

EXISTENCE RESULTS FOR IMPULSIVE BOUNDARY VALUE PROBLEM WITH INTEGRAL BOUNDARY CONDITIONS

MOUFFAK BENCHOHRA, FARIDA BERHOUN, AND JUAN J. NIETO

Laboratoire de Mathématiques, Université de Sidi Bel Abbès,
BP 89, 22000, Sidi Bel Abbès, Algérie

Laboratoire de Mathématiques, Université de Sidi Bel Abbès,
BP 89, 22000, Sidi Bel Abbès, Algérie

Departamento de Analisis Matematico, Facultad de Matematicas
Universidad de Santiago de Compostela, Santiago de Compostela, Spain

1. INTRODUCTION

This paper is concerned the existence and uniqueness of solutions for the second order impulsive boundary value problems with integral boundary conditions:

$$(1.1) \quad y''(t) = f(t, y(t)), \text{ for a.e. } t \in J := [0, 1], \quad t \neq t_i, \quad i = 1, \dots, m,$$

$$(1.2) \quad \Delta y|_{t=t_i} = I_i(y(t_i^-)), \quad i = 1, \dots, m;$$

$$(1.3) \quad \Delta y'|_{t=t_i} = \bar{I}_i(y(t_i^-)), \quad i = 1, \dots, m;$$

$$(1.4) \quad y(0) - k_1 y'(0) = \int_0^1 h_1(s, y(s)) ds;$$

$$(1.5) \quad y(1) + k_2 y'(1) = \int_0^1 h_2(s, y(s)) ds;$$

where $f, h_1, h_2 : J \times \mathbb{R} \rightarrow \mathbb{R}$ and $I_i, \bar{I}_i : \mathbb{R} \rightarrow \mathbb{R}$, are given functions, $t_i \in J$, $0 = t_0 < t_1 < \dots < t_m < t_{m+1} = 1$, k_1, k_2 are nonnegative constants, $\Delta y|_{t=t_i} = y(t_i^+) - y(t_i^-)$, $y(t_i^+) = \lim_{h \rightarrow 0^+} y(t_i + h)$ and $y(t_i^-) = \lim_{h \rightarrow 0^+} y(t_i - h)$ are the right and left hand limits of $y(t)$ at $t = t_i$, respectively. In what follows, we refer to problem (1.1)–(1.5) as (P).

The theory of impulsive differential equations (IDE) is an active area of research in recent years since they are adequate mathematical models of real phenomena in the physical, biological and social sciences. There has been a significant development in the theory of IDE; see for example the books [12, 31, 39, 43] and the papers [17, 23, 35, 36, 37, 40, 41, 44, 45, 46].

Boundary value problems with integral boundary conditions constitute a very interesting and important class of problems. These include two-point, three-point, multi-point and nonlocal boundary value problems as special cases. Integral boundary conditions appear in population dynamics [14] and cellular systems [1]. Moreover, boundary value problems with integral boundary conditions have been studied by a number of authors such as, for instance, Ahmad *et al.* [2], Arara and Benchohra [4], Bairamov and Karaman [5], Belarbi *et al.* [6, 7], Benchohra *et al.* [8, 10, 11, 13], Brown and Plum [16], Denche and Kourta [18, 19], Gallardo [21, 22], Infante [25], Jankowskii [26, 27], Karakostas and Tsamatos [28], Khan [29], Krall [30], Marhoune and Bouzit [33], Peculyte *et al.* [38] and the references therein. Other recent results involving integral boundary conditions are given in [3, 15, 20, 32, 34, 42].

We shall provide sufficient conditions ensuring some new existence and uniqueness results for problem (P) via an application of the Banach contraction principle and the nonlinear alternative of Leray-Schauder type. Our results extend and complement the previously cited results to those considered with integral boundary conditions.

2. PRELIMINARIES

In this section, we introduce notations, definitions and preliminary facts which are used throughout this paper.

Let $C(J, \mathbb{R})$ be the Banach space of all continuous functions from J into \mathbb{R} with the norm

$$\|y\|_{\infty} = \sup\{|y(t)| : 0 \leq t \leq 1\}.$$

$L^1(J, \mathbb{R})$ denote the Banach space of functions $y : J \rightarrow \mathbb{R}$ that are Lebesgue integrable with norm

$$\|y\|_{L^1} = \int_0^1 |y(t)| dt.$$

We consider the space

$$PC = \{y : [0, 1] \rightarrow \mathbb{R} : y_i \in C(J_i, \mathbb{R}), i = 1, \dots, m, \\ y(t_i^-) \text{ and } y(t_i^+) \text{ exist } i = 1, \dots, m, \text{ and } y(t_i^-) = y(t_i^+)\}.$$

PC is a Banach space with the norm

$$\|y\|_{PC} = \max \|y_i\|_{J_i} : i = 0, \dots, m,$$

where y_i is the restriction of y to $J_i = (t_i, t_{i+1}] \subset [0, 1]$, $i = 0, \dots, m$, and

$$\|y\|_{J_i} = \max_{t \in J_i} |y_i(t)|.$$

$AC^1((0, 1), \mathbb{R})$ is the space of differentiable functions $y : (0, 1) \rightarrow \mathbb{R}$, whose first derivative y' is absolutely continuous.

Definition 2.1. A map $f : J \times \mathbb{R} \rightarrow \mathbb{R}$ is said to be Carathéodory if

- $t \mapsto f(t, y)$ is measurable for each $y \in \mathbb{R}$;
- $y \mapsto f(t, y)$ is continuous for almost all $t \in J$;

In what follows, we assume that f is Carathéodory.

3. MAIN RESULT

We define a solution problem (P) as follows:

Definition 3.1. A function $y \in PC \cap \cup_{i=0}^m AC^1((t_i, t_{i+1}), \mathbb{R})$ is said to be a solution of (P) if $y''(t) = f(t, y(t))$, for a.e. $t \in J, t \neq t_i, i = 1, \dots, m$, and for each $i = 1, \dots, m$, $\Delta y|_{t=t_i} = I_i(y(t_i^-)), \Delta y'|_{t=t_i} = \bar{I}_i(y(t_i^-))$ and the boundary conditions (1.4)–(1.5) are satisfied.

For $a_i, b_i, i = 1, \dots, m$ real numbers, and $\sigma, \rho_1, \rho_2 : J \rightarrow \mathbb{R}$ be integrable functions, we consider the following linear problem

$$(3.1) \quad y''(t) = \sigma(t), \quad \text{a.e. } t \in J,$$

$$(3.2) \quad \Delta y|_{t=t_i} = a_i, \quad i = 1, \dots, m;$$

$$(3.3) \quad \Delta y'|_{t=t_i} = b_i, \quad i = 1, \dots, m;$$

$$(3.4) \quad y(0) - k_1 y'(0) = \int_0^1 \rho_1(s) ds;$$

$$(3.5) \quad y(1) + k_2 y'(1) = \int_0^1 \rho_2(s) ds.$$

We shall refer to (3.1)–(3.5) as (LP). We need the following auxiliary result to prove our existence results. Its proof can be found in Lemma 3.2 [9].

Lemma 3.2. $y \in PC$ is solution of the equation

$$(3.6) \quad y(t) = p(t) + \int_0^1 G(t, s)\sigma(s)ds + \sum_{i=1}^m W_i(t),$$

where

$$(3.7) \quad G(t, s) = \alpha \begin{cases} (k_1 + t)(1 - s + k_2), & 0 \leq t \leq s, \\ (k_1 + s)(1 - t + k_2), & s \leq t \leq 1 \end{cases}$$

$$p(t) = -\alpha \left\{ (1 - t + k_2) \int_0^1 \rho_1(s) ds + (k_1 + t) \int_0^1 \rho_2(s) ds \right\},$$

$$W_i(t) = -\alpha \begin{cases} (k_1 + t)[-a_i - (1 - t_i + k_2)b_i], & 0 \leq t \leq t_i, \\ (1 - t + k_2)[a_i - (t_i + k_1)b_i], & t_i \leq t \leq 1, \end{cases}$$

and $\alpha = \frac{-1}{(1+k_1+k_2)}$, if and only if y is solution of the boundary value problem (LP).

Remark 3.3. Note that for the solution (3.6) we have that:

- (i) p is the solution of $y'' = 0$ with the boundary conditions (3.4)–(3.5)
- (ii) $\int_0^1 G(t, s)\sigma(s)ds$ is the solution of $y'' = \sigma$ with homogeneous boundary conditions $y(0) = y(1) = 0$, and G is the Green's function for that problem.
- (iii) $\sum_{i=1}^m W_i(t)$ is the solution of $y'' = 0$ with homogeneous boundary conditions $y(0) = y(1) = 0$ and jumps (3.2)–(3.3).
- (iv) The function G is nonpositive and

$$G^* := \sup_{(t,s) \in J^2} |G(t, s)| \leq \frac{(1+k_1)(1+k_2)}{1+k_1+k_2}.$$

Our first result for problem (P) is based on the Banach contraction principle.

Theorem 3.4. Assume that f is Carathéodory, $h_1, h_2 : J \times \mathbb{R} \rightarrow \mathbb{R}$ are continuous and the following hypotheses hold

(H1) There exists $\bar{\alpha} > 0$ such that

$$|f(t, u) - f(t, v)| \leq \bar{\alpha}|u - v|, \quad \text{for each } u, v \in \mathbb{R} \text{ and a.e. } t \in J.$$

(H2) There exist constants $c, \bar{c} > 0$ such that

$$|h_1(t, u) - h_1(t, v)| \leq c|u - v|, \quad \text{for each } u, v \in \mathbb{R}, \text{ and each } t \in J$$

$$|h_2(t, u) - h_2(t, v)| \leq \bar{c}|u - v|, \quad \text{for each } u, v \in \mathbb{R} \text{ and each } t \in J$$

(H3) There exist constants $d, \bar{d} > 0$ such that

$$|I_i(u) - I_i(v)| \leq d|u - v| \quad \text{for each } u, v \in \mathbb{R},$$

$$|\bar{I}_i(u) - \bar{I}_i(v)| \leq \bar{d}|u - v| \quad \text{for each } u, v \in \mathbb{R}.$$

If

$$(3.8) \quad c + \bar{c} + \bar{\alpha}G^* + d + \bar{d}(1+k_2) < 1,$$

then problem (P) has a unique solution.

Proof. Transform problem (P) into a fixed point problem. Consider the operator, $N : PC \rightarrow PC$ defined by

$$(Ny)(t) = p(t) + \int_0^1 G(t, s)f(s, y(s))ds + \sum_{i=1}^m W_i(t, y(t_i)),$$

where

$$p(t) = -\alpha \left\{ (1-t+k_2) \int_0^1 h_1(s, y(s))ds + (k_1+t) \int_0^1 h_2(s, y(s))ds \right\},$$

and $G(t, s)$ is given by (3.7), and $W_i(t, y(t_i))$ is defined by

$$W_i(t, y(t_i)) = -\alpha \begin{cases} (k_1 + t)[-I_i(y(t_i)) - (1 - t_i + k_2)\bar{I}_i(y(t_i))], & 0 \leq t \leq t_i, \\ (1 - t + k_2)[I_i(y(t_i)) - (t_i + k_1)\bar{I}_i(y(t_i))], & t_i \leq t \leq 1. \end{cases}$$

Consider y, \bar{y} in PC . Then for each $t \in J$

$$\begin{aligned} |(Ny)(t) - (N\bar{y})(t)| &\leq -\alpha(1 - t + k_2) \int_0^1 |h_1(s, y(s)) - h_1(s, \bar{y}(s))| ds \\ &\quad - \alpha(1 + k_1) \int_0^1 |h_2(s, y(s)) - h_2(s, \bar{y}(s))| ds \\ &\quad + \int_0^1 |G(t, s)| |f(s, y(s)) - f(s, \bar{y}(s))| ds \\ &\quad + \sum_{i=1}^m |W_i(t, y(t_i)) - W_i(t, \bar{y}(t_i))|. \end{aligned}$$

For $0 \leq t \leq t_i$, we have

$$\begin{aligned} |(Ny)(t) - (N\bar{y})(t)| &\leq -\alpha(1 - t + k_2) \int_0^1 |h_1(s, y(s)) - h_1(s, \bar{y}(s))| ds \\ &\quad - \alpha(1 + k_1) \int_0^1 |h_2(s, y(s)) - h_2(s, \bar{y}(s))| ds \\ &\quad + \int_0^1 |G(t, s)| |f(s, y(s)) - f(s, \bar{y}(s))| ds \\ &\quad - \alpha(k_1 + 1) [|I_i(y(t_i)) - I_i(\bar{y}(t_i))|] \\ &\quad + (1 + k_2) [|\bar{I}_i(y(t_i)) - \bar{I}_i(\bar{y}(t_i))|] \\ &\leq \frac{1 + k_2}{1 + k_1 + k_2} \int_0^1 c |y(s) - \bar{y}(s)| ds \\ &\quad + \frac{1 + k_1}{1 + k_1 + k_2} \int_0^1 \bar{c} |y(s) - \bar{y}(s)| ds \\ &\quad + G^* \int_0^1 \bar{\alpha} |y(s) - \bar{y}(s)| ds \\ &\quad + \frac{1 + k_1}{1 + k_1 + k_2} [d \|y - \bar{y}\|_{PC} + \bar{d}(1 + k_2) \|y - \bar{y}\|_{PC}] \\ &\leq [c + \bar{c} + \bar{\alpha}G^* + d + \bar{d}(1 + k_2)] \|y - \bar{y}\|_{PC}. \end{aligned}$$

Similarly, we obtain the same result when $t_i \leq t \leq 1$. Then by (3.8) N is a contraction, so by Banach's principle N has a unique fixed point which is solution of problem (P). □

Now we give an existence result based on the nonlinear alternative of Leray-Schauder type. Let us introduce the following hypotheses which are assumed hereafter:

(H4) There exist a continuous nondecreasing function $g : [0, \infty) \rightarrow (0, \infty)$ and $q \in L^1(J, \mathbb{R}^+)$ such that

$$|f(t, u)| \leq q(t)g(|u|), \quad \text{for each } u \in \mathbb{R} \text{ and for } t \in J$$

(H5) There exist constants $c_1, c_2 > 0$ with $1 - 2c_1 - (1 + k_1k_2c_2) > 0$ such that

$$|I_i(u)| \leq c_1|u|, \quad \text{and } |\bar{I}_i(u)| \leq c_2|u|, \quad \text{for each } u \in \mathbb{R}, i = 1, \dots, m$$

(H6) There exist functions $g_1, g_2 : [0, \infty) \rightarrow [0, \infty)$ continuous, nondecreasing and $q_1, q_2 \in L^1(J, \mathbb{R}^+)$ such that

$$|h_1(t, u)| \leq q_1(t)g_1(|u|) \quad \text{for each } u \in \mathbb{R} \text{ and for } t \in J.$$

$$|h_2(t, u)| \leq q_2(t)g_2(|u|) \quad \text{for each } u \in \mathbb{R} \text{ and for } t \in J.$$

(H7) There exist a constant $M > 0$ such that

$$(3.9) \quad \frac{\bar{M}M}{g_1(M) \int_0^1 q_1(s)ds + g_2(M) \int_0^1 q_2(s)ds + G^*g(M) \int_0^1 q(s)ds} > 1,$$

where

$$\bar{M} = 1 - 2c_1 - (1 + k_1k_2c_2).$$

Theorem 3.5. *Suppose that f is Carathéodory, $h_1, h_2 : J \times \mathbb{R} \rightarrow \mathbb{R}$ are continuous, $I_1, I_2 : \mathbb{R} \rightarrow \mathbb{R}$ are continuous, and the hypotheses (H4)–(H7) are satisfied, then problem (P) has at least one solution.*

Proof. We shall show that the operator N defined in Theorem 3.4 is continuous and completely continuous.

Step 1. N is continuous.

Let $\{y_n\}$ be a sequence such that $y_n \rightarrow y$ in PC . Then for each $t \in J$

$$\begin{aligned} |(Ny_n)(t) - (Ny)(t)| &\leq -\alpha(1 - t + k_2) \int_0^1 |h_1(s, y_n(s)) - h_1(s, y(s))| ds \\ &\quad - \alpha(1 + k_1) \int_0^1 |h_2(s, y_n(s)) - h_2(s, y(s))| ds \\ &\quad + \int_0^1 |G(t, s)| |f(s, y_n(s)) - f(s, y(s))| ds \\ &\quad + \sum_{i=1}^m |W_i(t, y_n(t_i)) - W_i(t, y(t_i))| \\ &\leq \frac{1 + k_2}{1 + k_1 + k_2} \int_0^1 |h_1(s, y_n(s)) - h_1(s, y(s))| ds \\ &\quad + \frac{1 + k_1}{1 + k_1 + k_2} \int_0^1 |h_2(s, y_n(s)) - h_2(s, y(s))| ds \end{aligned}$$

$$\begin{aligned}
 &+ G^* \int_0^1 |f(s, y_n(s)) - f(s, y(s))| ds \\
 &+ \sum_{i=1}^m |W_i(t, y_n(t_i)) - W_i(t, y(t_i))|.
 \end{aligned}$$

Since the functions f is Carathéodory and h_1, h_2 are continuous, we have

$$\|N(y_n) - N(y)\|_{PC} \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Step 2. N maps bounded sets into bounded sets in PC .

Indeed, it is enough to show that there exists a positive constant ℓ such that for each $y \in B_r = \{y \in PC : \|y\|_{PC} \leq r\}$, one has $\|N(y)\|_{PC} \leq \ell$. Then for each $t \in J$, we have by (H4)–(H6)

$$\begin{aligned}
 |(Ny)(t)| &\leq |p(t)| + \int_0^1 |G(t, s)| |f(s, y(s))| ds + \sum_{i=1}^m |W_i(t, y(t_i))| \\
 &\leq p_r + G^* \int_0^1 g(r) q(s) ds \\
 &\quad - \alpha(k_1 + 1)[|I_i(y(t_i))| + (1 + k_2)|\bar{I}_i(y(t_i))|] \\
 &\leq p_r + G^* g(r) \int_0^1 q(s) ds \\
 &\quad + \frac{k_1 + 1}{1 + k_1 + k_2} c_1 |y(t_i)| + \frac{(k_1 + 1)(k_2 + 1)}{1 + k_1 + k_2} c_2 |y(t_i)| \\
 &\leq p_r + G^* g(r) \int_0^1 q(s) ds \\
 &\quad + \frac{(k_1 + 1)r}{1 + k_1 + k_2} (c_1 + (k_2 + 1)c_2) =: l,
 \end{aligned}$$

where

$$p_r = \frac{1}{1 + k_1 + k_2} \left\{ (1 + k_2)g_1(r) \int_0^1 q_1(s) ds + (1 + k_1)g_2(r) \int_0^1 q_2(s) ds \right\}.$$

Step 3. N maps bounded sets into equicontinuous sets of PC .

Let $\tau_1, \tau_2 \in [0, 1]$, $\tau_1 < \tau_2$, B_r be a bounded set of PC as in Step 2 and $y \in B_r$. Then

$$\begin{aligned}
 |(Ny)(\tau_2) - (Ny)(\tau_1)| &\leq |p(\tau_2) - p(\tau_1)| \\
 &\quad + \int_0^1 |G(\tau_2, s) - G(\tau_1, s)| |f(s, y(s))| ds \\
 &\quad + \sum_{i=1}^m |W_i(\tau_2, y(t_i)) - W_i(\tau_1, y(t_i))| \\
 &\leq -\alpha(\tau_1 - \tau_2) \left[\int_0^1 |h_1(s, y(s))| ds + \int_0^1 |h_2(s, y(s))| ds \right]
 \end{aligned}$$

$$\begin{aligned}
& + g(r) \int_0^1 q(s) |G(\tau_2, s) - G(\tau_1, s)| ds \\
& + \sum_{i=1}^m |W_i(\tau_2, y(t_i)) - W_i(\tau_1, y(t_i))|.
\end{aligned}$$

The right-hand side of the above inequality tends to zero as $\tau_2 - \tau_1 \rightarrow 0$. As a consequence of Steps 1 to 3 together with the version of Arzela-Ascoli theorem for a set of piecewise functions (see for instance [31]), we can conclude that $N : PC \rightarrow PC$ is continuous and completely continuous.

Step 4: (A priori bounds). We show now there exists an open set $U \subseteq PC$ with $y \neq \lambda N(y)$ for $\lambda \in (0, 1)$ and $y \in \partial U$. Let $y \in PC$ with $y = \lambda N(y)$ for some $0 < \lambda < 1$. Then for each $t \in J$, we have

$$y(t) = \lambda \left[p(t) + \int_0^1 G(t, s) f(s, y(s)) ds + \sum_{i=1}^m W_i(t, y(t_i)) \right]$$

so

$$|y(t)| \leq |p(t)| + \int_0^1 |G(t, s)| |f(s, y(s))| ds + \sum_{i=1}^m |W_i(t, y(t_i))|.$$

(H6) implies that

$$\begin{aligned}
|p(t)| & \leq -\alpha \left\{ (1 + k_2) \int_0^1 q_1(s) g_1(|y(s)|) ds + (k_1 + 1) \int_0^1 q_2(s) g_2(|y(s)|) ds \right\} \\
& \leq -\alpha \left\{ (1 + k_2) g_1(\|y\|_{PC}) \int_0^1 q_1(s) ds + (k_1 + 1) g_2(\|y\|_{PC}) \int_0^1 q_2(s) ds \right\} \\
& \leq g_1(\|y\|_{PC}) \int_0^1 q_1(s) ds + g_2(\|y\|_{PC}) \int_0^1 q_2(s) ds.
\end{aligned}$$

Using hypothesis (H5), we have

$$\begin{aligned}
\sum_{i=1}^m |W_i(t, y(t_i))| & \leq \sum_{0 \leq t \leq t_i} |W_i(t, y(t_i))| + \sum_{t_i \leq t \leq 1} |W_i(t, y(t_i))| \\
& \leq -\alpha [(k_1 + 1) |y(t_i)| (c_1 + (1 + k_2) c_2) \\
& \quad + (k_2 + 1) |y(t_i)| (c_1 + (1 + k_1) c_2)] \\
& \leq \|y\|_{PC} [2c_1 + (1 + k_1 k_2) c_2].
\end{aligned}$$

Then

$$\begin{aligned}
\|y\|_{PC} & \leq g_1(\|y\|_{PC}) \int_0^1 q_1(s) ds + g_2(\|y\|_{PC}) \int_0^1 q_2(s) ds \\
& \quad + G^* g(\|y\|_{PC}) \int_0^1 q(s) ds + \|y\|_{PC} [2c_1 + (1 + k_1 k_2) c_2],
\end{aligned}$$

this implies that

$$\frac{\overline{M}\|y\|_{PC}}{g_1(\|y\|_{PC}) \int_0^1 q_1(s)ds + g_2(\|y\|_{PC}) \int_0^1 q_2(s)ds + G^*g(\|y\|_{PC}) \int_0^1 q(s)ds} \leq 1,$$

then by (H7), we can affirm that there a constant $M > 0$ such that

$$\|y\|_{PC} \neq M.$$

Set

$$U = \{y \in PC : \|y\|_{PC} < M\}.$$

$N : \overline{U} \rightarrow PC$ is continuous and completely continuous. From the choice of U , there is no $y \in \partial U$ such that $y = \lambda N(y)$, for $\lambda \in (0, 1)$. As a consequence of the nonlinear alternative of Leray-Schauder type [24], we deduce that N has a fixed point in U which is a solution to problem (P). □

4. AN EXAMPLE

In this section we give an example to illustrate the usefulness of our main results. Let us consider the following impulsive boundary value problem,

$$(4.1) \quad y''(t) = \frac{e^{-t}(|y(t)| + 2)}{(9 + e^t)(1 + |y(t)|)}, \quad t \in J := [0, 1], \quad t \neq \frac{1}{2}$$

$$(4.2) \quad \Delta y|_{t=\frac{1}{2}} = \frac{|y(\frac{1}{2})|}{8(1 + |y(\frac{1}{2})|)}$$

$$(4.3) \quad \Delta y'|_{t=\frac{1}{2}} = \frac{|y(\frac{1}{2})|}{6(1 + |y(\frac{1}{2})|)}$$

$$(4.4) \quad y(0) - y'(0) = \int_0^1 \frac{|y(s)|}{6(1 + |y(s)|)} ds,$$

$$(4.5) \quad y(1) + y'(1) = \int_0^1 \frac{3|y(s)|}{20(1 + |y(s)|)} ds.$$

Set

$$f(t, x) = \frac{e^{-t}(x + 2)}{(9 + e^t)(1 + x)}, \quad (t, x) \in J \times [0, \infty),$$

$$I_1(x) = \frac{x}{8(1 + x)}, \quad x \in [0, \infty),$$

$$\overline{I}_1(x) = \frac{x}{6(1 + x)}, \quad x \in [0, \infty),$$

$$h_1(t, x) = \frac{x}{6(1 + x)}, \quad (t, x) \in [0, 1] \times [0, \infty),$$

$$h_2(t, x) = \frac{3x}{20(1 + x)}, \quad (t, x) \in [0, 1] \times [0, \infty).$$

Let $x, y \in [0, \infty)$ and $t \in J$. Then we have

$$\begin{aligned} |f(t, x) - f(t, y)| &= \frac{e^{-t}}{(9 + e^t)} \left| \frac{x+2}{1+x} - \frac{y+2}{1+y} \right| \\ &= \frac{e^{-t}|x-y|}{(9 + e^t)(1+x)(1+y)} \\ &\leq \frac{e^{-t}}{(9 + e^t)} |x-y| \\ &\leq \frac{1}{10} |x-y|. \end{aligned}$$

Hence the condition (H1) holds with $\bar{\alpha} = \frac{1}{10}$. Also, we have

$$|h_1(t, x) - h_1(t, y)| \leq \frac{1}{6} |x-y|, \quad \text{for each } x, y \in [0, \infty), \text{ and each } t \in [0, 1],$$

and

$$|h_2(t, x) - h_2(t, y)| \leq \frac{3}{20} |x-y|, \quad \text{for each } x, y \in [0, \infty) \text{ and each } t \in [0, 1],$$

which means that (H2) is satisfied. Moreover

$$|I_1(x) - I_1(y)| \leq \frac{1}{8} |x-y|,$$

and

$$|\bar{I}_1(x) - \bar{I}_1(y)| \leq \frac{1}{6} |x-y|.$$

Thus (H3) holds. From (3.7), the Green's function for the homogeneous problem is given by

$$G(t, s) = \begin{cases} \frac{(t+1)(s-2)}{3}, & 0 \leq s \leq t \\ \frac{(s+1)(t-2)}{3}, & t \leq s \leq 1. \end{cases}$$

We can easily see that

$$G^* = \sup_{(t,s) \in J \times J} |G(t, s)| < 1.$$

We shall check that condition (3.8) is satisfied with $c = \frac{1}{6}$ and $\bar{c} = \frac{3}{20}$. Indeed

$$c + \bar{c} + \bar{\alpha}G^* + d + \bar{d}(1 + k_2) < \frac{1}{6} + \frac{3}{20} + \frac{1}{10} + \frac{1}{8} + \frac{1}{3} = 1.$$

Then by Theorem 3.4 problem (4.1)–(4.5) has a unique solution on $[0, 1]$.

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