))} \frac{-1}{h_i h_j \theta_{i_-j_+}(\theta_{i_+j_-} + \theta_{i_-j_+})}$$
$$+ h_0 \Lambda_{h'}^{(r,m)} + h_0 \Gamma_{h'}^{(r,m)}.$$

Note that $\|\Lambda_{h'}^{(r,m)}\| \leq \sigma(t, \|T_{h'}[u_{h'}] - T_{h'}[v_{h'}]\|_{t^{(r)}})$. Let function $\varepsilon_{h'}^{(r)} : I_{h_0} \to \mathbb{R}_+$ be defined by

$$\varepsilon_{h'}^{(r)} = \max\{|z_{h'}(t^{(\tilde{r})}, x))|,\ \tilde{r} \leq r,\ x \in X_{h'}\},$$

where $0 \leq r \leq N$. From inequalities $B_{(i,+)}^{(r,m)} > 0$, $B_{(i,-)}^{(r,m)} > 0$, and from relation

$$B^{(r,m)} + B_{(i,+)}^{(r,m)} + B_{(i,-)}^{(r,m)}$$
$$+ \sum_{(i,j)\in J_{h',+}^{(r,m)}[z_{h'}]} f_{ij}(P^{(r,m)}[u_{h'}]) z^{(r+1,m+\theta_{i_+j_+}(h_i e_i + h_j e_j))} \frac{1}{h_i h_j \theta_{i_+j_+}(\theta_{i_-j_-} + \theta_{i_+j_+})}$$
$$\sum_{(i,j)\in J_{h',+}^{(r,m)}[z_{h'}]} f_{ij}(P^{(r,m)}[u_{h'}]) z^{(r+1,m-\theta_{i_-j_-}(h_i e_i + h_j e_j))} \frac{1}{h_i h_j \theta_{i_-j_-}(\theta_{i_-j_-} + \theta_{i_+j_+})}$$
$$\sum_{(i,j)\in J_{h',-}^{(r,m)}[z_{h'}]} f_{ij}(P^{(r,m)}[u_{h'}]) z^{(r+1,m+\theta_{i_-j_+}(-h_i e_i + h_j e_j))} \frac{-1}{h_i h_j \theta_{i_+j_-}(\theta_{i_+j_-} + \theta_{i_-j_+})}$$
$$\sum_{(i,j)\in J_{h',-}^{(r,m)}[z_{h'}]} f_{ij}(P^{(r,m)}[u_{h'}]) z^{(r+1,m+\theta_{i_+j_-}(h_i e_i - h_j e_j))} \frac{-1}{h_i h_j \theta_{i_-j_+}(\theta_{i_+j_-} + \theta_{i_-j_+})} = 0$$

we conclude that

$$\varepsilon_{h'}^{(r+1)} \leq \varepsilon_{h'}^{(r)} + h_0\sigma(t, \varepsilon_{h'}^{(r)}) + h_0\gamma(h'), \quad r = 0, 1, \dots, N-1.$$

Let us denote by $\omega : [0, a] \to \mathbb{R}_+$ the maximal solution of the Cauchy problem

$$\omega'(t) = \sigma(t, \omega(t)) + \gamma(h'), \quad \omega(0) = \alpha_0(h'). \tag{4.4}$$

Then

$$\lim_{h' \to 0} \omega_{h'}(t) = 0 \quad \text{ uniformly on } [0, a].$$

It is easy to see that $\omega_{h'}$ satisfies the recurrent inequality

$$\omega_{h'}^{(r+1)} \geq \omega_{h'}^{(r)} + h_0\sigma(t^{(r)}, \omega_{h'}^{(r)}) + h_0\gamma(h'), 0 \leq r \leq N-1.$$

This gives $\varepsilon_{h'}^{(r)} \leq \omega_{h'}^{(r)}$ for $0 \leq r \leq N$ and assertion is satisfied with $\alpha(h') = \omega_{h'}(a)$. This proves the theorem. □

Remark 4.4. If $E = [0, a] \times [-b, b]$ where $[-b, b] \subset \mathbb{R}^n$, $b = (b_1, \dots, b_n)$, $b_i > 0$ for $1 \leq i \leq n$, then the interpolating operator $T_{h'}$ presented in [4], Chapter 5, satisfies Assumption $H[T_{h'}]$. The construction given in [4] can be easy extended on the set E considered in the paper.

Remark 4.5. Suppose that Assumption H$[f, g, G]$ is satisfied with

$$\sigma(t, p) = Lp, \ (t, p) \in [0, a] \times \mathbb{R}_+ \quad \text{ where } L \in \mathbb{R}_+.$$

Then we have assumed that f, g and G satisfy the Lipschitz condition with respect to the functional variable. We obtain the following error estimates

$$\|u_{h'}^{(i,m)} - v_{h'}^{(i,m)}\| \leq \alpha_0(h')e^{La} + \tilde{\gamma}(h')\frac{e^{La} - 1}{L} \quad \text{ on } E_{h'} \text{ if } L > 0,$$

and

$$\|u_{h'}^{(i,m)} - v_{h'}^{(i,m)}\| \leq \alpha_0(h') + a\tilde{\gamma}(h') \text{ on } E_{h'} \text{ if } L = 0.$$

The above inequalities follows from (4.3) with $\alpha(h') = \omega_{h'}(a)$ where $\omega_{h'} : [0, a] \to \mathbb{R}_+$ is a solution of the problem

$$\zeta'(t) = L\zeta(t) + \tilde{\gamma}(h'), \quad \zeta(0) = \alpha_0(h').$$

Remark 4.6. There are the following consequences of Theorem 4.3. In classical theorems concerning explicit difference methods for (2.12), (2.13) it is assumed that (see [5])

$$1 - 2h_0 \sum_{i=1}^{n} \frac{1}{h_i^2} f_{ii}(t, x) + h_0 \sum_{(i,j) \in J} \frac{1}{h_i h_j} \left| f_{ij}(t, x) \right| \geq 0, \quad (t, x) \in E. \tag{4.5}$$

It is important in our considerations that we have omitted the above assumption.

5. NUMERICAL EXAMPLES

Results presented in that paper are applied to differential equation with deviated variable and to the differential integral problem. Let $n = 2$ and

$$\bar{Q} = \{(x, y) \in \mathbb{R}^2 : 4x^2 + y^2 \leq 1\}, \quad E = [0, 1] \times \bar{Q}, \quad E_0 = [-a_0, 0] \times \bar{Q}.$$

Initial boundary value problems considered in the present section have solutions on E.

Example 5.1. Consider the following differential equation containing deviated variables

$$\partial_t z(t, x, y) = \partial_{xx} z(t, x, y) - \frac{1}{2}\partial_{xy} z(t, x, y) + \partial_{yy} z(t, x, y) + z(t, \frac{x}{2}, y) + z(t, x, y)\bar{f}(t, x, y) - \hat{f}(t, x, y) + \tilde{f}(z(t, x, y)) \tag{5.1}$$

and the initial boundary condition

$$z(t, x, y) = e^{-1} \text{ on } \partial_0 E \cup E_0 \tag{5.2}$$

where

$\bar{f}(t, x, y) = 4x^2+y^2-t(10-4x^2-y^2+1+4t(16x^2+y^2-2xy)),\ \hat{f}(t, x, y) = e^{t(x^2+y^2-1)-1}$,

$$\tilde{f}(p) = \begin{cases} 0; & p \leq 0, \\ p|\ln p|; & p \in (0, e^{-1}), \\ p; & p \geq e^{-1}, \end{cases}$$

The solution of (5.1), (5.2) is known, it is $v(t, x, y) = e^{t(4x^2+y^2-1)-1}$.

Remark 5.2. Write

$$G(t, x, y, z) = z\left(t, \frac{x}{2}, y\right) + z(t, x, y)\bar{f}(t, x, y) - \hat{f}(t, x, y) + \tilde{f}(z).$$

Then

$$|G(t, x, y, z) - G(t, x, y, \bar{z})| \leq L\|z - \bar{z}\|_t + \|z - \bar{z}\|_t |\ln \|z - \bar{z}\|_t|$$

It follows that condition 2) of Assumption H$[f, g, G]$ is satisfied with

$$\sigma(t, p) = Lp + p|\ln p|.$$

Let us denote by $u_{h'} : E_{h'} \to \mathbb{R}$ the solution of implicit difference problem corresponding to (5.1), (5.2). Let $\tilde{u}_{h'} : E_{h'} \to \mathbb{R}$ be a numerical approximation of $u_{h'}$. Let numbers $\varepsilon_{h'}^{(r)}$ be the arithmetical means of the errors with fixed $t^{(r)}$. We give experimental values of the above defined errors for $h_0 = 0.01$, $h_1 = 0.01$, $h_2 = 0.01$ in the following table:

Table I

$t^{(r)}$	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90	1.0
$\varepsilon^{(r)}$	$9 \cdot 10^{-5}$	$5 \cdot 10^{-4}$	$8 \cdot 10^{-4}$	$1 \cdot 10^{-3}$	$1 \cdot 10^{-3}$	$1 \cdot 10^{-3}$	$1 \cdot 10^{-3}$	$1 \cdot 10^{-3}$	$1 \cdot 10^{-3}$	$1 \cdot 10^{-3}$

Note, that the function f and the steps of the mesh do not satisfy condition (4.5), which is necessary for the explicit method to be convergent. In our numerical example the average errors of the explicit method exceeded 10^{24}.

Example 5.3. Let us consider the differential integral equation

(5.3)

$$\partial_t z(t,x,y) = \partial_{xx} z(t,x,y) + \partial_{xy} z(t,x,y) + \partial_{yy} z(t,x,y) \\ + 2t\left(\int_0^y sz(t,x,s)ds - 4\int_0^x sz(t,s,y)ds\right) + z(t,x,y)\bar{g}(t,x,y) + \tilde{g}(t,x,y)$$

and the initial boundary condition

$$z(t,x,y) = e^{-1} \text{ on } \partial_0 E \cup E_0 \tag{5.4}$$

where

$$\bar{g}(t,x,y) = 4x^2 + y^2 - t(10 + 4t(16x^2 + y^2 + 2xy)), \quad \tilde{g} = e^{-(t+1)}(e^{4tx^2} - e^{(ty^2})$$

The solution of (5.3), (5.4) is known, it is exactly the same that in previous example.

Let us denote by $u_h : E_h \to \mathbb{R}$ the solution of implicit difference problem corresponding to (5.3), (5.4). In Table II we give experimental values of ε_h for $h_0 = 0.01$, $h_1 = 0.01$, $h_2 = 0.01$:

Table II

$t^{(r)}$	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90	1.0
$\varepsilon^{(r)}$	$9 \cdot 10^{-5}$	$5 \cdot 10^{-4}$	$8 \cdot 10^{-4}$	$1 \cdot 10^{-3}$	$1 \cdot 10^{-3}$	$1 \cdot 10^{-3}$	$1 \cdot 10^{-3}$	$1 \cdot 10^{-3}$	$1 \cdot 10^{-3}$	$1 \cdot 10^{-3}$

In the considered case condition (4.5) is not satisfied and the explicit method is not convergent. The average errors exceeded 10^{34}.

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