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MIXED NONLINEAR OSCILLATION OF SECOND ORDER FORCED DYNAMIC EQUATIONS

A. F. GÜVENİLİR, Y. ŞAHİNER, AND A. ZAFER

Department of Mathematics, Ankara University, 06100, Ankara, Turkey
Primary Mathematics Education Division, Department of Primary Education,
Faculty of Education, Hacettepe University, 06550, Ankara, Turkey
Department of Mathematics, Middle East Technical University, 06531
Ankara, Turkey

ABSTRACT. By using a technique similar to the one introduced by Kong [*J. Math. Anal. Appl.* 229 (1999) 258–270] and employing an arithmetic-geometric mean inequality, we establish oscillation criteria for second-order forced dynamic equations on time scales containing mixed nonlinearities of the form

$$\left(p(t)x^{\Delta}\right)^{\Delta} + q(t)x^{\sigma} + \sum_{i=1}^{n} q_i(t)|x^{\sigma}|^{\alpha_i - 1}x^{\sigma} = e(t), \quad t \geq t_0$$

where $p, q, q_i, e : \mathbb{T} \to \mathbb{R}$ are right-dense continuous with $p > 0$, σ is the forward jump operator, $x^{\sigma}(t) := x(\sigma(t))$, and the exponents satisfy

$$\alpha_1 > \cdots > \alpha_m > 1 > \alpha_{m+1} > \cdots \alpha_n > 0.$$

The results extend many well-known interval oscillation criteria from continuous case to arbitrary time scales.

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1. INTRODUCTION

In this paper, we consider the second order nonlinear dynamic equation

$$\left(p(t)x^{\Delta}\right)^{\Delta} + q(t)x^{\sigma} + \sum_{i=1}^{n} q_i(t)|x^{\sigma}|^{\alpha_i - 1}x^{\sigma} = e(t), \quad t \geq t_0 \tag{1.1}$$

on time scales, where $p, q, q_i, e : \mathbb{T} \to \mathbb{R}$ are right-dense continuous with $p > 0$, and the exponents satisfy

$$\alpha_1 > \cdots > \alpha_m > 1 > \alpha_{m+1} > \cdots \alpha_n > 0.$$

A *time scale* $\mathbb{T}$ is an arbitrary nonempty closed subset of the real numbers $\mathbb{R}$. The most well-known examples are $\mathbb{T} = \mathbb{R}$ and $\mathbb{T} = \mathbb{Z}$. For details, see the monograph [1].

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On a time scale $\mathbb{T}$, the forward and backward jump operators are defined by

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\} \quad \text{and} \quad \rho(t) = \sup\{s \in \mathbb{T} : s < t\},$$

where $\inf \emptyset := \sup \mathbb{T}$ and $\sup \emptyset := \inf \mathbb{T}$.

A point $t \in \mathbb{T}$, $t > \inf \mathbb{T}$, is said to be left-dense if $\rho(t) = t$, right-dense if $t < \sup \mathbb{T}$ and $\sigma(t) = t$, left-scattered if $\rho(t) < t$ and right-scattered if $\sigma(t) > t$. The graininess function μ for a time scale $\mathbb{T}$ is defined by $\mu(t) := \sigma(t) - t$. For any function $f : \mathbb{T} \to \mathbb{R}$, the notation $f^\sigma(t)$ denotes $f(\sigma(t))$. Let $a, b \in \mathbb{T}$ with $a \leq b$. The closed interval $[a, b]_{\mathbb{T}}$ is defined to be the set $\{t \in \mathbb{T} : a \leq t \leq b\}$. Other types of intervals are defined similarly.

By a proper solution of Eq. (1.1), we mean a function $x(t)$ which is nontrivial in the neighborhood of infinity and which satisfies the equation for $t \in [t_0, \infty)_{\mathbb{T}}$. As usual, such a solution $x(t)$ is said to be oscillatory if it is neither eventually positive nor eventually negative. The equation is called oscillatory if every proper solution is oscillatory.

In the case $\mathbb{T} = \mathbb{R}$, Eq. (1.1) becomes a second order differential equation

$$(p(t)x')' + q(t)x + \sum_{i=1}^{n} q_i(t)|x|^{\alpha_i - 1}x = e(t), \quad t \geq t_0 \tag{1.2}$$

while if $\mathbb{T} = \mathbb{Z}$, then it is a second order difference equation

$$\Delta(p(k)\Delta x(k)) + q(k)x(k+1) + \sum_{i=1}^{n} q_i(k)|x(k+1)|^{\alpha_i - 1}x(k+1) = e(k), \quad k \geq k_0. \tag{1.3}$$

There are many other special time scales useful in different point of view. In quantum calculus, the corresponding q-difference equation reads

$$\Delta_q\left(p(t)\Delta_q x(t)\right) + r(t)x(qt) + \sum_{i=1}^{n} r_i(t)|x(qt)|^{\alpha_i - 1}x(qt) = e(t), \quad t \geq t_0 \tag{1.4}$$

where $t_0, t \in q^{\mathbb{N}} := \{1, q, q^2, \ldots\}$, ($q > 1$ is a real number and n is a natural number), and

$$\Delta_q f(t) = [f(qt) - f(t)]/(qt - t).$$

Taking $\mathbb{T} = q^{\mathbb{N}}$, we see that $\sigma(t) = qt$ and hence $f^\Delta(t) = \Delta_q f(t)$.

The oscillation behavior of Eq. (1.2) has been studied by Sun and Wong [2] and Sun and Meng [3]. Because such equations arise in population dynamics, as in the growth of bacteria population with competitive species, further research is necessary. When $n = 1$ and $q(t) \equiv 0$, Eq. (1.2) becomes

$$(p(t)x'(t))' + q_1(t)|x(t)|^{\alpha - 1}x(t) = e(t), \quad t \geq t_0. \tag{1.5}$$

The results obtained in [2, 3] reduce to those of El-Sayed [4], Sun et al. [5], Nasr [6], and Sun and Wong [7]. To the best of our knowledge there is no oscillation criteria available in the literature for Eq. (1.3), not to mention for Eq. (1.5).

In the last decade there has been a great deal of research activity on the oscillation theory of dynamic equations on time scales. We refer the reader to the papers [8, 9, 10, 11, 12, 13], where the authors have usually considered equations of the form

$$\big(r(t)|x^{\Delta}(t)|^{\alpha-1}x^{\Delta}(t)\big)^{\Delta}+p(t)|x(\tau(t))|^{\beta-1}x(\tau(t))+q(t)|x(\theta(t))|^{\gamma-1}x(\theta(t))=f(t),$$

where $\beta,\gamma>\alpha$, excluding the equations with mixed nonlinearities.

Very recently, Agarwal and Zafer [8] has obtained interval oscillation criteria similar to the ones given by Sun and Wong [2] for equations with mixed nonlinearities

$$\big(r(t)\Phi_\alpha(x^{\Delta})\big)^{\Delta}+f(t,x^{\sigma})=e(t)$$

where

$$f(t,x)=q(t)\Phi_\alpha(x)+\sum_{i=1}^{n}q_i(t)\Phi_{\beta_i}(x),\quad \Phi_*(u)=|u|^{*-1}u.$$

In the present work, our aim is to extend the paper [3] to time scale calculus, and hence derive some new interval oscillation criteria for Eq. (1.3) and Eq. (1.4) in the special cases $\mathbb{T}=\mathbb{Z}$ and $\mathbb{T}=q^{\mathbb{N}}$, respectively. We also state similar oscillation criteria for q-difference equations, the quantum calculus case. We use a technique similar to the one introduced by Kong [14] and a well-known arithmetic-geometric mean inequality [15] to establish several interval oscillation criteria for Eq. (1.1).

2. LEMMAS

We need the following preparatory lemmas. The first two are given by Sun and Wong [2, Lemma 1], the last two are quite elementary via differential calculus, see [2, 16].

Lemma 2.1. *For any given n-tuple $\{\alpha_1,\alpha_2,\dots,\alpha_n\}$ satisfying*

$$\alpha_1>\cdots>\alpha_m>1>\alpha_{m+1}>\cdots>\alpha_n>0,$$

there corresponds an n-tuple $\{\eta_1,\eta_2,\dots,\eta_n\}$ such that

$$\sum_{i=1}^{n}\alpha_i\eta_i=1,\quad \sum_{i=1}^{n}\eta_i<1,\quad 0<\eta_i<1. \tag{2.1}$$

Lemma 2.2. *For any given n-tuple $\{\alpha_1,\alpha_2,\dots,\alpha_n\}$ satisfying*

$$\alpha_1>\cdots>\alpha_m>1>\alpha_{m+1}>\cdots>\alpha_n>0,$$

there corresponds an n-tuple $\{\eta_1,\eta_2,\dots,\eta_n\}$ such that

$$\sum_{i=1}^{n}\alpha_i\eta_i=1,\quad \sum_{i=1}^{n}\eta_i=1,\quad 0<\eta_i<1. \tag{2.2}$$

Lemma 2.3. *If $A \geq 0$, $B \geq 0$, and $\gamma > 1$ are real numbers, then*

$$A\,u^{\gamma} - \gamma(\gamma-1)^{1/\gamma-1}A^{1/\gamma}B^{1-1/\gamma}\,u + B \geq 0, \quad u \in [0,\infty).$$

Lemma 2.4. *If $C \geq 0$, $D \geq 0$, and $0 < \gamma < 1$ are real numbers, then*

$$Cu - Du^{\gamma} \geq -(1-\gamma)\gamma^{\gamma/(1-\gamma)}C^{\gamma/(\gamma-1)}D^{1/(1-\gamma)}, \quad u \in [0,\infty).$$

3. THE MAIN RESULTS

A function $H(t,s) : \mathbb{T}^2 \to \mathbb{R}$ is said to belong to $\mathcal{H}_{\mathbb{T}}$ if and only if it is a right-dense continuous, has continuous Δ-partial derivatives, and satisfies $H(t,t) = 0$ and $H(t,s) \neq 0$ for all $t \neq s$.

We denote by $H_1(t,s)$ and $H_2(t,s)$ the Δ-partial derivatives $H^{\Delta_t}(t,s)$ and $H^{\Delta_s}(t,s)$ of $H(t,s)$ with respect to t and s, respectively.

The theorems below extend the results obtained in [3] to arbitrary time scales and coincide with them when $H^2(t,s)$ is replaced by $H(t,s)$. Indeed, if one sets $H(t,s) = \sqrt{U(t,s)}$ then it follows that

$$H_1(t,s) = \frac{U_1(t,s)}{\sqrt{U(\sigma(t),s)} + \sqrt{U(t,s)}}, \quad H_2(t,s) = \frac{U_2(t,s)}{\sqrt{U(t,\sigma(s))} + \sqrt{U(t,s)}}.$$

When $\mathbb{T} = \mathbb{R}$, they become

$$\frac{\partial H(t,s)}{\partial t} = \frac{\partial U(t,s)/\partial t}{2\sqrt{U(t,s)}}, \quad \frac{\partial H(t,s)}{\partial s} = \frac{\partial U(t,s)/\partial s}{2\sqrt{U(t,s)}}$$

as in [3]. However, we choose to keep $H^2(t,s)$ instead of $U(t,s)$ for simplicity.

Theorem 3.1. *Suppose that for any given $T \in \mathbb{T}$, there exist a_1, b_1, a_2, $b_2 \in [T,\infty)_{\mathbb{T}}$ such that*

$$q_i(t) \geq 0 \quad \text{for} \quad t \in [a_1,b_1]_{\mathbb{T}} \cup [a_2,b_2]_{\mathbb{T}}, \quad (i = 1,2,\dots,n) \tag{3.1}$$

and

$$(-1)^k e(t) \geq 0 \; (\not\equiv 0) \quad \text{for} \quad t \in [a_k,b_k]_{\mathbb{T}}, \quad (k = 1,2). \tag{3.2}$$

Let $\{\eta_1,\eta_2,\dots,\eta_n\}$ be an n-tuple satisfying (2.1) *in Lemma* 2.1. *If there exist a function $H \in \mathcal{H}_{\mathbb{T}}$ and numbers $c_k \in (a_k,b_k)_{\mathbb{T}}$ such that*

$$\begin{aligned}&\frac{1}{H^2(c_k,a_k)}\int_{a_k}^{c_k}\Big[H^2(\sigma(s),a_k)Q(s) - p(s){H_1}^2(s,a_k)\Big]\Delta s\\ &+\frac{1}{H^2(b_k,c_k)}\int_{c_k}^{b_k}\Big[H^2(b_k,\sigma(s))Q(s) - p(s){H_2}^2(b_k,s)\Big]\Delta s > 0\end{aligned} \tag{3.3}$$

for $k = 1,2$, where

$$Q(t) = q(t) + k_0|e(t)|^{\eta_0}\prod_{i=1}^{n}q_i^{\eta_i}(t), \quad k_0 = \prod_{i=0}^{n}{\eta_i}^{-\eta_i}, \quad \eta_0 = 1 - \sum_{i=1}^{n}\eta_i,$$

then Eq. (1.1) *is oscillatory.*

Proof. To arrive at a contradiction, let us suppose that x is a nonoscillatory solution of (1.1). First, we assume that $x(t)$ is positive for all $t \geq t_1$, for some $t_1 \in [t_0, \infty)_{\mathbb{T}}$. Let $t \in [a_1, b_1]_{\mathbb{T}}$, where $a_1 \geq t_1$ is sufficiently large.

Define

$$w(t) = -p(t)\frac{x^{\Delta}(t)}{x(t)}.$$

It follows that

$$w^{\Delta}(t) = -\frac{(p(t)x^{\Delta}(t))^{\Delta}}{x(\sigma(t))} + p(t)\frac{(x^{\Delta}(t))^2}{x(t)x(\sigma(t))},$$

and hence

$$w^{\Delta}(t) = q(t) + \sum_{i=1}^{n} q_i(t)x^{\alpha_i - 1}(\sigma(t)) - \frac{|e(t)|}{x(\sigma(t))} + \frac{1}{p(t) - \mu(t)w(t)}w^2(t). \tag{3.4}$$

Note that

$$p(t) - \mu(t)w(t) = p(t)\Big[\frac{x(t) + \mu(t)x^{\Delta}(t)}{x(t)}\Big] = p(t)\frac{x(\sigma(t))}{x(t)} > 0.$$

By our assumptions (3.1) and (3.2), we have $q_i(t) \geq 0$ and $e(t) \leq 0$ on $[a_1, b_1]_{\mathbb{T}}$. Set

$$u_i = \frac{1}{\eta_i}q_i(t)x^{\alpha_i - 1}(\sigma(t)), \quad u_0 = \frac{1}{\eta_0}\frac{|e(t)|}{x(\sigma(t))}.$$

Then (3.4) becomes

$$w^{\Delta}(t) = q(t) + \sum_{i=0}^{n} \eta_i u_i + \frac{1}{p(t) - \mu(t)w(t)}w^2(t). \tag{3.5}$$

In view of (3.5) and the arithmetic-geometric mean inequality, see [15],

$$\sum_{i=0}^{n} \eta_i u_i \geq \prod_{i=0}^{n} u_i^{\eta_i},$$

we see that

$$\begin{aligned} w^{\Delta}(t) &\geq q(t) + k_0|e(t)|^{\eta_0}\prod_{i=1}^{n} q_i^{\eta_i}(t) + \frac{1}{p(t) - \mu(t)w(t)}w^2(t) \\ &= Q(t) + \frac{1}{p(t) - \mu(t)w(t)}w^2(t). \end{aligned} \tag{3.6}$$

Note that

$$\begin{aligned} H^2(t, \sigma(s))w^{\Delta}(s) &= (H^2(t,s)w(s))^{\Delta_s} - (H^2(t,s))^{\Delta_s}w(s) \\ &= (H^2(t,s)w(s))^{\Delta_s} - \big(H^{\Delta_s}(t,s)H(t,s) + H(t,\sigma(s))H^{\Delta_s}(t,s)\big)w(s) \\ &= (H^2(t,s)w(s))^{\Delta_s} - H^{\Delta_s}(t,s)\big(H(t,\sigma(s)) - \mu(s)H^{\Delta_s}(t,s)\big)w(s) \\ &\quad - H(t,\sigma(s))H^{\Delta_s}(t,s)w(s) \\ &= (H^2(t,s)w(s))^{\Delta_s} - 2H^{\Delta_s}(t,s)H(t,\sigma(s))w(s) + \mu(s)\big(H^{\Delta_s}(t,s)\big)^2 w(s). \end{aligned}$$

Using this identity in (3.6) we have

$$\begin{aligned} H^2(t,\sigma(s))Q(s) &\le (H^2(t,s)w(s))^{\Delta_s} - 2H^{\Delta_s}(t,s)H(t,\sigma(s))w(s) \\ &\quad + \mu(s)\big(H^{\Delta_s}(t,s)\big)^2 w(s) - \frac{H^2(t,\sigma(s))}{p(s)-\mu(s)w(s)} w^2(s) \\ &= (H^2(t,s)w(s))^{\Delta_s} + p(s)\Big(H^{\Delta_s}(t,s)\Big)^2 \\ &\quad - \Big(\frac{H(t,\sigma(s))w(s)}{\sqrt{p(s)-\mu(s)w(s)}} + \sqrt{p(s)-\mu(s)w(s)}H^{\Delta_s}(t,s)\Big)^2 \end{aligned}$$

Thus,

$$H^2(t,\sigma(s))Q(s) - p(s)\Big(H^{\Delta_s}(t,s)\Big)^2 \le (H^2(t,s)w(s))^{\Delta_s}. \tag{3.7}$$

It follows from (3.7) that

$$\frac{1}{H^2(b_1,c_1)} \int_{c_1}^{b_1} \Big[H^2(b_1,\sigma(s))Q(s) - p(s)H_1^2(b_1,s)\Big]\Delta s \le -w(c_1). \tag{3.8}$$

In a similar manner, one can easily obtain that

$$H^2(\sigma(t),s)Q(t) - p(t)\Big(H^{\Delta_t}(t,s)\Big)^2 \le (H^2(t,s)w(t))^{\Delta_t}, \tag{3.9}$$

and hence

$$\frac{1}{H^2(c_1,a_1)} \int_{a_1}^{c_1} \Big[H^2(\sigma(t),a_1)Q(t) - p(t)H_2^2(t,a_1)\Big]\Delta t \le w(c_1). \tag{3.10}$$

Finally, from (3.8) and (3.10) we have

$$\begin{aligned} &\frac{1}{H^2(c_1,a_1)} \int_{a_1}^{c_1} \Big[H^2(\sigma(t),a_1)Q(t) - p(t)H_2^2(t,a_1)\Big]\Delta t \\ &\qquad + \frac{1}{H^2(b_1,c_1)} \int_{c_1}^{b_1} \Big[H^2(b_1,\sigma(s))Q(s) - p(s)H_1^2(b_1,s)\Big]\Delta s \le 0, \end{aligned}$$

which contradicts (3.3). This completes the proof when $x(t)$ is eventually positive. The proof when $x(t)$ is eventually negative is analogous by repeating the arguments on the interval $[a_2,b_2]_{\mathbb{T}}$ instead of $[a_1,b_1]_{\mathbb{T}}$. □

Theorem 3.1 fails to apply if $e(t) \equiv 0$. In that case, we give the following theorem.

Theorem 3.2. *Suppose that for any given $T \in \mathbb{T}$, there exist $a, b \in [T,\infty)_{\mathbb{T}}$ such that*

$$q_i(t) \ge 0 \quad \text{for} \quad t \in [a,b]_{\mathbb{T}}, \quad (i = 1,2,\dots,n). \tag{3.11}$$

Let $\{\eta_1, \eta_2, \dots, \eta_n\}$ be an n-tuple satisfying (2.2) in Lemma 2.2. If there exist a function $H \in \mathcal{H}_{\mathbb{T}}$ and a number $c \in (a,b)_{\mathbb{T}}$ such that

$$\begin{aligned}&\frac{1}{H^2(c,a)}\int_a^c \left[H^2(\sigma(s),a)\tilde{Q}(s) - p(s){H_1}^2(s,a)\right]\Delta s\\ &+\frac{1}{H^2(b,c)}\int_c^b \left[H^2(b,\sigma(s))\tilde{Q}(s) - p(s){H_2}^2(b,s)\right]\Delta s > 0\end{aligned} \tag{3.12}$$

where

$$\tilde{Q}(t) = q(t) + k_1 \prod_{i=1}^{n} q_i^{\eta_i}(t), \qquad k_1 = \prod_{i=1}^{n} {\eta_i}^{-\eta_i}$$

then Eq. (1.1) with $e(t) \equiv 0$ is oscillatory.

Proof. Proceeding as in the proof of Theorem 3.1, we arrive at

$$w^{\Delta}(t) = q(t) + \sum_{i=1}^{n} \eta_i u_i + \frac{1}{p(t) - \mu(t)w(t)} w^2(t). \tag{3.13}$$

The arithmetic-geometric mean inequality we now need is

$$\sum_{i=1}^{n} \eta_i u_i \geq \prod_{i=1}^{n} u_i^{\eta_i}.$$

The remainder of the proof is exactly the same as that of Theorem 3.1. □

As in [2, 3], we can also remove the sign condition on the coefficients of the sublinear terms by requiring that $e(t)$ never vanishes on the intervals of interest.

Theorem 3.3. *Suppose that for any given $T \in \mathbb{T}$, there exist a_1, b_1, a_2, $b_2 \in [T,\infty)_{\mathbb{T}}$ such that*

$$q_i(t) \geq 0 \quad \textit{for} \quad t \in [a_1,b_1]_{\mathbb{T}} \cup [a_2,b_2]_{\mathbb{T}}, \quad (i = 1,2,\dots,m) \tag{3.14}$$

and

$$(-1)^k e(t) > 0 \quad \textit{for} \quad t \in [a_k,b_k]_{\mathbb{T}}, \quad (k=1,2). \tag{3.15}$$

If there exist a function $H \in \mathcal{H}_{\mathbb{T}}$, positive numbers λ_i and ϵ_i with

$$\sum_{i=1}^{m} \lambda_i + \sum_{i=m+1}^{n} \epsilon_i = 1$$

and numbers $c_k \in (a_k,b_k)_{\mathbb{T}}$ such that

$$\begin{aligned}&\frac{1}{H^2(c_k,a_k)}\int_{a_k}^{c_k} \left[H^2(\sigma(s),a_k)\hat{Q}(s) - p(s){H_1}^2(s,a_k)\right]\Delta s\\ &+\frac{1}{H^2(b_k,c_k)}\int_{c_k}^{b_k} \left[H^2(b_k,\sigma(s))\hat{Q}(s) - p(s){H_2}^2(b_k,s)\right]\Delta s > 0\end{aligned} \tag{3.16}$$

for $k = 1, 2$, *where*

$$\hat{Q}(t) = q(t) + \sum_{i=1}^{m} \mu_i \big(\lambda_i |e(t)|\big)^{1-\frac{1}{\alpha_i}} q_i^{\frac{1}{\alpha_i}}(t) - \sum_{i=m+1}^{n} \delta_i \big(\epsilon_i |e(t)|\big)^{1-\frac{1}{\alpha_i}} \hat{q}_i^{\frac{1}{\alpha_i}}(t)$$

with

$$\mu_i = \alpha_i(\alpha_i - 1)^{\frac{1}{\alpha_i}-1}, \quad \delta_i = \alpha_i(1-\alpha_i)^{\frac{1}{\alpha_i}-1} \quad \textit{and} \quad \hat{q}_i(t) = \max\{-q_i(t), 0\},$$

then Eq. (1.1) *is oscillatory.*

Proof. Suppose that Eq. (1.1) has a nonoscillatory solution. We may assume that $x(t)$ is eventually positive on $[a_1, b_1]_{\mathbb{T}}$ when a_1 sufficiently large. If $x(t)$ is eventually negative, then one can repeat the proof on the interval $[a_2, b_2]_{\mathbb{T}}$.

We rewrite Eq. (1.1) for $t \in [a_1, b_1]_{\mathbb{T}}$ as

$$\begin{aligned}\big(p(t)x^{\Delta}(t)\big)^{\Delta} + q(t)x(\sigma(t)) &+ \sum_{i=1}^{m} \Big[q_i(t)x^{\alpha_i}(\sigma(t)) - \lambda_i e(t)\Big] \\ &+ \sum_{i=m+1}^{n} \Big[q_i(t)x^{\alpha_i}(\sigma(t)) - \epsilon_i e(t)\Big] = 0.\end{aligned}$$

Applying Lemma 2.3 to each term in the first sum, we see that

$$\begin{aligned}\big(p(t)x^{\Delta}(t)\big)^{\Delta} + q(t)x(\sigma(t)) &+ \sum_{i=1}^{m} \mu_i \big(\lambda_i |e(t)|\big)^{1-\frac{1}{\alpha_i}} q_i^{\frac{1}{\alpha_i}}(t) x(\sigma(t)) \\ &+ \sum_{i=m+1}^{n} \Big[q_i(t)x^{\alpha_i}(\sigma(t)) - \epsilon_i e(t)\Big] \le 0.\end{aligned}$$

Set

$$w(t) = -p(t)\frac{x^{\Delta}(t))}{x(t)}.$$

In view of the last inequality, we have

$$\begin{aligned} w^{\Delta}(t) &= q(t) + \sum_{i=1}^{m} \mu_i \big(\lambda_i |e(t)|\big)^{1-\frac{1}{\alpha_i}} q_i^{\frac{1}{\alpha_i}}(t) \\ &\quad + \frac{1}{x(\sigma(t))} \sum_{i=m+1}^{n} \Big[q_i(t)x^{\alpha_i}(\sigma(t)) - \epsilon_i e(t)\Big] + \frac{w^2(t)}{p(t) - \mu(t)w(t)}. \end{aligned} \tag{3.17}$$

Noting that $q_i(t) = -(-q_i(t)) \ge -\hat{q}_i(t)$ and applying Lemma 2.4 for each term in the second sum in (3.17) with

$$u = x(\sigma(t)),\ D = \hat{q}_i(t),\ \lambda = \alpha_i \text{ and } C = -\lambda(1-\lambda)^{\frac{1}{\lambda}-1}\big(\epsilon |e(t)|\big)^{1-\frac{1}{\lambda}} \hat{q}_i^{\frac{1}{\lambda}}(t),$$

we obtain

$$w^{\Delta}(t) \ \ge \ \hat{Q}(t) + \frac{w^2(t)}{p(t) - \mu(t)w(t)}.$$

The remainder of the proof is the same as that of Theorem 3.1, hence it is omitted.

□

4. SPECIAL CASES

In this section, we restate the theorems obtained above for the particular time scales $\mathbb{T} = \mathbb{R}$, $\mathbb{T} = \mathbb{Z}$, and $\mathbb{T} = q^{\mathbb{N}}$. The results for $\mathbb{T} = \mathbb{R}$ coincide with the ones obtained in [3] when $H^2(t,s)$ is replaced by $H(t,s)$, see the note before Theorem 3.1 in Section 1. The interval oscillation criteria given for $\mathbb{T} = \mathbb{Z}$ and $\mathbb{T} = q^{\mathbb{N}}$ are completely new.

4.1. **Differential Equations.** Denote by $H_1(t,s)$ and $H_2(t,s)$ the usual partial derivatives of $H(t,s)$ with respect to the first and second variables, respectively. Note that $H(t,t) = 0$ for all t and $H(t,s) \neq 0$ for all $t \neq s$.

Theorem 4.1. *Suppose that for any given $T \geq t_0$, there exist real numbers a_1, b_1, a_2, b_2 satisfying $T \leq a_1 < b_1$, $T \leq a_2 < b_2$ such that*

$$q_i(t) \geq 0 \quad \text{for} \quad t \in [a_1, b_1] \cup [a_2, b_2], \quad (i = 1, 2, \ldots, n)$$

and

$$(-1)^k e(t) \geq 0 \; (\not\equiv 0) \quad \text{for} \quad t \in [a_k, b_k], \quad (k = 1, 2).$$

Let $(\eta_1, \eta_2, \ldots, \eta_n)$ be an n-tuple satisfying (2.1) *in Lemma* 2.1. *If there exist a function $H \in \mathcal{H}_{\mathbb{R}}$ and real numbers $c_k \in (a_k, b_k)$ such that*

$$\begin{aligned} &\frac{1}{H^2(c_k,a_k)} \int_{a_k}^{c_k} \Big[H^2(s,a_k) Q(s) - p(s) {H_1}^2(s,a_k) \Big] ds \\ &+ \frac{1}{H^2(b_k,c_k)} \int_{c_k}^{b_k} \Big[H^2(b_k,s) Q(s) - p(s) {H_2}^2(b_k,s) \Big] ds > 0 \end{aligned} \tag{4.1}$$

for $k = 1, 2$, where Q is the same as in Theorem 3.1, *then Eq.* (1.2) *is oscillatory.*

Theorem 4.2. *Suppose that for any given $T \geq t_0$, there exist real numbers a and b satisfying $T \leq a < b$ such that*

$$q_i(t) \geq 0 \quad \text{for} \quad t \in [a, b], \quad (i = 1, 2, \ldots, n).$$

Let $\{\eta_1, \eta_2, \ldots, \eta_n\}$, be an n-tuple satisfying (2.2) *in Lemma* 2.2. *If there exist a function $H \in \mathcal{H}_{\mathbb{R}}$ and a real number $c \in (a, b)$ such that*

$$\begin{aligned} &\frac{1}{H^2(c,a)} \int_{a}^{c} \Big[H^2(s,a) \tilde{Q}(s) - p(s) {H_1}^2(s,a) \Big] ds \\ &+ \frac{1}{H^2(b,c)} \int_{c}^{b} \Big[H^2(b,s) \tilde{Q}(s) - p(s) {H_2}^2(b,s) \Big] ds > 0 \end{aligned} \tag{4.2}$$

where $\tilde{Q}$ is the same as in Theorem 3.2, *then Eq.* (1.2) *with $e(t) \equiv 0$ is oscillatory.*

Theorem 4.3. *Suppose that for any given $T \geq t_0$, there exist real numbers a_1, b_1, a_2, b_2 satisfying $T \leq a_1 < b_1$, $T \leq a_2 < b_2$ such that*

$$q_i(t) \geq 0 \quad \text{for} \quad t \in [a_1, b_1] \cup [a_2, b_2], \quad (i = 1, 2, \ldots, m)$$

and

$$(-1)^k e(t) > 0 \quad \text{for} \quad t \in [a_k, b_k], \quad (k = 1, 2).$$

If there exist a function $H \in \mathcal{H}_{\mathbb{R}}$, positive numbers λ_i and μ_i with

$$\sum_{i=1}^{m} \lambda_i + \sum_{i=m+1}^{n} \epsilon_i = 1,$$

and numbers $c_k \in (a_k, b_k)$ such that

$$\begin{aligned} &\frac{1}{H^2(c_k, a_k)} \int_{a_k}^{c_k} \Big[H^2(s, a_k) \hat{Q}(s) - p(s) {H_1}^2(s, a_k) \Big] ds \\ &+ \frac{1}{H^2(b_k, c_k)} \int_{c_k}^{b_k} \Big[H^2(b_k, s) \hat{Q}(s) - p(s) {H_2}^2(b_k, s) \Big] ds > 0 \end{aligned} \tag{4.3}$$

for $k = 1, 2$, where $\hat{Q}$ is the same as in Theorem 3.3, then Eq. (1.2) is oscillatory.

4.2. **Difference Equations.** Let $[a, b]_{\mathbb{N}}$ denote a discrete interval, i.e.,

$$[a, b]_{\mathbb{N}} = \{a, a+1, a+2, \ldots, b\}, \quad a, b \in \mathbb{N}.$$

Note that $\mathcal{H}_{\mathbb{N}}$ denotes the functions defined on $\mathbb{N}^2$ and satisfying $H(j, j) = 0$ for all j and $H(j, i) \neq 0$ for all $j \neq i$.

Theorem 4.4. *Suppose that for any given natural number $T \geq t_0$, there exist natural numbers a_1, b_1, a_2, b_2 satisfying $T \leq a_1 < b_1$, $T \leq a_2 < b_2$ such that*

$$q_i(j) \geq 0 \quad \text{for} \quad j \in [a_1, b_1]_{\mathbb{N}} \cup [a_2, b_2]_{\mathbb{N}}, \quad (i = 1, 2, \ldots, n)$$

and

$$(-1)^k e(j) \geq 0 \ (\not\equiv 0) \quad \text{for} \quad j \in [a_k, b_k]_{\mathbb{N}}, \quad (k = 1, 2).$$

Let $\{\eta_1, \eta_2, \ldots, \eta_n\}$ be an n-tuple satisfying (2.1) in Lemma 2.1. If there exist a function $H \in \mathcal{H}_{\mathbb{N}}$ and numbers $c_k \in (a_k, b_k)_{\mathbb{N}}$ such that

$$\begin{aligned} &\frac{1}{H^2(c_k, a_k)} \sum_{j=a_k}^{c_k - 1} \Big[H^2(j+1, a_k) Q(j) - p(j) [H(j+1, a_k) - H(j, a_k)]^2 \Big] \\ &+ \frac{1}{H^2(b_k, c_k)} \sum_{j=c_k}^{b_k - 1} \Big[H^2(b_k, j+1) Q(j) - p(j) [H(b_k, j+1) - H(b_k, j)]^2 \Big] > 0 \end{aligned}$$

for $k = 1, 2$, where

$$Q(j) = q(j) + k_0 |e(j)|^{\eta_0} \prod_{i=1}^{n} q_i^{\eta_i}(j), \qquad k_0 = \prod_{i=0}^{n} {\eta_i}^{-\eta_i}, \qquad \eta_0 = 1 - \sum_{i=1}^{n} \eta_i,$$

then Eq. (1.3) *is oscillatory.*

Theorem 4.5. *Suppose that for any given natural number $T \geq t_0$, there exist natural numbers a and b satisfying $T \leq a < b$ such that*

$$q_i(j) \geq 0 \quad \textit{for} \quad j \in [a,b]_{\mathbb{N}}, \quad (i = 1, 2, \ldots, n).$$

Let $\{\eta_1, \eta_2, \ldots, \eta_n\}$ be an n-tuple satisfying (2.2) *in Lemma* 2.2. *If there exist a function $H \in \mathcal{H}_{\mathbb{N}}$ and a number $c \in (a,b)_{\mathbb{N}}$ such that*

$$\frac{1}{H^2(c,a)} \sum_{j=a}^{c-1} \left[H^2(j+1,a)\tilde{Q}(j) - p(j)[H(j+1,a) - H(j,a)]^2 \right]$$
$$+ \frac{1}{H^2(b,c)} \sum_{j=c}^{b-1} \left[H^2(b,j+1)\tilde{Q}(j) - p(j)[H(b,j+1) - H(b,j)]^2 \right] > 0$$

where

$$\tilde{Q}(j) = q(j) + k_1 \prod_{i=1}^{n} q_i^{\eta_i}(j), \qquad k_1 = \prod_{i=1}^{n} \eta_i^{-\eta_i}$$

then Eq. (1.3) *with $e(k) \equiv 0$ is oscillatory.*

Theorem 4.6. *Suppose that for any given natural number $T \geq t_0$, there exist natural numbers a_1, b_1, a_2, b_2 satisfying $T \leq a_1 < b_1$, $T \leq a_2 < b_2$ such that*

$$q_i(j) \geq 0 \quad \textit{for} \quad j \in [a_1,b_1]_{\mathbb{N}} \cup [a_2,b_2]_{\mathbb{N}}, \quad (i = 1, 2, \ldots, m)$$

and

$$(-1)^k e(j) > 0 \quad \textit{for} \quad j \in [a_k,b_k]_{\mathbb{N}}, \quad (k = 1, 2).$$

If there exist a function $H \in \mathcal{H}_{\mathbb{N}}$, positive numbers λ_i and ϵ_i with

$$\sum_{i=1}^{m} \lambda_i + \sum_{i=m+1}^{n} \epsilon_i = 1,$$

and numbers $c_k \in (a_k,b_k)_{\mathbb{N}}$ such that

$$\frac{1}{H^2(c_k,a_k)} \sum_{j=a_k}^{c_k-1} \left[H^2(j+1,a_k)\hat{Q}(j) - p(j)[H(j+1,a_k) - H(j,a_k)]^2 \right]$$
$$+ \frac{1}{H^2(b_k,c_k)} \sum_{c_k}^{b_k-1} \left[H^2(b_k,j+1)\hat{Q}(j) - p(j)[H(b_k,j+1) - H(b_k,j)]^2 \right] > 0$$

for $k = 1, 2$, where

$$\hat{Q}(j) = q(j) + \sum_{i=1}^{m} \mu_i \big(\lambda_i |e(j)|\big)^{1-\frac{1}{\alpha_i}} q_i^{\frac{1}{\alpha_i}}(j) - \sum_{i=m+1}^{n} \delta_i \big(\epsilon_i |e(j)|\big)^{1-\frac{1}{\alpha_i}} \hat{q}_i^{\frac{1}{\alpha_i}}(j)$$

with

$$\mu_i = \alpha_i(\alpha_i - 1)^{\frac{1}{\alpha_i}-1}, \quad \delta_i = \alpha_i(1-\alpha_i)^{\frac{1}{\alpha_i}-1} \quad \textit{and} \quad \hat{q}_i(j) = \max\{-q_i(j), 0\},$$

then Eq. (1.3) *is oscillatory.*

4.3. **q-Difference Equations.** Let $[a,b]_q$ denote a q-interval, i.e.,

$$[a,b]_q \equiv [q^a, q^b]_{q^{\mathbb{N}}} = \{q^a, q^{a+1}, q^{a+2}, \ldots, q^b\}, \quad a, b \in \mathbb{N}, \quad q \in \mathbb{R}, \quad q > 1.$$

$\mathcal{H}_q$ denotes the functions defined on $q^{\mathbb{N}} \times q^{\mathbb{N}}$ and satisfying $H(j,j) = 0$ for all j and $H(j,i) \neq 0$ for all $j \neq i$. Note that

$$H_1(t,s) = \frac{H(qt,s) - H(t,s)}{(q-1)t}, \quad H_2(t,s) = \frac{H(t,qs) - H(t,s)}{(q-1)s}.$$

Theorem 4.7. *Suppose that for any given natural number $T \geq t_0$, there exist natural numbers a_1, b_1, a_2, b_2 satisfying $T \leq a_1 < b_1$, $T \leq a_2 < b_2$ such that*

$$r_i(t) \geq 0 \quad \text{for} \quad t \in [a_1, b_1]_q \cup [a_2, b_2]_q, \quad (i = 1, 2, \ldots, n)$$

and

$$(-1)^k e(t) \geq 0 \ (\not\equiv 0) \quad \text{for} \quad t \in [a_k, b_k]_q, \quad (k = 1, 2).$$

Let $\{\eta_1, \eta_2, \ldots, \eta_n\}$ be an n-tuple satisfying (2.1) in Lemma 2.1. If there exist a function $H \in \mathcal{H}_q$ and numbers $q^{c_k} \in (a_k, b_k)_q$ such that

$$\frac{1}{H^2(q^{c_k}, q^{a_k})} \sum_{j=a_k}^{c_k-1} q^j \left[H^2(q^{j+1}, q^{a_k}) Q(q^j) - p(q^j) {H_1}^2(q^j, q^{a_k}) \right]$$
$$+ \frac{1}{H^2(q^{b_k}, q^{c_k})} \sum_{j=c_k}^{b_k-1} q^j \left[H^2(q^{b_k}, q^{j+1}) Q(q^j) - p(q^j) {H_2}^2(q^{b_k}, q^j) \right] > 0$$

for $k = 1, 2$, where

$$Q(t) = r(t) + k_0 |e(t)|^{\eta_0} \prod_{i=1}^{n} r_i^{\eta_i}(t), \qquad k_0 = \prod_{i=0}^{n} {\eta_i}^{-\eta_i}, \qquad \eta_0 = 1 - \sum_{i=1}^{n} \eta_i,$$

then Eq. (1.4) is oscillatory.

Theorem 4.8. *Suppose that for any given natural number $T \geq t_0$, there exist natural numbers a and b satisfying $T \leq a < b$ such that*

$$r_i(t) \geq 0 \quad \text{for} \quad t \in [a,b]_q, \quad (i = 1, 2, \ldots, n).$$

Let $\{\eta_1, \eta_2, \ldots, \eta_n\}$ be an n-tuple satisfying (2.2) in Lemma 2.2. If there exist a function $H \in \mathcal{H}_q$ and a number $q^c \in (a,b)_q$ such that

$$\frac{1}{H^2(q^c, q^a)} \sum_{j=a}^{c-1} q^j \left[H^2(q^{j+1}, q^a) \tilde{Q}(q^j) - p(q^j) {H_1}^2(q^j, q^a) \right]$$
$$+ \frac{1}{H^2(q^b, q^c)} \sum_{j=c}^{b-1} q^j \left[H^2(q^b, q^{j+1}) \tilde{Q}(q^j) - p(q^j) {H_2}^2(q^b, q^j) \right] > 0$$

where

$$\tilde{Q}(t) = r(t) + k_1 \prod_{i=1}^{n} r_i^{\eta_i}(t), \qquad k_1 = \prod_{i=1}^{n} {\eta_i}^{-\eta_i}$$

then Eq. (1.4) *with* $e(t) \equiv 0$ *is oscillatory.*

Theorem 4.9. *Suppose that for any given natural number* $T \geq t_0$, *there exist natural numbers* a_1, b_1, a_2, b_2 *satisfying* $T \leq a_1 < b_1$, $T \leq a_2 < b_2$ *such that*

$$r_i(t) \geq 0 \quad \textit{for} \quad t \in [a_1, b_1]_q \cup [a_2, b_2]_q, \quad (i = 1, 2, \ldots, m)$$

and

$$(-1)^k e(t) > 0 \quad \textit{for} \quad t \in [a_k, b_k]_q, \quad (k = 1, 2).$$

If there exist a function $H \in \mathcal{H}_q$, *positive numbers* λ_i *and* μ_i *with*

$$\sum_{i=1}^{m} \lambda_i + \sum_{i=m+1}^{n} \mu_i = 1,$$

and numbers $q^{c_k} \in (a_k, b_k)_q$ *such that*

$$\frac{1}{H^2(q^{c_k}, q^{a_k})} \sum_{j=a_k}^{c_k-1} q^j \left[H^2(q^{j+1}, q^{a_k}) \hat{Q}(q^j) - p(q^j) {H_1}^2(q^j, q^{a_k}) \right]$$
$$+ \frac{1}{H^2(q^{b_k}, q^{c_k})} \sum_{j=c_k}^{b_k-1} q^j \left[H^2(q^{b_k}, q^{j+1}) \hat{Q}(q^j) - p(q^j) {H_2}^2(q^{b_k}, q^j) \right] > 0$$

for $k = 1, 2$, *where*

$$\hat{Q}(t) = r(t) + \sum_{i=1}^{m} \mu_i \big(\lambda_i |e(t)|\big)^{1-\frac{1}{\alpha_i}} r_i^{\frac{1}{\alpha_i}}(t) - \sum_{i=m+1}^{n} \delta_i \big(\epsilon_i |e(t)|\big)^{1-\frac{1}{\alpha_i}} \hat{r}_i^{\frac{1}{\alpha_i}}(t)$$

with

$$\mu_i = \alpha_i (\alpha_i - 1)^{\frac{1}{\alpha_i} - 1}, \quad \delta_i = \alpha_i (1 - \alpha_i)^{\frac{1}{\alpha_i} - 1} \quad \textit{and} \quad \hat{r}_i(t) = \max\{-r_i(t), 0\},$$

then Eq. (1.4) *is oscillatory.*

5. EXAMPLES

We consider the case $n = 2$. The numbers in Lemma 2.1 become

$$\eta_1 = \frac{1 - \alpha_2(1 - \eta_0)}{\alpha_1 - \alpha_2}, \quad \eta_2 = \frac{\alpha_1(1 - \eta_0) - 1}{\alpha_1 - \alpha_2},$$

where η_0 is any positive number with $\alpha_1 \eta_0 < \alpha_1 - 1$. It follows from Lemma 2.2 that

$$\eta_1 = \frac{1 - \alpha_2}{\alpha_1 - \alpha_2}, \quad \eta_2 = \frac{\alpha_1 - 1}{\alpha_1 - \alpha_2}.$$

In all examples, we have taken $H(t, s) = t - s$, and by the choice of $\eta_0 = 1/4$, $\alpha_1 = 3/2$, $\alpha_2 = 1/2$, we have $k_0 = (1/4)^{-1/4}(5/8)^{-5/8}(1/8)^{-1/8}$ and $k_1 = (1/2)^{-1/2}(1/2)^{-1/2} = 2$. The summations and integrations are computed by using the computer algebra system Mathematica 6.0.

Example 5.1. Consider the forced differential equation

$$x''(t)+m\sin t\,x(t)+m_1\cos t\,|x(t)|^{1/2}x(t)+m_2\sin 2t\,|x(t)|^{-1/2}x(t)=A\cos 6t$$

where $A=0$ or $A=1$, and m, m_1 and m_2 are real numbers with $m_1,m_2>0$.

If $A=1$, we take $a_1=2n\pi+\pi/12$, $c_1=2n\pi+\pi/6$, $b_1=2n\pi+\pi/4=a_2$, $c_2=2n\pi+\pi/3$, and $b_2=2n\pi+5\pi/12$, $n\in\mathbb{N}$, and see that

$$Q(t)=m\sin t+k_0|\cos 6t|^{1/4}\big(m_1\cos t\big)^{5/8}\big(m_2\sin 2t\big)^{1/8}.$$

In case $A=0$, we take $a=2n\pi+\pi/12$, $c=2n\pi+\pi/6$ and $b=2n\pi+\pi/4$ and have

$$\tilde{Q}(t)=m\sin t+k_1(m_1\cos t)^{1/2}\big(m_2\sin 2t\big)^{1/2}.$$

Applying Theorem 4.1 and Theorem 4.2 we see that the above equation is oscillatory when $A=1, m=m_1=1, m_2>2.9\times 10^5$; when $A=1, m=m_2=1, m_1>12.5$; when $A=1, m_1=m_2=1, m>88$; when $A=1, m=-1, m_2=1, m_1>12.6$; when $A=1, m=-1, m_1=1, m_2>3.1\times 10^5$; when $A=0, m=m_1=1, m_2>118$; when $A=0, m=m_2=1, m_1>118$; when $A=0, m_1=m_2=1, m>87.5$; when $A=0, m=-1, m_2=1, m_1>121$; when $A=0, m=-1, m_1=1, m_2>121$.

Example 5.2. Consider for $\mathbb{T}=\mathbb{Z}$ the forced difference equation

$$\begin{aligned}\Delta^2x(k)+m\sin(\frac{\pi k}{12})x(k+1) \quad &+ \quad m_1\cos(\frac{\pi k}{12})|x(k+1)|^{1/2}x(k+1)\\ &+ \quad m_2\sin(\frac{\pi k}{6})|x(k+1)|^{-1/2}x(k+1)=A\sin(\frac{\pi k}{2})\end{aligned}$$

where $A=0$ or $A=1$, and m, m_1 and m_2 are real numbers with $m_1,m_2>0$. If $A=1$, let $a_1=2+24n$, $c_1=3+24n$, $b_1=4+24n=a_2$, $c_2=5+24n$ and $b_2=6+24n$, $n\in\mathbb{N}$, and see that

$$Q(j)=m\sin(\frac{\pi j}{12})+k_0|\sin(\frac{\pi j}{2})|^{1/4}\big(m_1\cos(\frac{\pi j}{12})\big)^{5/8}\big(m_2\sin(\frac{\pi j}{6})\big)^{1/8}.$$

In case $A=0$, we take $a=2+24n$, $c=3+24n$ and $b=4+24n$ and have

$$\tilde{Q}(j)=m\sin(\frac{\pi j}{12})+k_1\big(m_1\cos(\frac{\pi j}{12})\big)^{1/2}\big(m_2\sin(\frac{\pi j}{6})\big)^{1/2}.$$

From Theorem 4.4 and Theorem 4.5 it follows that the equation is oscillatory when $A=1,\ m_1,m_2>0, m>4/\sqrt{3}$; when $A=0,\ m+2\sqrt{3}(m_1m_2)^{1/2}>4$.

Example 5.3. Consider for $\mathbb{T}=q^{\mathbb{N}}=2^{\mathbb{N}}$, the forced 2-difference equation

$$\Delta_2^2x(t)+mx(2t) \quad + \quad m_1|x(2t)|^{1/2}x(2t)+m_2|x(2t)|^{-1/2}x(2t)=Ag(t)$$

where $A=0$ or $A=1$, m, m_1 and m_2 are real numbers with $m_1,m_2>0$, and

$$g(t)=\begin{cases}1, & t\in\{2^{24n+l},\ l=2,4,6,\ldots\}\\ -1, & t\in\{2^{24n+l},\ l=1,3,5,\ldots\}\\ 0, & t\in 2^{\mathbb{N}}\backslash\{2^{24n+l},\ l=1,2,3,\ldots\}.\end{cases}$$

If $A = 1$, we choose $a_1 = 2 + 24n$, $c_1 = 3 + 24n$, $b_1 = 4 + 24n = a_2$, $c_2 = 5 + 24n$ and $b_2 = 6 + 24n$, $n \in \mathbb{N}$. We have

$$Q(t) = m + k_0 m_1^{5/8} m_2^{1/8}.$$

In case $A = 0$, we take $a = 2 + 24n$, $c = 3 + 24n$, $b = 4 + 24n$, and have

$$\tilde{Q}(t) = m + k_1(m_1 m_2)^{1/2}.$$

We conclude from Theorem 4.7 and Theorem 4.8 that the above equation is oscillatory when $A = 1$, $m + k_0 m_1^{5/8} m_2^{1/8} > 0$; when $A = 0$, $m + k_1(m_1 m_2)^{1/2} > 0$.

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