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ON THE ω-LIMIT SETS OF PRODUCT MAPS

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ABSTRACT. Let $\omega(\cdot)$ denote the union of all ω-limit sets of a given map. As the main result of this paper we prove that, for given continuous interval maps $f_1, \ldots, f_m$, the union of all ω-limit sets of the product map $f_1 \times \cdots \times f_m$ and the cartesian product of the sets $\omega(f_1), \ldots, \omega(f_m)$ coincide.

This result enriches the theory of multidimensional permutation product maps, i.e., maps of the form

$$F(x_1, \ldots, x_m) = (f_{\sigma(1)}(x_{\sigma(1)}), \ldots, f_{\sigma(m)}(x_{\sigma(m)})),$$

where σ is a permutation of the set of indices $\{1, \ldots, m\}$. For any such map F, we prove that the set $\omega(F)$ is closed and we also show that $\omega(F)$ cannot be a proper subset of the center of the map F. These results solve open questions mentioned, e.g., in [F. Balibrea, J. S. Cánovas, A. Linero, *New results on topological dynamics of antitriangular maps*, Appl. Gen. Topol.].

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1. INTRODUCTION AND MAIN RESULTS

Let X be a compact metric space and let $C(X)$ be the set of continuous maps from X into itself. Put $I := [0, 1]$ and $I^m := I \times \cdots \times I$ (m times). For a given $\varphi \in C(X)$ and $x \in X$ we consider an *orbit* $\mathrm{Orb}_\varphi(x) := \{\varphi^n(x)\}_{n \geq 0}$, and we define the *$\omega$-limit set* $\omega_\varphi(x)$ of the point x with respect to the map φ as the set of limit points of $\mathrm{Orb}_\varphi(x)$. Finally,

$$\omega(\varphi) = \bigcup_{x \in X} \omega_\varphi(x)$$

is the *ω-limit set* of the map φ.

Consider now $f_1, \ldots, f_m \in C(X)$ and the equality

$$\omega(f_1 \times \cdots \times f_m) = \omega(f_1) \times \cdots \times \omega(f_m). \tag{1.1}$$

Clearly in some particular cases, for instance in the case of m identity maps, this equation is satisfied. On the other side the equality (1.1) need not be true in general,

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even when considering the product of two relatively simple restrictions of interval maps: see Section 6. Nevertheless, in the case of a finite product of interval maps, the equality holds as we present in the following theorem being the main result of this paper.

Theorem A. Let $f_1, \dots, f_m \in C(I)$. Then

$$\omega(f_1 \times \cdots \times f_m) = \omega(f_1) \times \cdots \times \omega(f_m)$$

and, consequently, $\omega(f_1 \times \cdots \times f_m)$ is closed.

In fact the authors were originally motivated by another open problem (see [3]) of the theory of permutation product maps. Here a map $F : I^m \to I^m$ is a *permutation product map* if it is of the form $F(x_1, \dots, x_m) = (f_{\sigma(1)}(x_{\sigma(1)}), \dots, f_{\sigma(m)}(x_{\sigma(m)}))$ for some $f_1, \dots, f_m \in C(I)$, where σ is a permutation of the set of indices $\{1, \dots, m\}$. When σ is the identity we obtain a *product map*. We denote the set of (m-dimensional) permutation product maps by $C_P(I^m)$. For example, in the two-dimensional case, permutation maps are either product maps $F(x,y) = (f(x), g(y))$, or maps of the type $F(x,y) = (g(y), f(x))$, sometimes called *antitriangular* maps, for instance, see [3]. The original motivation of this paper was to study the validity of the equality

$$\omega(F) = \omega(g \circ f) \times \omega(f \circ g). \tag{1.2}$$

for an antitriangular map F. Since, obviously, $F^2 = (g \circ f) \times (f \circ g)$, a close relation between (1.1) and (1.2) is given by the well-known fact $\omega(F) = \omega(F^n)$ for every positive integer n, see [5] (in particular $\omega(F) = \omega(F^2)$). Consequently, it is deduced from Theorem A that (1.2) holds.

Corollary B. Let $f, g \in C(I)$ and let $F(x,y) = (g(y), f(x))$. Then

$$\omega(F) = \omega(g \circ f) \times (f \circ g)$$

and, consequently, $\omega(F)$ is closed.

Moreover the same idea is easily generalizable to the case of general permutation product maps (see Corollary C), since for any $F \in C_P(I^m)$ there is l such that F^l is a product map. For instance, if

$$F(x_1, x_2, x_3, x_4, x_5) = (f_3(x_3), f_4(x_4), f_1(x_1), f_5(x_5), f_2(x_2)),$$

then

$$F^6(x_1, x_2, x_3, x_4, x_5) = (k_1(x_1), k_2(x_2), k_3(x_3), k_4(x_4), k_5(x_5)),$$

with $k_1 = (f_3 \circ f_1)^3$, $k_2 = (f_4 \circ f_5 \circ f_2)^2$, $k_3 = (f_1 \circ f_3)^3$, $k_4 = (f_5 \circ f_2 \circ f_4)^2$ and $k_5 = (f_2 \circ f_4 \circ f_5)^2$, and

$$\omega(F) = \omega(F^6) = \omega(k_1) \times \omega(k_2) \times \omega(k_3) \times \omega(k_4) \times \omega(k_5).$$

In general, for a given antitriangular map $F \in C_P(I^m)$, a permutation σ of order m is a decomposition of cyclic permutations σ_j of order o_j, $j = 1, \ldots, p$, with $p \le m$, $\sum_j o_j = m$, and σ_j acting on the indices $j_1, \ldots, j_{m_j}$, with $\{i_s\}_{s=1}^{m_i} \cap \{j_t\}_{t=1}^{m_j} = \emptyset$, for any $i \ne j$, $\quad i, j \in \{1, \ldots, p\}$. Then $F(x_1, \ldots, x_m) = (f_{\sigma(1)}(x_{\sigma(1)}), \ldots, f_{\sigma(m)}(x_{\sigma(m)}))$ implies that $F^{\mathrm{lcm}(o_j)}$ is a direct product of m interval maps, say $h_i \in C(I)$, $i = 1, 2, \ldots, m$, and we can apply Theorem A. Here $\mathrm{lcm}(o_j)$ denotes the least common multiple of all o_j, $j = 1, \ldots, p$. Hence we have:

Corollary C. If $F \in C_P(I^m)$ is given by

$$F(x_1, \ldots, x_m) = (f_{\sigma(1)}(x_{\sigma(1)}), \ldots, f_{\sigma(m)}(x_{\sigma(m)})),$$

then

$$\omega(F) = \omega(h_1) \times \omega(h_2) \times \omega(h_3) \times \ldots \times \omega(h_m),$$

where each h_j, $j = 1, \ldots, m$, is an appropriate composition of maps from the family $\{f_1, \ldots, f_m\}$, and, consequently, $\omega(F)$ is closed.

Thus, we have found a large class of two-dimensional maps whose ω-limit sets are closed. This property is not held for general two-dimensional maps on I^2. For instance, Kolyada in [11] found such examples of so-called triangular maps $T(x, y) = (f(x), g_x(y))$ of the square I^2 ($f, g_x \in C(I)$ for all $x \in I$). Also, Hero in [9], using the technique of inverse limits, constructed an example of a plane homeomorphism for which the ω-limit set is not closed. In the case of interval or, in fact, graphs maps, $\omega(f)$ is always closed (see [16, 10, 17, 7]), but this is not generally true either for one-dimensional continua. An example of a continuous map on a dendrite whose ω-limit set is not closed can be found in [7].

We also prove (Corollary D) that the center of a permutation product map is contained in its ω-limit set. As expected, this property is not true for general two-dimensional maps or even for triangular maps of the square I^2 (see again [11]).

The key tool is the following. For any direct product of interval maps we have

$$\mathrm{C}(f_1 \times \ldots \times f_m) = \mathrm{C}(f_1) \times \ldots \times \mathrm{C}(f_m)$$

as a consequence of the fact $\mathrm{C}(g) = \overline{\mathrm{P}(g)}$ if $g \in C(I)$ (see [8]), where $\mathrm{C}(\cdot)$ and $\mathrm{P}(\cdot)$ denote the center and the set of periodic points of a map, respectively. Since $\mathrm{C}(f_j) \subseteq \omega(f_j)$, and $\mathrm{C}(F^n) = \mathrm{C}(F)$ for all $F \in C_P(I^m)$ and $n \ge 1$ (see [5]), Corollary C implies:

Corollary D. Let $F \in C_P(I^m)$. Then $\mathrm{C}(F) \subseteq \omega(F)$.

The paper is organized as follows. In Section 3 and 4 we study some properties of ω-limit sets of one-dimensional maps. We use these results to prove Theorem A in Section 5. Section 6 provides a counterexample to Theorem A after replacing the interval I by a Cantor set M. But firstly we introduce some basic notions.

2. DEFINITIONS AND NOTATIONS

For any $\varphi \in C(X)$, an *n-th iteration* is defined inductively by $\varphi^n = \varphi \circ \varphi^{n-1}$, $n \in \mathbb{N}$, where φ^0 is the identity map on X.

For a subset $M \subseteq X$, we use $\overline{M}$ to denote its closure. We say that $x \in X$ is a *periodic point* of a *period* $n \in \mathbb{N}$ if $\varphi^n(x) = x$ and $\varphi^i(x) \neq x$ for $0 < i < n$. If $n = 1$, x is called a *fixed point*. We use $\mathrm{P}(\varphi)$ to denote the set of periodic points of φ. By $\mathrm{R}(\varphi)$ we denote the set of *recurrent points* of φ, i.e., the set of points $x \in X$ satisfying $x \in \omega_\varphi(x)$. Finally, the center $C(\varphi)$ of the map φ is defined as the closure of the set of recurrent points.

Clearly, whenever $x \in \mathrm{R}(\varphi)$ and $U \subseteq X$ is any open neighborhood x, one can find an infinite increasing sequence of return times $\{m_i\}$ such that $\varphi^{m_i}(x) \in U$ for $i = 1, 2, \ldots$. Consequently, recurrent points can be classified depending on the properties of the sequence of return times. For instance, if, for any neighborhood U of x, $\{m_i\}$ is relatively dense in $\mathbb{N}$, that is, the distances $|m_{i+1} - m_i|$ are bounded, then x is *uniformly recurrent*. If, for any U, there is an integer m such that $m_i = m \cdot i$, the point x is *regularly recurrent*. Denote by $x \in \mathrm{UR}(\varphi)$, $x \in \mathrm{RR}(\varphi)$ a uniformly recurrent and regularly recurrent point, respectively. Obviously, any regularly recurrent point is also uniformly recurrent.

A set $M \subseteq X$ is *φ-invariant* (or only *invariant*) if $\varphi(M) \subseteq M$. Further, M is *minimal* with respect to φ if it is nonempty, closed and φ-invariant and if no proper subset of M has these three properties. It is well known that there is a connection between minimal sets and uniformly recurrent points: any point of a given minimal set is uniformly recurrent and, conversely, the ω-limit set of any uniformly recurrent point is minimal.

In what follows we use extensively the notion of periodic interval (now $X = I$ and $f \in C(I)$). In the literature this notion is defined in a number of related, but not exactly equivalent ways, so it is convenient to distinguish among them. We say that a compact interval $J \subseteq I$ is *weakly periodic* (respectively, *periodic*, *strongly periodic*) of *period* $r \in \mathbb{N}$ if $f^r(J) = J$ and r is minimal with respect to this property (respectively, $f^r(J) = J$ and the intervals $\{f^i(J)\}_{i=0}^{r-1}$ have pairwise disjoint interiors, $f^r(J) = J$ and the intervals $\{f^i(J)\}_{i=0}^{r-1}$ are pairwise disjoint). In any case $\mathrm{Orb}_f(J)$, the *orbit* of J, denotes the union $J \cup f(J) \cup \cdots \cup f^{r-1}(J)$.

Remark 2.1. If J is weakly periodic of period r, then either J is strongly periodic of period r, or r is even and $J \cup f^{r/2}(J)$ is strongly periodic of period $r/2$. This result has been proved in [4], and for the sake of completeness, we include here its proof.

It clearly suffices to show that if $1 \leq k < r$ satisfies $f^k(J) \cap J \neq \emptyset$ then $k = r/2$. Suppose not. Then we can assume that $k < r/2$, which in view of the minimality of r implies (after rewriting $g = f^k$) that the intervals J, $g(J)$ and $g^2(J)$ are pairwise different. As neither $g(J)$ can be strictly contained in J nor it can strictly contain J (because $g^r(J) = J$) we can for example assume that $g(J)$ is to the right of J, that is, there are points of $g(J)$ to the right of J, and J is to the left of $g(J)$, that is, there are points of J to the left of $g(J)$.

We claim that $g^2(J)$ is then to the right of $g(J)$. Actually, if $g^2(J)$ is to the left of $g(J)$, then either $g^2(J)$ is to the right of J and so $g^r(J \cup g(J))$ is strictly contained in $J \cup g(J)$, or $g^2(J)$ is to the left of J and then $g^r(J \cup g(J))$ strictly contains $J \cup g(J)$; in both cases we arrive at a contradiction.

Repeating the argument we find that $g^{j+1}(J)$ is to the right of $g^j(J)$ for every j, which is impossible because $g^r(J) = J$. $\square$

Following [14], we say that an ω-limit set ω of $f \in C(I)$ has a *periodic decomposition* of period r if there exists a weakly periodic interval $J \subseteq I$ of period r such that $\omega \subseteq \mathrm{Orb}_f(J)$, and such that for $0 \le i < r$ the convex hulls $\mathrm{conv}(\omega_i) := \mathrm{conv}(f^i(J) \cap \omega)$ have at most endpoints in common.

For a given $f \in C(I)$, consider the system $\{\omega_f(x) : x \in I\}$ of its ω-limit sets partially ordered by inclusion. It is well known that each ω-limit set of f is a subset of some *maximal* ω-limit set (see [15]). Following the terminology of [15], we say that a maximal ω-limit set of f is of the *second kind* (or a *basic set*) if it contains periodic points and is infinite; otherwise we call it a maximal set of f of the *first kind*. We denote by $\mathcal{A}_1(f)$ (respectively, $\mathcal{A}_2(f)$) the system of *infinite* maximal ω-limit sets of the first kind (respectively, of the second kind). The sets from $\mathcal{A}_1(f)$ are also called *solenoidal* sets.

Finally, we say that an interval $J \subset I$ is *wandering* if all its iterates are pairwise disjoint and the orbits of its points do not converge to any periodic orbit.

3. ON SOLENOIDAL SETS

In this section we study ω-limit sets of the first kind. The starting point of this analysis is given by the fact that they exhibit a "periodic" behavior.

Proposition 3.1. [15] *Let $f \in C(I)$ and let $\omega = \omega_f(x)$ be an infinite maximal ω-limit set of f. Then ω is of the first kind if and only if ω has periodic decompositions of arbitrarily high periods.*

From Proposition 3.1, Remark 2.1 and well-known results on solenoidal sets mentioned e.g. in [5] it follows:

Proposition 3.2. *For any solenoidal set $\omega \subset I$ there are a strictly increasing sequence of positive integers $\{r_m\}_{m=1}^{\infty}$ and a decreasing sequence of strongly periodic intervals*

$\{J_m\}$ of periods $\{r_m\}$ such that

$$\omega \subseteq K = \bigcap_{m=1}^{\infty} \bigcup_{i=0}^{r_m-1} f^i(J_m).$$

Moreover, ω is a perfect set and:

(i) *Any one-point connected component of K consists of a regularly recurrent point belonging to ω.*

(ii) *If J is a nondegenerate component of K, then at least one of its endpoints belongs to ω and none of its interior points belongs to ω. Moreover, J is a wandering interval.*

In the next two lemmas we fix a point u belonging to a solenoidal set $\omega = \omega_f(x)$. Without loss of generality we assume that a subsequence of $\{f^n(x)\}$ approaches u from the left, i.e. for any $\varepsilon > 0$ there exists $n \in \mathbb{N}$ such that

$$f^n(x) \in [u-\varepsilon, u]. \tag{3.1}$$

Let $\{r_m\}$ and $\{J_m\}$ be the corresponding sequences of numbers and strongly periodic intervals and define K as in Proposition 3.2. We can assume that

$$u \in J_m \tag{3.2}$$

for every m. In what follows, $\operatorname{Int} M$ denotes the interior of M as a subset of $\mathbb{R}$, that is, $\operatorname{Int} I = (0,1)$.

Lemma 3.3. *For every $\varepsilon > 0$ there is an r-periodic interval J such that $u \in J$ and* $\operatorname{Int} J \subseteq \bigcup_{i=0}^{\infty} f^{ri}([u-\varepsilon, u])$.

Proof. Since the sequence $\{f^n(x)\}$ approaches u from the left, there is $s \in \mathbb{N}$ such that $f^s([u-\varepsilon,u]) \cap [u-\varepsilon,u] \neq \emptyset$. Notice that $f^{si}([u-\varepsilon,u]) \cap f^{s(i+1)}([u-\varepsilon,u]) \neq \emptyset$ for all integers i and each $f^{si}([u-\varepsilon,u])$ is a connected set. Consequently, $M := \bigcup_{i=0}^{\infty} f^{si}([u-\varepsilon,u])$ is an interval.

Moreover, $\{f^{si}(M)\}_{i=0}^{\infty}$ is a decreasing sequence of intervals and we claim that its intersection $S := \bigcup_{i=0}^{\infty} f^{si}(M)$ is a nondegenerate interval. From $\{f^{si}(u)\} \subseteq M$ we take a subsequence $\{f^{sn_k}(u)\}_k$ convergent to $w \in I$. Since $u \in M$, we have $f^{sn_k}(u) \in f^{sn_k}(M)$ for all k. If S is not an interval, then $\bigcap_{i\geq 0} \overline{f^{si}(M)}$ consists of exactly one point w. On the other hand, it is clear that $f^s(w) = w$, so w is a periodic point, which contradicts that $w \in \omega_f(u) \subset \omega$ with ω of the first kind. This ends the claim.

Let $J' = \overline{S}$. Clearly $f^s(J') = J'$. Let r' be the minimal positive integer satisfying $f^{r'}(J') = J'$. Now, according to Remark 2.1 , we have that either J' is r'-periodic (then we define $r = r'$ and $J = J'$), or r' is even, $\operatorname{Int} J' \cap \operatorname{Int} f^{r'/2}(J') \neq \emptyset$, and

$J' \cup f^{r'/2}(J')$ is $r'/2$-periodic (then we define $r = r'/2$ and $J = J' \cup f^{r'/2}(J')$). Notice that in both cases $r|s$.

The statement $\operatorname{Int} J \subseteq \bigcup_{i=0}^{\infty} f^{ri}([u-\varepsilon, u])$ is clear from the construction, so it only rests to show $u \in J$.

To begin with this observe that $u \in K$ and $\omega_f(u) \subseteq \omega$. Moreover, by the construction of J, every limit point of $\{f^{ri}(u)\}$ must belong to J. If $\{u\}$ is a one-point connected component of K, then u is regularly recurrent by Proposition 3.2. Thus, $\{f^{ri}(u)\}$ accumulates at u and $u \in J$.

If $\{u\}$ is not a one-point connected component of K, then, again by Proposition 3.2, it is an endpoint of a nondegenerate component $[u, v]$ of K, which is a wandering interval containing no points of ω in its interior. We emphasize that $u < v$ because we assume condition (3.1). Now, if $\{f^{ri}(u)\}$ accumulates at u, we obtain $u \in J$, so we can assume that $\{f^{ri}(u)\}$ accumulates at $v \in J$. Since the orbit of v is infinite (because it belongs to a wandering interval) and $f^s(J) = J$, some iterate of v belongs to $\operatorname{Int} J$. Then there exists an iterate of u in $\operatorname{Int} J$ and also $f^k(x) \in \operatorname{Int} J$ for some positive integer k. As a consequence of the periodicity of J we get $f^n(x) \in \operatorname{Orb}(J)$ for any $n \geq k$. Since $u \in \omega_f(f^k(x))$ and $\operatorname{Orb}(J)$ is a closed invariant set by f, we conclude that u belongs to some interval $f^t(J)$, $0 \leq t < r$. If $u \notin J$, then $t > 0$ and $f^t(J)$ is to the left of J. In this case, if $[a, b]$ is the (possibly degenerate) interval connecting J and $f^t(J)$, then the f^r-invariance of J and $f^t(J)$ implies that $f^r(a) \leq a$, $f^r(b) \geq b$, hence $[a, b]$ contains some fixed point of f^r. This is impossible because $[u, v]$ is a wandering interval covering $[a, b]$. □

Lemma 3.4. *For every $\varepsilon > 0$ there are $0 < \delta < \frac{\varepsilon}{2}$ and $k \in \mathbb{N}$ such that*

$$[u-\delta, u] \subseteq \bigcap_{n=0}^{\infty} f^{kn}([u-\varepsilon, u]).$$

Proof. Let r and J be as in Lemma 3.3. Let m be such that $r_m > 2r$. From the $r-$periodicity of J, r_m-periodicity of J_m and (3.2) we deduce that all three intervals J_m, $f^r(J_m)$, and $f^{2r}(J_m)$ intersect J. These intervals are pairwise disjoint, so one of them is contained in J (recall that $r_m > 2r$). From here, by Lemma 3.3, it is clear that $J_m \subseteq J$ and r divides r_m since $f^{r_m}(J_m) = J_m \subseteq J$ implies $f^{r_m}(J) = J$.

We claim that u cannot be the left endpoint of J_m. The reason is that, by Proposition 3.2, J_m contains infinitely many points from ω. Hence $\{f^n(x)\}$ eventually falls into the orbit of J_m so it cannot approach u from the left; we also use the strong periodicity of J_m. This contradicts assumption (3.1) and finishes the claim.

Let $0 < \delta < \frac{\varepsilon}{2}$ be such that $[u-\delta, u] \subseteq J_m$. We next show that this number δ is adequate to our purposes (for an appropriate positive integer k). To simplify the notation we next write $g = f^r$, when the terms "orbit", "fixed point", and so on, refer to g. We emphasize that the orbit of J_m is contained in J. Taking into account that $r|r_m$, $r < r_m$ and $r_m > 2$, we deduce that some interval Q of the orbit of J_m is mapped to the right by g, that is, $g(Q) > Q$, and another interval $\tilde{Q}$ of the orbit of J_m is mapped to the left (that is, $g(\tilde{Q}) < \tilde{Q}$). Let $L < R$ be two such consecutive intervals, that is, $g(L) > L, g(R) < R$ and $\operatorname{conv}(L \cup R) \cap \operatorname{Orb}(J_m) = L \cup R$. Then there exists a fixed point p lying between L and R. In particular, $p \in \operatorname{Int} J$.

Notice that, by Lemma 3.3, we have $\operatorname{Int} J \subseteq \bigcup_{i=0}^{\infty} g^i([u-\varepsilon, u])$. Since one of the intervals of the orbit of J_m is contained in $\operatorname{Int} J$, we get that the whole orbit of J_m is contained in $\bigcup_{i=0}^{\infty} g^i([u-\varepsilon, u])$. Since $p \in \operatorname{Int} J$, we get $p \in g^{i'}([u-\varepsilon, u])$ for some i' and then $p \in g^i([u-\varepsilon, u])$ for every $i \geq i'$. Let $[c, d] = \operatorname{conv}(\operatorname{Orb}_g(J_m))$. Now two possibilities arise.

a) We firstly assume that two intervals from the orbit of J_m lying on the same side of p are mapped by g to different sides of p (remember that $r_m > 2r$). There is no loss of generality in assuming that these two intervals, call them R_1 and R_2, are to the right of p. Let v be the right endpoint of $\operatorname{conv}(R_1 \cup R_2)$. Find a point $w \in L$ and a number l such that $g^l(w) = c$. Similarly, there are a point $z \in R$ and a number t such that $g^t(z) = d$. Then $g^l([w, p]) \supseteq [c, p] \supseteq [w, p]$ and $g^t([p, z]) \supseteq [p, d] \supseteq [p, z]$, thus $g^{sl}([w, p]) \supseteq [c, p]$ and $g^{st}([p, z]) \supseteq [p, d]$ for every $s \geq 1$. From this, $g^{lt}([w, z]) \supseteq [c, d]$. Moreover, using that L and R are neighboring, $p \in \operatorname{conv}(L \cup R)$, and the assumption on R_1 and R_2, we obtain $\operatorname{conv}(L \cup R) \subseteq g(\operatorname{conv}(R_1 \cup R_2)) \subseteq g([p, v])$.

Next, we show that $v \in g^j([u-\varepsilon, u])$ for some $j \geq i'$. To see this, by the periodicity of J_m, $v \in \operatorname{Orb}(J_m)$ implies the existence of a point $z' \in \operatorname{Orb}(J_m)$ such that $g^{i'}(z') = v$. Moreover, $z' \in \bigcup_{i=0}^{\infty} g^i([u-\varepsilon, u])$, so $z' \in g^i([u-\varepsilon, u])$ for some $i \geq 0$. Hence $v \in g^j([u-\varepsilon, u])$ for some $j \geq i'$.

Finally,

$$\begin{aligned} g^{j+1+lt}([u-\varepsilon, u]) &\supseteq g^{1+lt}([p, v]) \supseteq g^{lt}(\operatorname{conv}(L \cap R)) \supseteq g^{lt}([w, z]) \\ &\supseteq [c, d] \supseteq J_m \supseteq [u-\delta, u], \end{aligned}$$

and similarly $g^i([u-\varepsilon, u]) \supseteq [c, d] \supseteq J_m \supseteq [u-\delta, u]$ for every $i \geq j+1+lt$. Now it suffices to take $k = r(j+1+lt)$ to finish this part of the proof.

b) Now we assume that all intervals to the left of p are mapped to the right of p, and conversely. Then the convex hulls of the intervals from the orbit of J_m to the same side of p are mapped by g^2 onto themselves. For instance assume that J_m (hence u) is to the left of p and find l' such that $g^{2l'}(u)$ belongs to an interval from the orbit of J_m to the left of L (this is possible because $r_m > 2r$). This number l' can be taken as large as necessary and, consequently, we can assume that $g^{2l'}([u-\varepsilon, u])$ contains p (and then also L). Find t such that $g^{2t}(L)$ is the interval from the orbit of J_m containing the endpoint c. Then

$$g^{2l'+2t}([u-\varepsilon, u]) \supseteq g^{2t}(\operatorname{conv}(L \cap \{p\})) \supseteq [c, p] \supseteq J_m \supseteq [u-\delta, u],$$

and similarly $g^{2l}([u-\varepsilon,u]) \supseteq [c,p] \supseteq J_m \supseteq [u-\delta,u]$ for every $l \geq l'+t$, since $g^2([c,p]) \supseteq [c,p]$. Thus, if j is an even number large enough, the statement of the lemma is valid for $k = rj$. □

We are ready to prove the main result of this section.

Proposition 3.5. *Let $f \in C(I)$ and let $x \in I$. Assume that $\omega_f(x)$ is solenoidal. Then for every $u \in \omega_f(x)$ there is a sequence $\{k_j\}_{j=1}^{\infty}$ of positive integers with the following property: Let $\{s_j\}$ be a sequence of positive integers and put $n_j = \sum_{i=1}^{j} s_i k_i$ for every j. Then there is $z \in I$ such that $\{f^{n_j}(z)\}$ converges to u.*

Proof. Take any $\varepsilon_0 > 0$ and construct a sequence $\{\varepsilon_j\}_{j=0}^{\infty}$ such that, for every $j \geq 1$, if $\varepsilon = \varepsilon_{j-1}$, then ε_j and k_j are the numbers δ and k from Lemma 3.4, respectively. Then $\varepsilon_j \to 0$.

Let $\{s_j\}$ be a sequence of positive integers and write $n_j = \sum_{i=1}^{j} s_i k_i$. From Lemma 3.4 we have $f^{s_i k_i}([u-\varepsilon_{i-1},u]) \supseteq [u-\varepsilon_i,u]$ for every i.

Find a closed subinterval $I_0 \subseteq [u-\varepsilon_0,u]$ so that $f^{n_1}(I_0) = f^{s_1k_1}(I_0) = [u-\varepsilon_1,u]$. Observe that

$$f^{n_2}(I_0) = f^{s_2k_2+s_1k_1}(I_0) = f^{s_2k_2}([u-\varepsilon_1,u]) \supseteq [u-\varepsilon_2,u].$$

Then, inside I_0 we can find a closed subinterval I_1 holding $f^{n_2}(I_1) = [u-\varepsilon_2,u]$ and such that $f^{n_1}(I_1) \subseteq [u-\varepsilon_1,u]$. Proceeding by induction, we are able to find a decreasing sequence of subintervals

$$I_0 \supseteq I_1 \supseteq I_2 \supseteq \cdots \supseteq I_j \supseteq I_{j+1} \supseteq \cdots$$

satisfying $f^{n_j}(I_{j-1}) = [u-\varepsilon_j,u]$ and $f^{n_i}(I_{j-1}) \subseteq [u-\varepsilon_i,u]$ for all $j \geq 1$ and $i < j$. Hence $\{I_j\}$ is a decreasing sequence of nonempty compact sets. Let $z \in \bigcap_j I_j \neq \emptyset$. Then $f^{n_j}(z) \to u$. □

4. ON BASIC SETS

Now we are going to analyze a useful property of ω-limit sets of the second kind. Recall that $\varphi \in C(X)$ is *topologically mixing* if for given two nonempty open sets $U, V \subseteq X$ there is $n_0 \in \mathbb{N}$ such that $\varphi^n(V) \cap U \neq \emptyset$ for all $n \geq n_0$. We say that a φ-invariant set $Y \subseteq X$ is *topologically mixing* if $\varphi|_Y$ is topologically mixing. Following Proposition 3.1, if $f \in C(I)$ and $\omega \in \mathcal{A}_2(f)$, then it admits a maximal finite decomposition into periodic portions.

Lemma 4.1. *Given $f \in C(I)$ and $x \in I$, assume that $\omega := \omega_f(x) \in \mathcal{A}_2(f)$. Let $\{\omega_i\}_{i=0}^{s-1}$ be a maximal periodic decomposition of ω. Then, for any $i = 0, \ldots, s-1$, $f^s|_{\omega_i}$ is topologically mixing.*

Proof. Clearly, it suffices to show that one periodic portion (say ω_0) of ω is topologically mixing. From [6, Theorem 4.1] we know that $f^s|_{\omega_0}$ is almost conjugated to some mixing interval map $g : J \to J$. This means that there exists a monotone continuous surjective map $\varphi : \omega_0 \to J$ satisfying the following properties:

(i) $g \circ \varphi = \varphi \circ f^s$,
(ii) for each $z \in J$, $\varphi^{-1}(z)$ contains at most two points of the set ω_0.

Now we are ready to show that $f^s|_{\omega_0}$ is topologically mixing. Let U, V be any nonempty open sets in ω_0. After taking if necessary smaller sets we can assume that both $\varphi(U)$ and $\varphi(V)$ are open sets in J and, moreover, if $\varphi(x) = \varphi(y)$ for some points $x, y \in \omega_0$, then either $x, y \in U$, or $x, y \notin U$ (we use (ii)).

Since g is mixing, $g^i(\varphi(V)) \cap \varphi(U) \neq \emptyset$ for any integer i greater than some $n_0 \in \mathbb{N}$. Then $\varphi((f^s)^i(V)) \cap \varphi(U) \neq \emptyset$ for any $i \geq n_0$ by (i). But this also implies $(f^s)^i(V) \cap U \neq \emptyset$ for any $i \geq n_0$, because if $x \in (f^s)^i(V)$ and $y \in U$ are such that $\varphi(x) = \varphi(y)$, then $y \in U$ forces $x \in U$ as well. We have shown that $f^s|_{\omega_0}$ is topologically mixing. □

5. PROOF OF THEOREM A

Let $f_1, \dots, f_m \in C(I)$. We have trivially

$$\omega(f_1 \times \cdots \times f_m) \subseteq \omega(f_1) \times \cdots \times \omega(f_m).$$

Hence it suffices to show that if $(x_1, \dots, x_m) \in I^m$, then

$$\omega_{f_1}(x_1) \times \cdots \times \omega_{f_m}(x_m) \subseteq \omega(f_1 \times \cdots \times f_m). \tag{5.1}$$

There is no loss of generality in assuming that all ω-limit sets to the left of (5.1) are maximal. Now two possibilities arise. First assume that some of these ω-limit sets, say $\omega_{f_1}(x_1)$, is of the second kind. By Lemma 4.1, $\omega_{f_1}(x_1)$ has a finite number (say s) of topologically mixing portions. Since

$$\omega(f_1 \times \cdots \times f_m) = \omega(f_1^s \times \cdots \times f_m^s)$$

and $\omega_{f_r}(x_r) = \bigcup_{i=0}^{s-1} \omega_{f_r^s}(f_r^i(x_r))$, $r = 1, \dots, m$, it is not restrictive to assume that $\omega_{f_1}(x_1)$ itself is topologically mixing. Then (5.1) is exactly Theorem 12 from [1].

Thus, since a direct product of mixing maps is mixing, it only rests to consider the case when every set $\omega_{f_r}(x_r)$, $r = 1, \dots, m$, is either finite (a periodic orbit) or solenoidal. In both situations, if the points $u_r \in \omega_{f_r}(x_r)$ are given, then there are sequences $\{k_{r,j}\}_{j=1}^{\infty}$ of positive integers with the property described in Proposition 3.5 for u and $\{k_j\}$ (if $\omega_{f_r}(x_r)$ is periodic of period s, then $k_{r,j} = sj$ does the job). This property allows us, after defining $n_j = \sum_{i=1}^{j} k_{1,i} k_{2,i} \cdots k_{m,i}$, to find points $z_1, \dots, z_m$ such that each sequence $\{f_r^{n_j}(z_r)\}$ converges to u_r, $r = 1, \dots, m$. Hence $(u_1, \dots, u_m) \in$

$\omega_{f_1\times\cdots\times f_m}(z_1,\ldots,z_m)$. This implies (5.1) and finishes the proof of Theorem A in general.

6. A COUNTEREXAMPLE

The following map was used in [3] as a counterexample to the equality $\mathrm{UR}(f\times f)=\mathrm{UR}(f)\times\mathrm{UR}(f)$. We next show that it can be used to contradict a similar equality concerning ω-limit points - namely $\omega(f\times g)=\omega(f)\times\omega(g)$ (see the introduction for the complete explanation).

Example 6.1. There are a Cantor set $M\subset I$ and a continuous map $f:M\to M$ such that

$$\omega(F)\neq\omega(f)\times\omega(f)$$

for the product map $F=f\times f$.

It is well known (see [13]) that if the number $\alpha\in(0,1)$ is appropriately chosen, then $h\in C(I)$ defined by $h(x)=\max\{\alpha,1-|2x-1|\}$ is of type 2^∞ in the Sharkovsky ordering and has a unique h-minimal Cantor set $M\subset I$, which is a solenoidal set for h. More precisely, there is a decreasing sequence of strongly periodic intervals $\{J_m\}$ of periods $\{2^m\}$ such that $M\subseteq K=\bigcap_{m=1}^\infty\bigcup_{i=0}^{2^m-1}h^i(J_m)$. Moreover, the interval $[u,v]$ of constancy of h is one of the components of K, with both u and v belonging to M.

Let f denote the restriction of h to M. We have $\omega(f)=M$ due to the minimality of M. Then it suffices to show that $(u,v)\notin\omega(F)$.

Suppose that $(u,v)\in\omega_F(x,y)$ for some $(x,y)\in M\times M$. Then there is a sequence $n_i\to\infty$ such that $h^{n_i}(x)\to u$ and $h^{n_i}(y)\to v$.

Fix arbitrarily the number m. Then u and v belong to the same interval $h^k(J_m)$ of the h-orbit of J_m. Since both $h^{n_i}(x)$ and $h^{n_i}(y)$ belong to $h^k(J_m)$, and $x,y\in M$ (so, $x\in\omega_h(h^{n_i}(x))$ and $y\in\omega_h(h^{n_i}(y))$), we get that there is an interval $h^l(J_m)$ containing both x and y. Since m was arbitrarily chosen, we conclude that the interval having x and y as its endpoints is wandering for h. Hence $|h^n(x)-h^n(y)|\to 0$ as $n\to\infty$, which is impossible. $\square$

Notice that a slight variation of the above example provides an antitriangular map $G(x,y)=(g(y),f(x))$ from M^2 into itself for which $\omega(G)\neq\omega(g\circ f)\times\omega(f\circ g)$; just take f as above, put $g=\mathrm{Id}$ and realize that $G^2=F$. It could be an interesting problem to search for conditions ensuring the equality mentioned in Theorem A.

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